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ON THE DIMENSION OF AN IRREDUCIBLE TENSOR REPRESENTATION OF THE GENERAL LINEAR GROUP GL(d)

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1. As has long been known, the irreducible tensor representations of GL(d) of rank n may be labeled by means of the irreducible representations of S_n, i.e., by means of the Young diagrams $[\lambda]$, where $\lambda_1+\lambda_2+\cdots+\lambda_r=n$. We denote such a tensor representation by $\langle\lambda\rangle$. Using Young's raising operator R_{ij} we can write [1, p. 42]

$$(1.1)\qquad [\lambda] = \prod(1-R_{ij})[\lambda]_1\cdot[\lambda_2]\cdot\cdots\cdot[\lambda_r] = |\,[\lambda_i-i+j]\,|^{\cdot}$$

where the dot denotes the inducing process. For example, $[3]\cdot[2]$ is that representation of S_5 induced by the identity representation of its subgroup $S_3\times S_2$. Analogously, we may write [1, p. 59]

$$(1.2)\qquad \langle\lambda\rangle = |\langle\lambda_i-i+j\rangle|^{\times}$$

where the components of the determinant are multiplied to yield Kronecker product representations of GL(d).

The *dimension* f^λ of $[\lambda]$ is easily calculated from (1.1) if we observe that the dimension of $[\lambda_1]\cdot[\lambda_2]\cdot\cdots\cdot[\lambda_r]$ is $n!/\lambda_1!\,\lambda_2!\ldots\lambda_r!$ so that

$$(1.3)\qquad f^\lambda = n!\left|\frac{1}{(\lambda_i-i+j)!}\right| = n!/H^\lambda$$

where H^λ is the product of the hook lengths of $[\lambda]$. The purpose of the present note is to show how an exactly analogous procedure leads to the expression $G^\lambda(d)/H^\lambda$ for the dimension $\delta^\lambda(d)$ of $\langle\lambda\rangle$. Incidentally, we suppose that $d\geq n$.

2. To begin with, we observe that the dimension of the symmetric tensor representation of GL(d) denoted by $\langle\lambda_i\rangle$ is

$$(2.1)\qquad d(d+1)(d+2)\cdots(d+\lambda_i-1)/\lambda_i!$$

which is just the number of ways of choosing λ_i things from d, with repetitions allowed [2, p. 46] so that

$$(2.2)\qquad \delta^\lambda(d) = \left|\frac{((d-1)+(\lambda_i-i+j))!}{(d-1)!\,(\lambda_i-i+j)!}\right|.$$

Proceeding as in the case of f^λ, let us factor out the *denominators* $(d-1)!\,(\lambda_i-i+r)!$ in the last column ($j=r$) of the determinant, and the *numerators* in the first column ($j=1$) to yield the expression

$$(2.3)\qquad \delta^\lambda(d) = \left[\frac{\prod_{i=1}^r (d+\lambda_i-i)!}{((d-1)!)^r\prod_{i=1}^r(\lambda_i-i+r)!}\right]\Delta$$

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where each term of the determinant Δ contains $r-1$ factors in the numerator with denominator 1.

EXAMPLE. Let us suppose that $[\lambda]=[3, 2]$ so that

$$\delta^{3,2}(d) = \begin{vmatrix} \dfrac{(d+2)!}{(d-1)!\,3!} & \dfrac{(d+3)!}{(d-1)!\,4!} \\[2ex] \dfrac{d!}{(d-1)!\,1!} & \dfrac{(d+1)!}{(d-1)!\,2!} \end{vmatrix}$$

$$= \left[\frac{(d+2)!\,d!}{((d-1)!)^2 4!\,2!}\right] \begin{vmatrix} 4 & d+3 \\ 2 & d+1 \end{vmatrix}$$

after dividing out as described above. If we subtract the first from the second column of the modified determinant we obtain a common factor $(d-1)$ which is to be divided out, and then subtract the second row from the first so that

$$\delta^{3,2}(d) = \frac{d(d+1)(d+2)\cdot d}{4!\,2!} \begin{vmatrix} 4 & d-1 \\ 2 & d-1 \end{vmatrix}$$

$$= \frac{d(d+1)(d+2)\cdot(d-1)d}{4\cdot 3\cdot 2} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= G^{3,2}(d)/H^{3,2}$$

where we have written $G^\lambda(d)=\prod_{i,j}(d+j-i)$.

It is important to see what is going on in the above example. By subtracting *columns* of Δ we have obtained common factors which can be divided out from the columns to yield the factors of $G^\lambda(d)$ associated with the (i, j)-nodes of $[\lambda]$ for which $i>j$, leaving the determinant Δ'. By subtracting *rows* of Δ' we have then obtained factors which can be divided out from the rows to yield the *missing* hooks [1, p. 44] in $[\lambda]$, which, cancelling the corresponding factors of $\prod_{i=1}^{r}(\lambda_i-i+r)!$ in the denominator, yield H^λ. Thus we have

$$\delta^\lambda = G^\lambda(d)/H^\lambda.$$

It should be noted that the procedure for factoring the determinant (2.2) so far as the denominators are concerned, applies also to (1.3). The added complication in the case of δ^λ arises via the numerators to which we have applied a similar procedure.

REFERENCES

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