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NATURALLY ORDERED REGULAR SEMIGROUPS WITH MAXIMUM INVERSES

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Introduction

Let S be a regular semigroup. An inverse subsemigroup S° of S is called an *inverse transversal* if S° contains a unique inverse of each element of S. An inverse transversal S° of S is called *multiplicative* if $x^\circ xyy^\circ$ is an idempotent of S° for every $x, y \in S$, where x° denotes the unique inverse of $x \in S$ in S°. In Section 1, we obtain a necessary and sufficient condition in order for inverse transversals to be multiplicative.

It is well known that the set $E(S)$ of idempotents of a regular semigroup S can be partially ordered by setting $e\omega f \Leftrightarrow ef = fe = e$ for any $e, f \in E(S)$. This partial order is called the natural order on $E(S)$. A partially ordered semigroup $S(\cdot, \leqq)$ is called *naturally ordered* if the order $\leqq$ extends the natural order ω on $E(S)$, i.e. $ef = fe = e$ implies $e \leqq f$. In this case, there is no assumption that $e \leqq f$ implies $ef = fe = e$.

The interesting results on a naturally ordered regular semigroup S with a greatest idempotent u have been obtained by Blyth and McFadden in [1] as follows.

(a) Every $x \in S$ has a maximum inverse x° in S.

(b) The set $S^\circ = \{x^\circ : x \in S\}$ of maximum inverses of S forms a multiplicative inverse transversal of S, and $S^\circ = uSu$.

Let S be a naturally ordered regular semigroup in which each element x has a maximum inverse x°. The Green's relation $\mathscr{R}$ [resp. $\mathscr{L}$] is called *regular* on S if $x \leqq y$ implies $xx^\circ \leqq yy^\circ$ [resp. $x^\circ x \leqq y^\circ y$] for any $x, y \in S$.

A structure theorem on a naturally ordered regular semigroup with maximum inverses has been obtained by Blyth and McFadden ([2, Theorem 6.2]). In this case, there are the assumptions that the set S° of maximum inverses is a multiplicative inverse transversal of S and that $\mathscr{R}$ and $\mathscr{L}$ are regular on S. In Section 2, we show that if $\mathscr{R}$ and $\mathscr{L}$ are regular on a naturally ordered regular semigroup S, then the set S° of maximum inverses of S is a multiplicative inverse transversal of S.

An idempotent u of a regular semigroup S is called medial if $x = xux$ for every $x \in \langle E(S) \rangle$, where $\langle E(S) \rangle$ denotes the subsemigroup of S generated by the set $E(S)$ of idempotents of S. A medial idempotent u is called normal if $u\langle E(S)\rangle u$ is a semilattice. In [5], McAlister and McFadden have shown that a regular semigroup S with a normal medial idempotent u can be naturally ordered in such a way that u is the greatest idempotent of S.

In Section 3, we show that if a regular semigroup S has a multiplicative inverse transversal S°, and if x° is the unique inverse of $x \in S$ in S°, then S can be naturally ordered in such a way that x° is the maximum inverse of x, and that $\mathcal{R}$ and $\mathcal{L}$ are regular on S.

1. Multiplicative inverse transversals

Let S be a regular semigroup with an inverse transversal S°. If $x \in S$, the unique element of $V(x) \cap S^\circ$ is denoted by x°, and $x^{\circ\circ}$ denotes $(x^\circ)^\circ$, where $V(x)$ denotes the set of inverses of x. A subset Q of S is called a *quasi-ideal* if $QSQ \subseteq Q$.

We restate some results about regular semigroups with inverse transversals which will be used in this paper:

Let S be a regular semigroup with an inverse transversal S°. Then:

(1°) If S° is multiplicative, then S° is a quasi-ideal of S ([**4**, Lemma 1.2]).

(2°) If S° is a quasi-ideal of S, then

(i) $e^\circ \in E(S^\circ)$ and $gg^\circ = g$ [resp. $f^\circ f = f$] imply $e^\circ g = e^\circ g^\circ$ [resp. $fe^\circ = f^\circ e^\circ$] ([**4**, Lemma 1.6]),

(ii) $x\mathcal{R}y$ [resp. $x\mathcal{L}y$] implies $xx^\circ = yy^\circ$ [resp. $x^\circ x = y^\circ y$] for every $x, y \in S$ ([**4**, Proposition 1.7]), and

(iii) $axb = ax^{\circ\circ}b$ for every $a, b \in S^\circ$ and for every $x \in S$ ([**7**, Proposition 1.7]).

Lemma 1.1. *Let S be a regular semigroup with an inverse transversal S° which is a quasi-ideal of S. Suppose that $e^\circ \in E(S^\circ)$ for every $e \in E(S)$. If $x' \in V(x)$ for any $x \in S$, then $(xx')^\circ = x^{\circ\circ}x^\circ$, $(x'x)^\circ = x^\circ x^{\circ\circ}$ and $(x')^\circ = x^{\circ\circ}$.*

Proof. Let $x \in S$ and let $x' \in V(x)$. Since $xx'\mathcal{R}x$, $xx'(xx')^\circ = xx^\circ$. By the assumption, $(xx')^\circ \in E(S^\circ)$, so that $xx^\circ(xx')^\circ = xx'(xx')^\circ(xx')^\circ = xx'(xx')^\circ = xx^\circ$ and $(xx')^\circ xx^\circ = (xx')^\circ xx'(xx')^\circ = (xx')^\circ$ which shows $xx^\circ \mathcal{L}(xx')^\circ$. Thus we have $x^{\circ\circ}x^\circ = (xx^\circ)^\circ xx^\circ = (xx')^{\circ\circ}(xx')^\circ = (xx')^\circ$. Similarly $x^\circ x^{\circ\circ} = (x'x)^\circ$. By using the above facts, we have

$$\begin{aligned} x^{\circ\circ}x'x^{\circ\circ} &= x^{\circ\circ}x'xx'x^{\circ\circ}x^\circ x^{\circ\circ} = x^{\circ\circ}x'xx'(xx')^\circ x^{\circ\circ} = x^{\circ\circ}x'xx^\circ x^{\circ\circ} \\ &= x^{\circ\circ}(x'x)^{\circ\circ}x^\circ x^{\circ\circ} = x^{\circ\circ}x^\circ x^{\circ\circ}x^\circ x^{\circ\circ} = x^{\circ\circ} \end{aligned}$$

and $x'x^{\circ\circ}x' = x'xx'x^{\circ\circ}x^\circ x^{\circ\circ}x' = x'xx'(xx')^\circ x^{\circ\circ}x' = x'xx^\circ x^{\circ\circ}x' = x'x(x'x)^\circ x' = x'$. Consequently $(x')^\circ = x^{\circ\circ}$.

If S° is multiplicative, for any $e \in E(S)$, $e^\circ = e^\circ ee^\circ = e^\circ eee^\circ \in E(S^\circ)$. Thus we obtain:

Corollary 1.2. *Let S be a regular semigroup with a multiplicative inverse transversal S°. If $x' \in V(x)$ for $x \in S$, then $(xx') = x^{\circ\circ}x^\circ$, $(x'x)^\circ = x^\circ x^{\circ\circ}$* and *$(x')^\circ = x^{\circ\circ}$.*

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Theorem 1.3. *Let S be a regular semigroup with an inverse transversal S°. Then S° is multiplicative if and only if S° is a quasi-ideal of S and $e^\circ \in E(S^\circ)$ for every $e \in E(S)$.*

Proof. Suppose that S° is a quasi-ideal of S and $e^\circ \in E(S^\circ)$ for every $e \in E(S)$. Let $x, y \in S$ and put $e = xx^\circ$ and $f = y^\circ y$. Then $fe \in S^\circ S S^\circ \subseteq S^\circ$, so that $(fe)^{\circ\circ} = fe$. Since $e(fe)^\circ f \in V(fe)$, by Lemma 1.1 $fe = (fe)^{\circ\circ} = (e(fe)^\circ f)^\circ$. Since $e(fe)^\circ f \in E(S)$, by the assumption $(e(fe)^\circ f)^\circ \in E(S^\circ)$. Consequently $y^\circ y x x^\circ \in E(S^\circ)$. Thus S° is multiplicative.

From (1°) the converse is true.

2. Ordered regular semigroups with maximum inverses

Let S be a partially ordered regular semigroup in which each element has the maximum inverse. If $x \in S$, the maximum element of $V(x)$ is denoted by x°, and $x^{\circ\circ}$ denotes $(x^\circ)^\circ$. The set of maximum inverse of S is denoted by S°, i.e. $S^\circ = \{a \in S : a = x^\circ$ for some $x \in S\}$. The $\mathscr{R}$-class [resp. $\mathscr{L}$-class] containing $x \in S$ is denoted by R_x [resp. L_x].

Proposition 2.1. *Let S be a partially ordered regular semigroup with maximum inverses. Then:*

(1) *ee° [resp. $e^\circ e$] is the maximum idempotent of R_e [resp. L_e] for every $e \in E(S)$.*

(2) *$(x^\circ x)^\circ x^\circ = x^\circ (xx^\circ)^\circ = x^\circ$ for every $x \in S$.*

(3) *xx° [resp. $x^\circ x$] is the maximum idempotent of R_x [resp. L_x] for every $x \in S$.*

(4) *$x \mathscr{R} y$ [resp. $x \mathscr{L} y$] implies $xx^\circ = yy^\circ$ [resp. $x^\circ x = y^\circ y$] for every $x, y \in S$.*

(5) *$(xx^\circ)^\circ = x^{\circ\circ} x^\circ$ and $(x^\circ x)^\circ = x^\circ x^{\circ\circ}$ for every $x, y \in S$.*

Proof. (1) Let $e, f \in E(S)$ with $e \mathscr{R} f$. Since $f \in V(e)$, $f \leqq e^\circ$. Thus $f = ef \leqq ee^\circ$. Since $e \mathscr{R} ee^\circ$, ee° is the maximum idempotent of R_e.

(2) Let $x \in S$. Since $x^\circ x \in V(x^\circ x)$, $x^\circ x \leqq (x^\circ x)^\circ$, so that $x^\circ \leqq (x^\circ x)^\circ x^\circ$. Conversely, since $(x^\circ x)^\circ x^\circ \in V(x)$, $(x^\circ x)^\circ x^\circ \leqq x^\circ$. Thus $(x^\circ x)^\circ x^\circ = x^\circ$.

(3) Let $x \in S$. Then by (2) $xx^\circ (xx^\circ)^\circ = xx^\circ$. Therefore, by (1) xx° is the maximum idempotent of $R_{xx^\circ} = R_x$.

(4) By (3), this is clear.

(5) Let $x \in S$. Since $x^\circ \mathscr{R} x^\circ x$, by (3) $x^\circ x^{\circ\circ} = x^\circ x (x^\circ x)^\circ$. By (2) we have $x^\circ x (x^\circ x)^\circ = (x^\circ x)^\circ x^\circ x (x^\circ x)^\circ = (x^\circ x)^\circ$, so that $x^\circ x^{\circ\circ} = (x^\circ x)^\circ$.

The following fact is very useful.

If S is a naturally ordered semigroup, then $e \leqq f$ for $e, f \in E(S)$ implies $e = efe$ ([1, Theorem 1.1]).

Proposition 2.2. *Let S be a naturally ordered regular semigroup with maximum inverses. Then:*

(1) *$e^{\circ\circ}$ is an inverse of e for every $e \in E(S)$.*

(2) *$e^{\circ\circ}$ is an idempotent for every $e \in E(S)$.*

(3) *If* e° *is an idempotent for* $e \in E(S)$, *then* $e^\circ = e^{\circ\circ}$.

(4) $x^{\circ\circ\circ} = x^\circ$ *for every* $x \in S$.

Proof. (1) Let $e \in E(S)$. Since $e \in V(e)$ and $e \in V(e^\circ)$, $e \leqq e^\circ$ and $e \leqq e^{\circ\circ}$, so that $e \leqq e^\circ e^{\circ\circ}$ and $e \leqq e^{\circ\circ}e^\circ$. By the above fact, $e = ee^\circ e^{\circ\circ}e = ee^{\circ\circ}e^\circ e$. Thus we have $e^{\circ\circ}ee^{\circ\circ} = e^{\circ\circ}e^\circ ee^\circ e^{\circ\circ}eee^{\circ\circ}e^\circ ee^\circ e^{\circ\circ} = e^{\circ\circ}e^\circ ee^\circ e^{\circ\circ} = e^{\circ\circ}$ and $ee^{\circ\circ}e = ee^{\circ\circ}e^\circ eee^\circ e^{\circ\circ}e = e$. Consequently $e^{\circ\circ} \in V(e)$.

(2) By (1), $ee^{\circ\circ} \in V(e)$, so that $ee^{\circ\circ} \leqq e^\circ$. Since $e^\circ ee^{\circ\circ}e^\circ = e^\circ ee^{\circ\circ}e^\circ ee^\circ = e^\circ ee^\circ = e^\circ$ and $ee^{\circ\circ}e^\circ ee^{\circ\circ} = ee^{\circ\circ}$, $ee^{\circ\circ} \in V(e^\circ)$, so that $ee^{\circ\circ} \leqq e^{\circ\circ}$. Thus we have $e^{\circ\circ}e^{\circ\circ} = e^{\circ\circ}ee^{\circ\circ}e^{\circ\circ} \leqq e^{\circ\circ}e^\circ e^{\circ\circ} = e^{\circ\circ} = e^{\circ\circ}ee^{\circ\circ} \leqq e^{\circ\circ}e^{\circ\circ}$. Consequently $(e^{\circ\circ})^2 = e^{\circ\circ}$

(3) Since $e^{\circ\circ} \in V(e)$, $e^{\circ\circ} \leqq e^\circ$. If e° is an idempotent, then $e^\circ \in V(e^\circ)$, so that $e^\circ \leqq e^{\circ\circ}$. Consequently $e^\circ = e^{\circ\circ}$.

(4) Let $x \in S$. By (5) of Proposition 2.1, $(xx^\circ)^\circ = x^{\circ\circ}x^\circ$ is an idempotent, so that by (3) $(xx^\circ)^\circ = (xx^\circ)^{\circ\circ}$. Thus we have $x^{\circ\circ}x^\circ = (xx^\circ)^\circ = (xx^\circ)^{\circ\circ} = x^{\circ\circ}x^{\circ\circ\circ}$, and similarly $x^\circ x^{\circ\circ} = x^{\circ\circ\circ}x^{\circ\circ}$. Consequently $x^{\circ\circ\circ} = x^{\circ\circ\circ}x^{\circ\circ}x^{\circ\circ\circ} = x^\circ x^{\circ\circ}x^\circ = x^\circ$.

In the following Lemmas 2.3, 2.4 and 2.5, S denotes a naturally ordered regular semigroup with maximum inverses and suppose that $\mathscr{R}$ and $\mathscr{L}$ are regular on S.

Lemma 2.3. $V(x) \cap S^\circ = \{x^\circ\}$ *for every* $x \in S$.

Proof. Let $x \in S$ and let $a \in V(x) \cap S^\circ$. Then $a = y^\circ$ for some $y \in S$. By (4) of Proposition 2.2, $a = y^\circ = y^{\circ\circ\circ} = a^{\circ\circ}$. Since $a \in V(x)$, $a \leqq x^\circ$. Thus $aa^\circ \leqq x^\circ x^{\circ\circ}$ since $\mathscr{R}$ is regular on S. Conversely, $x \in V(a)$ implies $x \leqq a^\circ$, so that $x^\circ x \leqq a^{\circ\circ}a^\circ = aa^\circ$ since $\mathscr{L}$ is regular on S. Thus we have $x^\circ x^{\circ\circ} = x^\circ xx^\circ x^{\circ\circ} = x^\circ x(x^\circ x)^\circ \leqq aa^\circ(aa^\circ)^\circ = aa^\circ$. Consequently $x^\circ x^{\circ\circ} = aa^\circ$. Similarly $x^{\circ\circ}x^\circ = a^\circ a$. Therefore $a^\circ = a^\circ axaa^\circ = x^{\circ\circ}x^\circ xx^\circ x^{\circ\circ} = x^{\circ\circ}$, so that $a = a^{\circ\circ} = x^{\circ\circ\circ} = x^\circ$.

Lemma 2.4. $e^\circ \in E(S^\circ)$ for every $e \in E(S)$.

Proof. Let $e \in E(S)$. By Proposition 2.2, $e^{\circ\circ} \in V(e) \cap E(S^\circ)$. Since $e^\circ \in V(e) \cap S^\circ$, by Lemma 2.3 $e^\circ = e^{\circ\circ}$. Consequently e° is an idempotent of S°.

Lemma 2.5. $(xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ(xyy^\circ)^\circ$ *for every* $x, y \in S$.

Proof. Let $x, y \in S$. Then $xy\mathscr{L}x^\circ xy$, so that $(xy)^\circ xy = (x^\circ xy)^\circ x^\circ xy$. Since

$$xy(xy)^\circ \mathscr{R} xy(x^\circ xy)^\circ x^\circ, xy(xy)^\circ = xy(xy)^\circ(xy(xy)^\circ)^\circ = xy(x^\circ xy)^\circ x^\circ(xy(x^\circ xy)^\circ x^\circ)^\circ.$$

Since

$$xx^\circ, xy(x^\circ xy)^\circ x^\circ \in E(S)$$

and

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$$xy(x^\circ xy)^\circ x^\circ = xx^\circ xy(x^\circ xy)^\circ x^\circ = xy(x^\circ xy)^\circ x^\circ xx^\circ,\ xy(x^\circ xy)^\circ x^\circ \leqq xx^\circ,$$

so that $xy(xy)^\circ = xy(x^\circ xy)^\circ x^\circ(xy(x^\circ xy)^\circ x^\circ)^\circ \leqq xx^\circ(xx^\circ)^\circ = xx^\circ$. Thus $x^\circ xy(xy)^\circ \leqq x^\circ$. Since $(x^\circ xy)^\circ x^\circ \in V(xy)$, $(x^\circ xy)^\circ x^\circ \leqq (xy)^\circ$. Then we have $xy(xy)^\circ = xy(x^\circ xy)^\circ x^\circ xy(xy)^\circ \leqq xy(x^\circ xy)^\circ x^\circ \leqq xy(xy)^\circ$. Consequently $xy(xy)^\circ = xy(x^\circ xy)^\circ x^\circ$, so that $(xy)^\circ = (xy)^\circ xy(xy)^\circ = (xy)^\circ xy(x^\circ xy)^\circ x^\circ = (x^\circ xy)^\circ x^\circ xy(x^\circ xy)^\circ x^\circ = (x^\circ xy)^\circ x^\circ$. Similarly we obtain $(xy)^\circ = y^\circ(xyy^\circ)^\circ$.

Theorem 2.6. *Let S be a naturally ordered regular semigroup in which every element x has a maximum inverse x°, and on which $\mathscr{R}$ and $\mathscr{L}$ are regular. Then the set $S^\circ = \{x^\circ : x \in S\}$ of maximum inverses of S is a multiplicative inverse transversal of S.*

Proof. We show that S° is a quasi-ideal of S. Let $a, b \in S^\circ$ and let $x \in S$. Then $a = a^{\circ\circ}$ and $b = b^{\circ\circ}$. By Lemma 2.5, we have $axb(axb)^\circ aa^\circ = axb(a^\circ axb)^\circ a^\circ aa^\circ = axb(a^\circ axb)^\circ a^\circ = axb(axb)^\circ = aa^\circ axb(axb)^\circ$, so that $axb(axb)^\circ \leqq aa^\circ$. Similarly $(axb)^\circ axb \leqq b^\circ b$. Therefore we have

$$(axb)^{\circ\circ}(axb)^\circ axb = (axb(axb)^\circ)^\circ axb(axb)^\circ axb \leqq (aa^\circ)^\circ aa^\circ axb$$

$$= axb = axb(axb)^\circ axb \leqq (axb)^{\circ\circ}(axb)^\circ axb,$$

consequently $axb = (axb)^{\circ\circ}(axb)^\circ axb$. Similarly we obtain $axb = axb(axb)^\circ(axb)^{\circ\circ}$. Thus we have $axb = (axb)^{\circ\circ}(axb)^\circ axb(axb)^\circ(axb)^{\circ\circ} = (axb)^{\circ\circ} \in S^\circ$, which shows S° is a quasi-ideal of S. Since S is a regular semigroup, a quasi-ideal of S is a subsemigroup. By Lemma 2.3, S° is an inverse transversal. By Lemma 2.4 and Theorem 1.3, S° is multiplicative.

3. The ordering on regular semigroups

Throughout this section S denotes a regular semigroup with a multiplicative inverse transversal S°. If $x \in S$, the unique element of $V(x) \cap S^\circ$ is denoted by x°, and $x^{\circ\circ}$ denotes $(x^\circ)^\circ$. Let $I = \{e \in S : ee^\circ = e\}$ and $\Lambda = \{f \in S : f^\circ f = f\}$. Then, I [resp. Λ] is a left [resp. right] normal band, i.e. $efg = egf$ [resp. $efg = feg$] for every $e, f, g \in I$ [resp. Λ] [4]. The following fact has been obtained by Blyth and McFadden in [2]:

S is algebraically isomorphic to

$$W = \{[e, a, f) \in I \times S^\circ \times \Lambda : e^\circ = aa^{-1}, f^\circ = a^{-1}a\},$$

where multiplication in W is defined by

$$(e, a, f)(g, b, h) = (eafga^{-1}, afgb, b^{-1}fgbh).$$

We shall define a relation $\leqq$ on I by $e \leqq g$ if and only if $ge = e$ or $ge = e^\circ$ for any $e, g \in I$, and similarly on Λ, using the same symbol $\leqq$; $f \leqq h$ if and only if $fh = f$ or $fh = f^\circ$ for any $f, h \in \Lambda$.

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Lemma 3.1. *The above defined relation on I [resp. Λ] is an order relation.*

Proof. It is evident that $e \leqq e$ for every $e \in I$.

Let $e, g \in I$ with $e \leqq g$ and $g \leqq e$. The following three cases can be considered: (1) $ge = e$ and $eg = g$. (2) $ge = e$ and $eg = g^\circ$. (3) $ge = e^\circ$ and $eg = g^\circ$. For (1), since I is left normal, $e = ge = ege = eeg = eg = g$. For (2), we have $g = gg^\circ = geg = gge = ge = e$. For (3), we have $e^\circ g^\circ = e^\circ g = e^\circ eg = e^\circ e^\circ = e^\circ$ and similarly $e^\circ g^\circ = g^\circ e^\circ = g^\circ$, so that $e^\circ = g^\circ$. Thus we have $e = ee^\circ = ege = g^\circ e = e^\circ e = e^\circ$ and similarly $g = g^\circ$. Consequently $e = g$.

Let $e, g, m \in I$ with $e \leqq g$ and $g \leqq m$. Then the following four cases can be considered: (1) $ge = e$ and $mg = g$. (2) $ge = e^\circ$ and $mg = g$. (3) $ge = e$ and $mg = g^\circ$. (4) $ge = e^\circ$ and $mg = g^\circ$. For each case, we can prove that $e \leqq m$. In each case, the proof is simple but tedious, so the proofs are omitted.

I [resp. Λ] is clearly naturally ordered under the above defined order $\leqq$. Since $I \cap \Lambda = E(S^\circ)$, we have $e^\circ \omega f^\circ$ in $E(S)$ if and only if $e^\circ \leqq f^\circ$ in I [resp. Λ] for $e^\circ, f^\circ \in E(S^\circ)$.

Lemma 3.2. *$I(\cdot, \leqq)$ [resp. $\Lambda(\cdot, \leqq)$] is a naturally ordered left [resp. right] normal band, and e° is the maximum inverse of $e \in I$ [resp. Λ].*

Proof. Let $e, g \in I$ with $e \leqq g$ and let $m \in I$. If $ge = e$, then $mgme = mmge = me$ and $gmem = gemm = em$, so that $me \leqq mg$ and $em \leqq gm$. If $ge = e^\circ$, then $mgme = mmge = me^\circ = me$ and $gmem = gemm = e^\circ m = e^\circ m^\circ = (em)^\circ$, so that $me \leqq mg$ and $em \leqq gm$. Let $e \in I$ and let $g \in V(e) \cap I$. Then $e^\circ = g^\circ$, so that $e^\circ g = g^\circ g = g^\circ$. Consequently $g \leqq e^\circ$.

Lemma 3.3. *Let $e, g \in I$ with $e \leqq g$ and let $f, h \in \Lambda$ with $f \leqq h$. Then $fe \omega hg$.*

Proof. The following four cases can be considered: (1) $ge = e$ and $fh = f$. (2) $ge = e^\circ$ and $fh = f$. (3) $ge = e$ and $fh = f^\circ$. (4) $ge = e^\circ$ and $fh = f^\circ$. Since S° is multiplicative, $fe, hg \in E(S^\circ)$. Then, for (1), we have $hgfe = fehg = fhgehg = fhge^\circ hg = fhghge^\circ = fhge = fe$, so that $fe \omega hg$. For each other case, we can similarly prove that $fe \omega hg$.

It is well-known that an inverse semigroup S° can be partially ordered by setting, for any $a, b \in S^\circ$, $a \leqq b$ if and only if $a = eb$ for some $e \in E(S^\circ)$. We use the cartesian ordering on $W = \{(e, a, f) \in I \times S^\circ \times \Lambda : e^\circ = aa^{-1}, f^\circ = a^{-1}a\}$: $(e, a, f) \leqq (g, b, h)$ if and only if $e \leqq g$ in I, $a \leqq b$ in S° and $f \leqq h$ in Λ.

Theorem 3.4. *Under the cartesian ordering*

$$W = \{(e, a, f) \in I \times S^\circ \times \Lambda : e^\circ = aa^{-1}, f^\circ = a^{-1}a\}$$

is a naturally ordered regular semigroup in which each element (e, a, f) has the maximum inverse $(f^\circ, a^{-1}, e^\circ)$, and on which $\mathcal{R}$ and $\mathcal{L}$ are regular.

Proof. Let $(e, a, f), (g, b, h) \in W$ with $(e, a, f) \leqq (g, b, h)$ and let $(m, c, n) \in W$. Then $e \leqq g$, $a \leqq b$ and $f \leqq h$, so that $a^{-1} \leqq b^{-1}$ and by Lemma 3.3 $fm \leqq hm$ in S°. Therefore $afmc \leqq bhmc$, $afma^{-1} \leqq bhmb^{-1}$ and $c^{-1}fmc \leqq c^{-1}hmc$ in S°, so that $afma^{-1} \leqq bhmb^{-1}$ in I and $c^{-1}fmc \leqq c^{-1}hmc$ in Λ. Thus $eafma^{-1} \leqq gbhmb^{-1}$ in I and $c^{-1}fmcn \leqq c^{-1}hmcn$

in Λ. Consequently $(e,a,f)(m,c,n) \leqq (g,b,h)(m,c,n)$. Similarly we can show $(m,c,n)(e,a,f) \leqq (m,c,n)(g,b,h)$.

Let $(e,a,f),(g,b,h) \in E(W)$ with $(e,a,f)(g,b,h)=(g,b,h)(e,a,f)=(e,a,f)$. Then $gbheb^{-1}=e$, $afgb=a$ and $b^{-1}fgbh=f$. Thus $ge=e$ and $fh=f$, so that $e \leqq g$ and $f \leqq h$. By Theorem 1.3, $a,b \in E(S^\circ)$. Therefore $ba=ab=afgbb=afgb=a$, so that $a \leqq b$. Consequently W is naturally ordered.

Let $(e,a,f) \in W$ and let $(g,b,h) \in V((e,a,f))$. By Corollary 1.2, $b=a^{-1}$, so that $g \leqq g^\circ = bb^{-1} = a^{-1}a = f^\circ$ and similarly $h \leqq e^\circ$. Consequently $(g,b,h) \leqq (f^\circ, a^{-1}, e^\circ) = (e,a,f)^\circ$. Thus each element $(e,a,f) \in W$ has the maximum inverse $(f^\circ, a^{-1}, e^\circ)$.

Let $(e,a,f),(g,b,h) \in W$ with $(e,a,f) \leqq (g,b,h)$. Then $a \leqq b$, so that $aa^{-1} \leqq bb^{-1}$. Since $(e,a,f)(e,a,f)^\circ = (e, aa^{-1}, aa^{-1})$ and $(g,b,h)(e,b,h)^\circ = (g, bb^{-1}, bb^{-1})$, $(e,a,f)(e,a,f)^\circ \leqq (g,b,h)(g,b,h)^\circ$. Thus $\mathscr{R}$ is regular on W. Similarly $\mathscr{L}$ is regular on W.

We shall define a relation on S by, for any $x,y \in S$, $x \leqq y$ if and only if $xx^\circ \leqq yy^\circ$ in I, $x^{\circ\circ} \leqq y^{\circ\circ}$ in S° and $x^\circ x \leqq y^\circ y$ in Λ. Then $x \leqq y$ in S implies $(xx^\circ, x^{\circ\circ}, x^\circ x) \leqq (yy^\circ, y^{\circ\circ}, y^\circ y)$ in W. Conversely, $(e,a,f) \leqq (g,b,h)$ in W implies $eaf(eaf)^\circ = e \leqq g = gbh(gbh)^\circ$ in I, $(eaf)^{\circ\circ} = a \leqq b = (gbh)^{\circ\circ}$ in S° and $(eaf)^\circ eaf = f \leqq h = (gbh)^\circ gbh$ in Λ, so that $eaf \leqq gbh$ in S. Thus the isomorphism $\theta: W \to S$ defined by $(e,a,f)\theta = eaf$ is isotone.

Thus we obtain:

Theorem 3.5. *Let S be a regular semigroup with a multiplicative inverse transversal S°, and let $V(x) \cap S^\circ = \{x^\circ\}$ for every $x \in S$. Then S can be naturally ordered in such a way that x° is the maximum inverse of any element x of S and that $\mathscr{R}$ and $\mathscr{L}$ are regular on S.*

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