Can. J. Math., Vol. XXXIV, No. 4, 1982, pp. 961–968

ON POLYNOMIALS WITH CURVED MAJORANTS

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A well-known result of Chebyshev is that if $p_n \in P_n$, (P_n is the set of polynomials of degree at most n) and

(1) $|p_n(x)| \leqq 1, \quad -1 \leqq x \leqq 1$

then $a_n(p_n)$, the leading coefficient of p_n, satisfies

(2) $|a_n(p_n)| \leqq 2^{n-1}$

with equality holding only for $p_n = \pm T_n$, where T_n is the Chebyshev polynomial of degree n. (See [**6**, p. 57].) This is an example of an extremal problem in which the norm of a given linear operator on P_n is sought. Another example is A. A. Markov's result that (1) implies that

(3) $\max_{-1 \leqq x \leqq 1} |p_n'(x)| \leqq n^2.$

There are also results for the linear functionals $p_n^{(k)}(x_0)$, x_0 real, $k = 1, \ldots, n-1$ ([**8**]).

Suppose $\varphi(x) \geqq 0$ on $[-1, 1]$ and (1) is generalized to

$$|p_n(x)| \leqq \varphi(x), \quad -1 \leqq x \leqq 1,$$

as suggested by Rahman [**4**] (polynomials with curved majorants), what can then be said about the analogue of (3) or similar extremal problems?

Chebyshev himself established the analogue of (2) in the case that

$$\varphi(x) = q_m(x) > 0, q_m \in P_m, \ m \leqq n,$$

a result which was generalized by A. A. Markov (see [**1**]) to

$$\varphi(x) = \sqrt{q_k(x)},$$

where $q_k \in P_k$, $q_k > 0$ on $[-1, 1]$ and $k \leqq 2n$. According to Rahman [**4**], Turán proposed estimation of the derivative with the assumption $\varphi(x) = (1 - x^2)^{1/2}$, i.e., when the graph of $p_n(x)$ is contained in the closed unit disc. Important progress in Turán's program was made by Rahman [**4**], [**5**] and Pierre and Rahman [**3**]. We wish to present two more results concerning polynomials with curved majorants.

1. Let K_n denote the real polynomials, $p(x)$, of degree at most n, satisfying

$$|p(x)| \leqq (1 - x^2)^{1/2}, \quad -1 \leqq x \leqq 1.$$

We prove

Received August 12, 1981.

https://doi.org/10.4153/CJM-1982-069-9 Published online by Cambridge University Press

THEOREM 1. *If $p \in K_n$, $n \geqq 2$, then for $-1 \leqq x \leqq 1$*

$$\max_{p \in K_n} |p(x)| = \begin{cases} (1 - x^2)^{1/2}, \ |x| \leqq \cos \dfrac{\pi}{2(n-1)} \\ (1 - x^2)|U_{n-2}(x)|, \ \cos \dfrac{\pi}{2(n-1)} < |x| \leqq 1 \end{cases}$$

where $U_k(x)$ is the Chebyshev polynomial of the second kind.

Theorem 1 is an immediate consequence of the following result. Let C_n denote the (real) polynomials of degree at most n, $P(x)$, satisfying

$$|P(x)| \leqq (1 - x^2)^{-1/2}, \quad -1 < x < 1.$$

THEOREM 2. *If $P \in C_n$ then*

$$\max_{P \in C_n} P(x) = \begin{cases} (1 - x^2)^{-1/2}, 0 \leqq x \leqq \cos \dfrac{\pi}{2(n+1)} \\ U_n(x), \ \cos \dfrac{\pi}{2(n+1)} < x \leqq 1. \end{cases}$$

For if $p \in K_n$, $n \geqq 2$ then $p(x) = (1 - x^2)P(x)$ for some $P \in C_{n-2}$. Therefore, we turn to a proof of Theorem 2.

Proof. (i) Suppose

$$(4) \quad \cos \frac{\pi}{2(n+1)} < x \leqq 1.$$

Let

$$\xi_j = \cos \frac{(2j-1)\pi}{2(n+1)}, \quad j = 1, \ldots, (n+1)$$

be the zeros of $T_{n+1}(x)$, the Chebyshev polynomial (of the first kind) of degree $n + 1$. The Lagrange interpolation formula for $P \in C_n$ gives

$$P(x) = T_{n+1}(x) \sum_{j=1}^{n+1} \frac{P(\xi_j)}{(x - \xi_j)T'_{n+1}(\xi_j)}$$
$$= \frac{T_{n+1}(x)}{n+1} \sum_{j=1}^{n+1} \frac{P(\xi_j)(-1)^{j-1}(1 - \xi_j^2)^{1/2}}{x - \xi_j}.$$

Note that (4) implies that each of the denominators in the last sum is positive, as is $T_{n+1}(x)$. Thus, since $P \in C_n$ we obtain

$$P(x) \leqq \frac{T_{n+1}(x)}{n+1} \sum_{j=1}^{n+1} \frac{1}{x - \xi_j} = \frac{T_{n+1}(x)}{(n+1)} \frac{T'_{n+1}(x)}{T_{n+1}(x)} \leqq \frac{T'_{n+1}(x)}{n+1} = U_n(x).$$

Finally, observe that $U_n(x) \in C_n$.

(ii) Suppose

$$0 \leqq x \leqq \cos \frac{\pi}{2(n+1)}.$$

https://doi.org/10.4153/CJM-1982-069-9 Published online by Cambridge University Press

Let S_k denote the sine polynomials of degree at most k, $S(t)$, satisfying

$$|S(t)| \leqq 1$$

for all t. Note that if $P \in C_n$ then $(\sin t)P(\cos t) = S(t) \in S_{n+1}$. Thus, to complete the proof of the theorem it suffices to show for $k = 2, 3, \ldots,$

$$\max_{S \in S_k} S(\theta) = 1, \frac{\pi}{2k} \leqq \theta \leqq \frac{\pi}{2}.$$

That is, we need only show that given $\theta \in [\pi/(2k), \pi/2]$ there exists $S \in S_k$ such that $S(\theta) = 1$, and, indeed, it is therefore enough to show that given $\theta \in [\pi/(2k), \pi/(2(k-1))]$ there is an $S \in S_k$ such that $S(\theta) = 1$, for all $k > 1$, since $S_j \subset S_k$, $j = 2, \ldots, k - 1$. To this end we use the following result.

LEMMA. *If $k > 1$ and $\lambda > 0$ then $T(t) = \sin kt + \lambda \sin (k-1)t$, attains its maximum modulus in $[0, \pi]$ at exactly one point which, furthermore, lies in $(\pi/(2k), \pi/2(k-1))$ and at which $T(t)$ is positive.*

Proof. Consider the derivative

$$T'(t) = k \cos kt + \lambda(k-1) \cos (k-1)t.$$

It is positive at $\pi/(2k)$ and negative at $\pi/(2(k-1))$. Similarly, a sign change occurs from $(2j-1)\pi/(2k)$ to $(2j-1)\pi/(2(k-1))$, $j = 2, 3, \ldots, k-1$. If $T'(t)$ has 2 distinct zeros in $(\pi/(2k), \pi/(2(k-1)))$ then it has 3 zeros (counting multiplicities) there and hence, at least $k+1$ zeros in $(0, \pi)$ which is impossible. Thus, we conclude that$T'(t)$ has only one zero in $(\pi/(2k), \pi(2(k-1)))$. This point is clearly a local maximum of $T(t)$ (the only such point in the interval), and the value of $T(t)$ at this point is bigger than its endpoint values

$$T\left(\frac{\pi}{2k}\right) = 1 + \lambda \cos \frac{\pi}{2k}, T\left(\frac{\pi}{2(k-1)}\right) = \cos \frac{\pi}{2(k-1)} + \lambda.$$

Next observe that $T(t)$ is monotone increasing from zero to $T(\pi/2k)$ for $0 \leqq t \leqq \pi/(2k)$. Also that

$$T'\left(\frac{\pi}{2(k-1)}\right) < 0,$$

$$T'\left(\frac{3\pi}{2k}\right) < 0,$$

hence, $T'(t)$ must be negative throughout $[\pi/(2k-1)), (3\pi)/(2k)]$ for otherwise T' has at least two zeros in that interval and at least $k+1$ zeros in $(0, \pi)$ which is impossible. Thus $T(t)$ decreases from $T(\pi/$

$(2(k-1)))$ to $T((3\pi)/(2k))$. We can now conclude our proof of the lemma by showing that

$$|T(t)| < \max\left(T\left(\frac{\pi}{2k}\right), T\left(\frac{\pi}{2(k-1)}\right)\right), \frac{3\pi}{2k} \leqq t \leqq \pi.$$

$$|T(t)| = |\mathrm{Im}\,(e^{ikt} + \lambda e^{i(k-1)t})| \leqq |e^{ikt} + \lambda e^{i(k-1)t}|$$
$$= (1 + \lambda^2 + 2\lambda \cos t)^{1/2} \leqq \left(1 + \lambda^2 + 2\lambda \cos \frac{3\pi}{2k}\right)^{1/2}.$$

Case I. $0 < \lambda \leqq 1$. We show that

$$\left(1 + \lambda^2 + 2\lambda \cos \frac{3\pi}{2k}\right)^{1/2} < 1 + \lambda \cos \frac{\pi}{2k}.$$

Namely,

$$\left(1 + \lambda \cos \frac{\pi}{2k}\right)^2 - \left(1 + \lambda^2 + 2\lambda \cos \frac{3\pi}{2k}\right)$$
$$= \lambda\left(\sin^2 \frac{\pi}{2k}\right)\left(8 \cos \frac{\pi}{2k} - \lambda\right) \geqq \lambda\left(\sin^2 \frac{\pi}{2k}\right)\left(8 \cos \frac{\pi}{4} - 1\right) > 0.$$

Case II. $\lambda > 1$. We show that

$$\left(1 + \lambda^2 + 2\lambda \cos \frac{3\pi}{2k}\right)^{1/2} < \lambda + \cos \frac{\pi}{k} \leqq \lambda + \cos \frac{\pi}{2(k-1)}.$$

Namely, consider

$$(5) \quad \left(\lambda + \cos \frac{\pi}{k}\right)^2 - \left(1 + \lambda^2 + 2\lambda \cos \frac{3\pi}{2k}\right) = 4\lambda \sin \frac{5}{4}\frac{\pi}{k} \sin \frac{\pi}{4k} - \sin^2 \frac{\pi}{k}.$$

(5) is positive if $k = 2$, and if $k > 2$ it is greater than the positive quantity

$$\sin \frac{\pi}{k}\left(4 \sin \frac{\pi}{4k} - \sin \frac{\pi}{k}\right).$$

The lemma is proved.

Now suppose $\theta \in (\pi/(2k), \pi/(2(k-1)))$. Consider

$$S(t) = \frac{(k-1)\cos (k-1)\theta \sin kt - k \cos k\theta \sin (k-1)t}{(k-1)\cos (k-1)\theta \sin k\theta - k \cos k\theta \sin (k-1)\theta}$$
$$= \frac{(k-1)\cos (k-1)\theta}{(k-1)\cos (k-1)\theta \sin k\theta - k\cos k\theta \sin (k-1)\theta}$$
$$\times \left(\sin kt - \frac{k \cos k\theta}{(k-1)\cos (k-1)\theta} \sin (k-1)t\right).$$

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The lemma (applied with $\lambda = (-k \cos k\theta)/((k-1)\cos(k-1)\theta)$ implies that, since $S'(\theta) = 0$,

$$1 = S(\theta) = \max_{0 \leqq t \leqq \pi} |S(t)|,$$

which establishes the theorem.

2. Schur [**7**, p. 285] proves a result which is easily seen to be equivalent to the following:

If $p \in P_n$ satisfies

$$|p(x)| \leqq \frac{1}{|x|}, \quad 0 < |x| \leqq 1$$

then

$$\max_{-1 \leqq x \leqq 1} |p(x)| \leqq \begin{cases} n, & n \text{ odd} \\ n+1, & n \text{ even.} \end{cases}$$

This suggested to us the complex result that follows.

THEOREM 3. *Suppose $q(z)$ is a (complex) polynomial of degree at most n which satisfies*

$$(6) \qquad |q(z)| \leqq \frac{1}{|1-z|}, \quad |z| \leqq 1$$

then

$$(7) \qquad \max_{|z| \leqq 1} |q(z)| \leqq (n+1)/2.$$

Equality holds only for

$$e^{i\alpha} \frac{1 - z^{n+1}}{2(1-z)} = e^{i\alpha} q^*(z),$$

α an arbitrary real number.

Proof. We begin by observing that when $n = 0$ the result is obvious. Suppose henceforth that $n \geqq 1$.

$$t(\theta) = |q(e^{i\theta})|^2$$

is a trigonometric polynomial of degree at most n satisfying, for all θ,

$$(8) \qquad 0 \leqq t(\theta) \leqq \frac{1}{2(1-\cos\theta)}.$$

The same is true of

$$s(\theta) = |q^*(e^{i\theta})|^2 = \frac{1}{4}\left|\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right|^2 = \frac{1}{4}\frac{1 - \cos(n+1)\theta}{1 - \cos\theta},$$

which, additionally, satisfies

$$(9)\qquad s\left(\frac{j\pi}{n+1}\right) = \begin{cases} 0, j = 2, 4, \ldots, 2n, \\ \dfrac{1}{2\left(1 - \cos\dfrac{j\pi}{n+1}\right)}, j = 1, 3, \ldots, 2n+1. \end{cases}$$

We consider two cases.

(i) Suppose $|\theta| \geqq \pi/(n+1)$ or

$$\frac{|\theta|}{2} \geqq \frac{\pi}{2(n+1)}$$

which implies that

$$\sin\frac{|\theta|}{2} > \frac{1}{n+1}$$

or, after squaring both sides,

$$\frac{1-\cos\theta}{2} > \frac{1}{(n+1)^2},$$

hence,

$$\frac{1}{2(1-\cos\theta)} < \frac{(n+1)^2}{4}$$

Thus, in this case,

$$t(\theta) < \frac{(n+1)^2}{4},$$

and so

$$(10)\qquad \max_{|\theta| \geqq (\pi)/n+1} |q(e^{i\theta})| < \frac{(n+1)}{2}.$$

(ii) Suppose $|\theta| < \pi/(n+1)$. We wish to show that in this case $t(\theta) \leqq s(\theta)$. To this end we use the following.

LEMMA. *If a (real) trigonometric polynomial of degree at most n, $v(\theta)$, satisfies*

$$(-1)^i v(\theta_i) \geqq 0, \quad i = 0, \ldots, 2n+1,$$

where

$$\theta_0 < \theta_1 < \ldots < \theta_{2n+1} < \theta_0 + 2\pi$$

then $v = 0$.

Proof. There is no loss in generality in assuming that $v(\theta_0) \neq 0$ (since if v is zero at every θ_i the lemma is trivial) and we do so. We now note the following:

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1. If $v(\theta_i) \neq 0$, then $\operatorname{sgn} v(\theta_i) = (-1)^i$.

2. If $v(\theta_i) \neq 0$, $v(\theta_{i+1}) = \ldots = v(\theta_{i+j-1}) = 0$, $v(\theta_{i+j}) \neq 0$ then v has j zeros (counting multiple zeros as many times as their multiplicities) in (θ_i, θ_{i+j}). For, v has at least $j - 1$ zeros in (θ_i, θ_{i+j}) and if j is even $v(\theta_i)$ and $v(\theta_{i+j})$ are of like sign, hence, v has an even number of zeros in (θ_i, θ_{i+j}), so at least j of them, while if j is odd $v(\theta_i)$ and $v(\theta_{i+j})$ differ in sign, hence, v has an odd number of zeros in (θ_i, θ_{i+j}), so at least j zeros.

Suppose the non-zero $v(\theta_i)$ occur for the indices $i = n_0(= 0), n_1, \ldots, n_m$ $(\leqq 2n + 1)$. Each interval $(\theta_{n_j}, \theta_{n_{j+1}})$, $j = 0, \ldots, m - 1$ contains at least $n_{j+1} - n_j$ zeros, as we have just shown. Thus, v has

$$\sum_{j=0}^{m-1} (n_{j+1} - n_j) = n_m$$

zeros in (θ_0, θ_{n_m}). If $n_m = 2n + 1$ then $v = 0$. If $n_m < 2n + 1$, the interval $(\theta_{n_m}, \theta_{2n+1}]$ contains the zeros $\theta_{n_m+1}, \ldots, \theta_{2n+1}$, $2n + 1 - n_m$ in number, giving a total of $2n + 1$ zeros in $(\theta_0, \theta_{2n+1}]$, and again $v = 0$. This establishes the lemma.

We claim next that for $|\theta| < \pi/(n + 1)$ we have $t(\theta) \leqq s(\theta)$. Let

$$\theta_j = \frac{j\pi}{n+1}, j = 1, \ldots, 2n + 1$$

and suppose that there exists θ_0, $|\theta_0| < \pi/(n + 1)$, such that

(11) $t(\theta_0) > s(\theta_0)$.

In view of (8) and (9) we also have

$$\begin{aligned} t(\theta_1) &\leqq s(\theta_1) \\ t(\theta_2) &\geqq s(\theta_2) \\ &\cdot \\ &\cdot \\ &\cdot \\ t(\theta_{2n+1}) &\leqq s(\theta_{2n+1}). \end{aligned}$$

Consider $v(\theta) = t(\theta) - s(\theta)$. It satisfies the lemma, hence $t = s$, contradicting (11). This establishes our claim.

Now,

$$s(\theta) \leqq (n + 1)^2/4$$

for all θ, with equality only for $\theta = 0$. Therefore,

(12) $$t(\theta) \leqq \frac{(n+1)^2}{4}, \quad |\theta| \leqq \frac{\pi}{n+1}$$

with equality possible only for $\theta = 0$. Recalling the maximum principle for analytic functions we see that (10) and (12) prove (7). Finally, if $t(0) = s(0) = (n + 1)^2/4$, then the lemma yields $t = s$. Thus, every zero of s is a zero of t and $q(z) = cq^*(z)$, which can only hold if $|c| = 1$. Theorem 3 is proved.

Remark. We have also shown that, if (6) holds then

$$(13) \quad |q(e^{i\theta})| \leqq |q^*(e^{i\theta})|$$

when $|\theta| < \pi/(n + 1)$, or $\theta = \theta_j$, $j = 1, 3, 5, \ldots, 2n + 1$. But (13) certainly does not hold for $\theta = \theta_j$, $j = 2, 4, \ldots, 2n$. Also, a result of [**2**] implies that, for q subject to (6)

$$\max |q(0)| = \left[\cos \frac{\pi}{2(n + 1)}\right]^{n+1}, n \geqq 1.$$

The problem of maximizing the linear functional, $q(z)$, (z arbitrary in $|z| \leqq 1$), among q satisfying (6) seems difficult.

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