ON THE BASIS PROBLEM FOR VECTOR VALUED FUNCTION SPACES

H. W. ELLIS

1. Introduction. In a recent paper **(2)** Halperin and the author considered separable Banach spaces L^λ of real valued functions on general measure spaces and proved the existence of 1-regular (§2) Haar or σ-Haar bases when λ was the classical p-norm or any levelling length function **(3)** and, more generally, of K-regular Haar or σ-Haar bases when λ was a continuous length function satisfying certain additional conditions **(2**, Theorem 3.2**)**.

In the present note, separable spaces $L^\lambda(S; X)$, $V^\lambda(S; X)$ of functions valued in a normed vector space X on a general measure space S are considered and the existence of a $3KK'$-regular basis is established when $L^\lambda(V^\lambda)$ has a K-regular Haar or σ-Haar basis and X has a K'-regular basis.

2. Terminology. S will denote an arbitrary space of points P with a countably additive, non-negative measure $\gamma(E)$ defined for a complemented, countably additive family of sets; λ will be an arbitrary length function; $L^\lambda(S)$ will denote the Banach space of real valued functions $f(P)$ on S with $\lambda(f)$ defined and finite; X will denote an arbitrary normed vector space with real scalars; $|| \ ||$ the norm in X; $L^\lambda(S; X)$ the space of Bochner measurable functions **(4)** $f(P)$ valued in X on S with $\lambda(f) = \lambda[f(P)] = \lambda(||f(P)||)$ defined and finite. (If X is complete $L^\lambda(S; X)$ is a Banach space **(3)**.)

Upper case letters will be used for arbitrary measurable sets, lower case letters will always denote measurable sets of finite measure; $f_E(P)$ will denote the function equal to $f(P)$ in E and vanishing elsewhere and $\lambda(E)$ will be an abbreviation for $\lambda(1_E)$.

Definition. A basis $\{x_i\}$ in X will be called *K-regular*, if

2.1 $$\left|\left|\sum_1^n a_i x_i\right|\right| \leqslant K||x||, \quad 1 \leqslant K < \infty, \qquad n = 1, 2, \dots$$

for every $x = \sum a_i x_i \in X$.

The referee has pointed out that, if X is a Banach space, Banach's boundedness theorem **(1**, p. 80**)** shows that any basis in M is a K-regular basis for some K.

3. Bases in $L^\lambda(S; X)$. Suppose that X has a K-regular basis $\{x_i\}$. If $f(P)$ is valued in X,

3.1 $$f(P) = \sum a_i(P)\, x_i,$$

Received May 26, 1955.

https://doi.org/10.4153/CJM-1956-048-7 Published online by Cambridge University Press

the $a_i(P)$ being determined uniquely for each i and all P. We shall show that if $f(P) \in L^\lambda(S; X)$ then the real valued functions $a_i(P)$ $(i = 1, 2, \ldots)$ belong to $L^\lambda(S)$. This involves showing that each $a_i(P)$ is measurable with $\lambda[a_i(P)] < \infty$. We note that the uniqueness of the representation $x = \sum a_i x_i \in X$ implies that if $f(P)$ is constant in E so is each $a_i(P)$. Since

$$3.2 \quad |a_i(P)|\, ||x_i|| \leqslant \left|\left| \sum_{j=1}^{i} a_j(P)x_j \right|\right| + \left|\left| \sum_{j=1}^{i-1} a_j(P)x_j \right|\right| \leqslant 2K||f(P)||,$$

and $f(P)$ is the almost uniform limit of countably valued functions, each $a_i(P)$ is the almost uniform limit of measurable (countably valued) functions and is therefore measurable. Thus $\lambda(a_i)$ is defined for each i. Using (3.2) and properties (L 2) and (L 4) of length functions **(3)**,

$$\lambda[a_i(P)] = \lambda(|a_i(P)|) \leqslant 2K\lambda(||f(P)||)/||x_i|| < \infty.$$

Lemma 3.1. *If $f(P) \in L^\lambda(S; X)$, where λ is a continuous length function and $L^\lambda(S)$ is separable, if X has a K'-regular basis $\{x_i\}$, and if* (3.1) *holds, then*

$$3.3 \qquad \lim_{n\to\infty} \lambda\left[f(P) - \sum_{1}^{n} a_i(P)x_i\right] = 0.$$

Proof. First suppose that $\gamma(S) < \infty$. Given $\epsilon > 0$, let e denote the set of points P for which

$$\left|\left| f(P) - \sum_{1}^{n} a_i(P)x_i \right|\right| < \epsilon$$

for all $n > N$. Then $\gamma(S - e) \to 0$ as $N \to \infty$,

$$\lambda\left[f(P) - \sum_{1}^{n} a_i(P)x_i\right] < \epsilon\gamma(S) + (1 + K')\,\lambda(f_{S-e}),$$

and $\lambda(f_{S-e}) \to 0$ as $N \to \infty$ by **(2**, Lemma 3.2**)**.

If S is arbitrary there exists e' by **(2**, Lemma 3.2 (iii)**)** with $\lambda(f - f_{e'})$ arbitrarily small,

$$\lambda\left[f(P) - \sum_{1}^{n} a_i(P)x_i\right] \leqslant (1 + K')\,\lambda(f - f_{e'}) + \lambda\left[\left(f - \sum_{1}^{n} a_i(P)x_i\right)_{e'}\right]$$

and the right side can be made arbitrarily small by choice of e' and n.

Lemma 3.2. *Let $f(P) \in L^\lambda(S; X)$, let $L^\lambda(S)$ have a Haar or σ-Haar basis and X have a K'-regular basis $\{x_i\}$ and suppose that* (3.1) *holds. Then* (3.3) *holds.*

Proof. The assumption that $L^\lambda(S)$ has a σ-Haar basis implies that $L^\lambda(S)$ is separable and that $S = E + \bigcup e_n$ where $\lambda(f_E) = 0$ for every $f(P) \in L^\lambda(S)$ and where the σ-Haar basis functions correspond to a σ-Haar system of sets $H_\sigma(\bigcup e_n)$ which forms a countable basis **(2)** in S. Then for arbitrary $f(P) \in L^\lambda(S)$, where

https://doi.org/10.4153/CJM-1956-048-7 Published online by Cambridge University Press

$$f_N(P) = \begin{cases} f(P) \text{ in } \bigcup_1^N e_n, \\ 0 \text{ elsewhere,} \end{cases}$$

$$\lambda(f) = \lambda(f_{S-E}) = \sup_{N\to\infty} \lambda(f_N),$$

by (L 5) for length functions so that λ is continuous and Lemma 3.1 applies. A similar argument applies if $L^\lambda(S)$ has a Haar basis.

COROLLARY. *Under the hypotheses of Lemma* 3.2, $\lambda[a_n(P)] \to 0$ *as* $n \to \infty$.

If $\{\phi_j(P)\}$ is a basis in $L^\lambda(S)$, $a_i(P) = \sum_j a_{ij}\phi_j(P)$, $(i = 1, 2, \ldots)$, the coefficients a_{ij} being uniquely determined. Thus with each $f(P) \in L^\lambda(S; X)$ can be associated a unique double series

3.4 $$f(P) \backsim \sum_i[\sum_j a_{ij}\phi_j(P)]\, x_i.$$

We shall show that the $\{x_i\phi_j(P)\}$, ordered suitably into a single sequence, form a basis in $L^\lambda(S; X)$.

In the proof of the next lemma the Bochner integral would be used if X were a Banach space. To extend the proof to an arbitrary normed vector space X, we generalize the Bochner integral for certain functions by defining, where $x_i \in X$, $g_i(P) \in L(S)$ $(i = 1, 2, \ldots, n)$,

$$(X)\int_S\Big[\sum_1^n x_i\, g_i(P)\Big]d\gamma(P) = \sum_1^n x_i \int_S g_i(P)d\gamma(P).$$

We shall use the fact that

3.5 $$\left|\left|(X)\int_S\Big[\sum_1^n x_i\, g_i(P)\Big]d\gamma(P)\right|\right| \leqslant \int_S\left|\left|\sum_1^n x_i\, g_i(P)\right|\right| d\gamma(P).$$

This is easily shown if the g_i are finitely valued, constant in the same sets, and the general result is then obtained by standard arguments.

LEMMA 3.3. *If* $\{\phi_i(P)\}$ *is a K-regular Haar or* σ*-Haar basis in* $L^\lambda(S)$, *if* $\{x_i\}$ *is a K′-regular basis in X, then for all m, n,*

3.6 $$\lambda\Big[\sum_{i=1}^m \sum_{j=1}^n a_{ij}x_i\phi_j(P)\Big] \leqslant KK'\lambda(f).$$

Proof. By (**2**, Lemma 2.1, Corollary 1)

$$\sum_{j=1}^n a_{ij}\phi_j(P) = \sum_{r=1}^N\left(\Big[\gamma(e_r)^{-1}\int_{e_r} a_i(P)\, d\gamma(P)\Big] \text{in } e_r\right)$$

for some sequence of Haar or σ-Haar sets e_r depending only on n where

$$\bigcup_1^N e_r = S - E$$

(E defined as in Lemma 3.2) if $L^\lambda(S)$ has a Haar basis and where

https://doi.org/10.4153/CJM-1956-048-7 Published online by Cambridge University Press

$$\sum_1^n a_{ij}\phi_j(P)$$

vanishes outside $\bigcup e_r$ if $L^\lambda(S)$ has a σ-Haar basis. Let $||f(P)|| = \sum b_j\phi_j(P)$. Then

$$\begin{aligned}
\lambda\left\{\sum_{i=1}^m \sum_{j=1}^n a_{ij}x_i\phi_j(P)\right\} &= \lambda\left\{\sum_{i=1}^m x_i \sum_{r=1}^N \left(\left[\gamma(e_r)^{-1}\int_{e_r} a_i(P)\,d\gamma(P)\right]\text{in } e_r\right)\right\} \\
&= \left\{\sum_{r=1}^N \left(\left[\gamma(e_r)^{-1}(X)\int_{e_r}\left(\sum_{i=1}^m a_i(P)x_i\right)d\gamma(P)\right]\text{ in } e_r\right)\right\} \\
&\leqslant \lambda\left\{\sum_{r=1}^N \left(\left[\gamma(e_r)^{-1}\int_{e_r}\left|\left|\sum_{i=1}^m a_i(P)x_i\right|\right|d\gamma(P)\right]\text{ in } e_r\right)\right\} \\
&\leqslant K'\lambda\left\{\sum_{r=1}^N \left(\left[\gamma(e_r)^{-1}\int_{e_r}||f(P)||\,d\gamma(P)\right]\text{in } e_r\right)\right\} \\
&= K'\lambda\left\{\sum_{j=1}^n b_j\phi_j(P)\right\} \\
&\leqslant KK'\lambda\{f(P)\}.
\end{aligned}$$

The sequence

$$\sum_1^n \sum_1^n a_{ij}x_i\phi_j(P)$$

is KK'-regular in $L^\lambda(S;X)$ and suggests that the $x_i\phi_j(P)$ be ordered so as to give partial sums differing as little as possible from square or rectangular sums. To this end we order then as follows: $x_1\phi_1, x_1\phi_2, x_2\phi_1, x_2\phi_2, \ldots, x_{n-1}\phi_{n-1}, x_n\phi_1, x_n\phi_2, \ldots, x_n\phi_{n-1}, x_1\phi_n, x_2\phi_n, \ldots, x_n\phi_n, \ldots$. Given $f(P) \in L^\lambda(S;X)$, consider 3.4 and let $S_N = S_N(f)$ denote the sum of the first N terms $a_{ij}x_i\phi_j(P)$ with the above ordering. Let

$$S_{M,N} = \sum_{i=1}^M \sum_{j=1}^N a_{ij}x_i\phi_j(P).$$

Then

3.7 $$\begin{aligned}
S_N &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}x_i\phi_j(P) + \sum_{j=1}^{N-(n-1)^2} a_{nj}x_n\phi_j(P), \quad (n-1)^2 < N \leqslant n(n-1) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n-1} a_{ij}x_i\phi_j(P) + \sum_{i=1}^{N-n(n-1)} a_{in}x_i\phi_n(P), \quad n(n-1) < N \leqslant n^2.
\end{aligned}$$

3.8 $$\begin{aligned}
S_N &= S_{n-1,n-1} + S_{n,N-(n-1)^2} - S_{n-1,N-(n-1)^2}, \quad (n-)^2 < N \leqslant n(n-1); \\
&= S_{n,n-1} + S_{N-n(n-1),n} - S_{N-n(n-1),n-1}, \quad n(n-1) < N \leqslant n^2.
\end{aligned}$$

From 3.7, 3.8 and Lemma 3.3 we obtain for every $f \in L^\lambda(S;X)$ and all N

3.9 $$\lambda[S_N(f)] \leqslant 3KK'\lambda(f).$$

THEOREM 3.1. *If $L^\lambda(S)$ has a K-regular Haar or σ-Haar basis $\{\phi_j(P)\}$ and if X has a K'-regular basis $\{x_i\}$ then the $x_i\phi_j(P)$ with the ordering of the preceding paragraph form a $3KK'$-regular basis in $L^\lambda(S; X)$.*

Proof. We shall give the proof where $L^\lambda(S)$ has a σ-Haar basis, the proof where there is a Haar basis being similar and simpler. We can suppose that the σ-Haar basis functions correspond to a σ-Haar system of sets $H_\sigma(S)$ dense in S (cf. Lemma 3.2). As in Lemma 3.2 each $f \in L^\lambda(S; X)$ is the strong limit of functions f_e (vanishing outside sets of finite measure). Each f_e is the almost uniform limit of finitely valued functions and an easy computation using (2, Lemma 3.2) shows that these functions converge strongly to f_e in $L^\lambda(S; X)$. Finally each set of constancy of an arbitrary finitely valued function g can be approximated arbitrarily closely by finite collections of sets of $H_\sigma(S)$ with the corresponding functions converging strongly to g. We conclude that finitely valued functions with sets of constancy in $H_\sigma(S)$ are dense in $L^\lambda(S; X)$. Since

$$\lambda[f(P) - S_N(f)] \leqslant \lambda[f(P) - h(P)] + \lambda[h(P) - S_N(h)] + \lambda[S_N(f) - S_N(h)]$$
$$\leqslant (1 + 3KK')\,\lambda(f - h) + \lambda[h - S_N(h)],$$

for any f, h in $L^\lambda(S; X)$ it will be sufficient to prove that $\lambda[f - S_N(f)] \to 0$ as $n \to \infty$ where $f(P)$ is a finitely valued function with sets of constancy in $H_\sigma(S)$. Then, if $f(P) = \sum a_i(P)x_i$, each $a_i(P)$ is finitely valued with the same sets of constancy as $f(P)$ and by (2, Lemma 2.1, Corollary 3) there exists n_0 with

$$\sum_{j=1}^{n} a_{ij}\phi_j(P) = a_i(P), \qquad n > n_0,\ i = 1, 2, \ldots.$$

Then

$$\begin{aligned}\lambda[f - S_N(f)] &\leqslant \lambda\Big[f - \sum_1^{n-1} a_i(P)x_i\Big] + \lambda\Big(\sum_{i=1}^{n-1} x_i\Big[a_i(P) - \sum_{j=1}^{n-1} a_{ij}\phi_j(P)\Big]\Big)\\ &\quad + \lambda\Big[x_n \sum_{j=1}^{N-(n-1)^2} a_{nj}\phi_j(P)\Big] \qquad (n-1)^2 < N \leqslant n(n-1),\\ &= \lambda_1 + \lambda_2 + \lambda_3\\ &\leqslant \lambda\Big[f - \sum_1^{n} a_i(P)x_i\Big] + \lambda\Big(\sum_{i=1}^{N-n(n-1)} x_i\Big[a_i(P) - \sum_{j=1}^{n} a_{ij}\phi_j(P)\Big]\Big)\\ &\quad + \lambda\Big(\sum_{i=N-n(n-1)+1}^{N} x_i\Big[a_i(P) - \sum_{j=1}^{n-1} a_{ij}\phi_j(P)\Big]\Big) \quad n(n-1) < N \leqslant n^2,\\ &= \lambda_1' + \lambda_2' + \lambda_3';\end{aligned}$$

$\lambda_1 \to 0$, $\lambda_1' \to 0$ as $n \to \infty$ by Lemma 3.2, λ_2, λ_2' and λ_3' vanish if $n - 1 > n_0$ and $\lambda_3 \leqslant K\|x_n\|\,\lambda[a_n(P)] \to 0$ as $n \to \infty$ by Lemma 3.2, Corollary.

Write $\sum'$ for sums of terms $a_{ij}x_i\phi_j(P)$ ordered as in Theorem 3.1. We have established the existence of a convergent series $\sum' a_{ij}x_i\phi_j(P)$ with sum $f(P)$

for every $f(P) \in L^\lambda(S; X)$. The $x_i\phi_j(P)$ with the specified ordering will be a basis in $L^\lambda(S; X)$ if there is only one such series for each $f(P)$ and that this is true is a consequence of the uniqueness of the respective series for elements of X and $L^\lambda(S)$ in terms of the bases $\{x_i\}$, $\{\phi_j(P)\}$.

The referee has observed that a K-regular basis in X is a basis in the completion of X, so that there would be no loss of generality in assuming throughout that X is a Banach space. This would permit the use of the Bochner integral in Lemma 3.3.

4. Bases in $V^\lambda(S; X)$. In §3 the assumption that λ is a length function implies that $L^\lambda(S)$ is a Banach space. The above arguments remain valid for more general function spaces. Consider a general normed vector space of measurable functions $f(P)$ with norm $\lambda(f) = \lambda(|f(P)|)$. The definition of a norm implies properties (L 1), (L 3) and (L 4) of length functions for λ. Property (L 2) has played a fundamental role in the proofs in §3. However if the normed vecor space is not required to be complete the results in §3 can be obtained with (L 5) replaced by weaker assumptions.

Suppose that for every measurable function u with $0 \leqslant u(P) \leqslant \infty$ for almost all P, $\lambda(u)$ is defined with $0 \leqslant \lambda(u) \leqslant \infty$ and satisfies (L 1)-(L 4) for length functions,

(L 5′) If e is fixed, $e' \subset e, \lambda(f) < \infty$, then $\lambda(f_e) - \lambda(f_{e'}) \to 0$ as $\gamma(e - e') \to 0$,

and

(L 6) $\lambda(u) = \sup_e \lambda(u_e)$ (i.e. λ is continuous).

Let $V^\lambda(S)$ denote the space of real valued functions $f(P)$ with $|f(P)|$ measurable and $\lambda(f) = \lambda(|f|) < \infty$ and let $V^\lambda(S; X)$ be the analogue of $L^\lambda(S; X)$. $V^\lambda(S; X)$ is a normed vector space. If $V^\lambda(S)$ is separable and $\lambda(f) < \infty$, the argument of **(2**, Lemma 3.2**)** gives:

(i) $\lambda(f_e) \to 0$ as $\gamma(e) \to 0$, and

(ii) there exists e with $\lambda(f - f_e)$ arbitrarily small. With L^λ replaced by V^λ Lemmas 3.1-3.3 and Theorem 3.1 then hold.

References

1. Stefan Banach, *Théorie des opérations linéaires* (Warsaw, 1932).

2. H. W. Ellis and Israel Halperin, *Haar functions and the basis problem for Banach spaces*, J. London Math. Soc., *31* (1956), 28–39.

3. ———, *Function spaces determined by levelling length functions*, Can. J. Math., *5* (1953), 576–592.

4. Einar Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publications, Vol. XXXI (New York, 1948).

Queen's University