MOTIONS OF MATRIX RINGS

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Introduction. Metric spaces in which the distances are not real numbers have been studied by several people (**2**, **3**, **4**, **7**, **9**). Any ring R together with a mapping, $X \to \phi(X)$, of R into a lattice A with 0 and 1 satisfying

$$\phi(X) = \phi(0) \quad \text{if and only if } X = 0, \tag{1}$$

$$\phi(X + Y) \subset \phi(X) \cup \phi(Y), \tag{2}$$

and

$$\phi(X \cdot Y) = \phi(X) \cap \phi(Y), \tag{3}$$

is called a "lattice-valued ring," where the operations union, $\cup$, and intersection, $\cap$, are the usual lattice operations. The mapping ϕ is called a "valuation" and A is a "valuation lattice." If R is a lattice-valued ring and a mapping d is defined by

$$d(X, Y) = \phi(X - Y),$$

which maps $R \times R$ into A, then d is called a distance function on R. It is easily seen that d satisfies

$$d(X, Y) = \phi(0) \quad \text{if and only if } X = Y, \tag{4}$$

$$d(X, Y) = d(Y, X), \tag{5}$$

and

$$d(X, Y) \cup d(Y, Z) \supset d(X, Z). \tag{6}$$

The ring R together with mapping ϕ and distance function d is called a "lattice metric space."

If we take a ring R with identity together with a mapping $X \to \phi(X)$ of R into a lattice L, which satisfies (1) and (2) above but instead of (3) the following:

$$\phi(-X) = \phi(X), \tag{3'}$$

we then call R a "*weak lattice-valued ring*," and L a "*weak valuation lattice*." If d is defined by

$$d(X, Y) = \phi(X - Y)$$

$X, Y \in R$, then d is a distance function satisfying (4), (5), and (6) which maps $R \times R$ into L. The ring R together with the mapping ϕ and distance function d is again called a "lattice metric space."

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In this paper we determine the motions of the ring of all linear transformations on an arbitrary vector space over a division ring. Since the ring of row-finite matrices with elements from a division ring is isomorphic to the ring of all linear transformations over the division ring, we can consider motions of the ring of row-finite matrices.

Let R be a division ring, R' the ring of row-finite matrices with elements from R, and L the lattice of right ideals of R'. R satisfies (1), (2), and (3′) and if we define

$$d(X, Y) = \phi(X - Y)$$

$X, Y \in R'$, then d is a distance function satisfying (4), (5), and (6) which maps $R \times R$ into L. The principal result is the following theorem.

THEOREM 1. *If R is a division ring and R' the ring of row-finite infinite matrices over R, the mapping $X \rightarrow F(X)$, $X \in R'$, is a motion of R' with respect to the distance function d if and only if $F(X) = XA + B$, where A and B are fixed elements of R' and A is non-singular.*

1. Definitions.

Consider an arbitrary ring R and I_R the lattice of right ideals of R.

DEFINITION 1. *Let ϕ be a mapping from R into I_R such that if $A \in R$, $\phi(A)$ is the principal right ideal in I_R generated by A.* (We shall denote $\phi(A)$ by $[A]_r$.)

It is easily shown that the mapping ϕ satisfies (1), (2), and (3′) above and clearly I_R contains a first, the null ideal, and a last, the whole ring, element; thus R with the mapping ϕ and "distance function" d is a lattice metric space.

DEFINITION 2. *A one-to-one mapping of R onto R which preserves distances is a motion of R relative to the distance function d.* Thus, if $A, B \in R$ and f is a motion of R, then

$$[f(A) - f(B)]_r = [A - B]_r.$$

2. Row-finite infinite matrices.

Let V be a left vector space of infinite dimension over a division ring R. The ring L of linear transformations on V is isomorphic to the ring R' of row-finite matrices with elements from R. (See **5**, Chap. IX.) To determine the group of motions of L it is sufficient to study the motions of R'.

DEFINITION 3. *Any infinite matrix A is "row-finite" provided each row of A has only a finite number of non-zero elements.*

Remark. For $A \in R'$,

$$[A]_r = (Y : Y = AX, \text{ for all } X \in R').$$

Proof. Any element of $[A]_r$ is in the form $AX + nA$, $X \in R'$, $n \in N$, where N is the ring of integers. R' contains an identity I; thus

$$AX + nA = AX + AnI = A(X + nI).$$

Let E_{ij}, i, j any ordinal numbers, be the matrix with 1 in the ith row and jth column and zeros elsewhere and consider any motion f which sends zero into zero. For any ordinal j

$$[E_{1j}]_r = [f(E_{1j})]_r$$

and hence $f(E_{1j})$ has non-zero elements only in the first row. Let

$$f(E_{1j}) = \begin{pmatrix} a_{j1}\, a_{j2} \ldots a_{jn} \ldots \\ O \end{pmatrix}, \qquad j \text{ any ordinal,}$$

and define a matrix A such that the jth row of A is identical with the first row of $f(E_{1j})$. Thus

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & \ldots \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ a_{n1} & a_{n2} & \ldots & a_{nn} & \ldots \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \end{pmatrix}$$

and it is clear that $f(E_{1j}) = E_{1j}A$, j any ordinal. In any row of A there are only a finite number of non-zero elements; hence $A \in R'$.

LEMMA 1. *For f, $f(E_{ij}) = E_{ij}A$, i and j being any ordinal numbers.*

Proof. Let $f(E_{ij}) = (x_{ks})$. Since $f(E_{ij})$ is a right multiple of E_{ij} it is clear that $x_{ks} = 0$ for $k \neq i$. Now f is a motion; hence

$$[f(E_{ij}) - f(E_{1j})]_r = [E_{ij} - E_{1j}]_r,$$

$$[(x_{ks}) - E_{1j}A]_r = [E_{ij} - E_{1j}]_r,$$

so there exists a $T \in R'$ such that

$$(x_{ks}) - E_{1j}A = (E_{ij} - E_{1j})T.$$

In matrix $(E_{ij} - E_{1j})T$ the ith row is the negative of the first row; hence the same is true in $x_{ks} - E_{1j}A$. Thus

$$x_{is} = a_{js} \qquad \text{for any ordinal } s,$$

and hence

$$f(E_{ij}) = E_{ij}A.$$

COROLLARY 1. *For any $\alpha \in R$,*

$$f(\alpha E_{ij}) = \alpha E_{ij}A = \alpha f(E_{ij}) \qquad i, j \text{ any ordinals.}$$

Proof. Let $(x_{ks}) = f(\alpha E_{ij})$. Then using the same procedure as in the proof of Lemma 1 we obtain

$$x_{is} = \alpha a_{js} \qquad \text{for any ordinal } s,$$

and hence

$$f(\alpha E_{ij}) = \alpha E_{ij}A = \alpha f(E_{ij}).$$

Let $I_i + E_{w_1w_1} + E_{w_2w_2} + \ldots + E_{w_iw_i}$, where $w_1, w_2, \ldots, w_i$ is any finite set of ordinal numbers. Thus, I_i is a matrix with i 1's arbitrarily down the diagonal and zeros elsewhere.

LEMMA 2. *For f, $f(I_i) = I_iA$, with $i = 1, 2, \ldots, n$.*

Proof. $I_1 = E_{w_1w_1}$; hence $f(I_1) = I_1A$. We complete the proof by finite induction. Suppose

$$f(I_t) = I_tA$$

and consider I_{t+1}. Let $f(I_{t+1}) = (x_{ks})$. Then

$$[f(I_{t+1}) - f(E_{w_{t+1}w_{t+1}})]_r = [I_{t+1} - E_{w_{t+1}w_{t+1}}]_r = [I_t]_r,$$

and hence there exists a $T \in R'$ such that

$$(x_{ks}) - E_{w_{t+1}w_{t+1}}A = I_tT.$$

Therefore

$$x_{t+1\,s} = a_{w_{t+1}s}, \quad s \text{ any ordinal.}$$

Also,

$$[f(I_{t+1}) - f(I_t)]_r = [I_{t+1} - I_t]_r = [E_{w_{t+1}w_{t+1}}]_r.$$

Hence there exists a $T \in R'$ such that

$$(x_{ks}) - I_tA = E_{w_{t+1}w_{t+1}}T.$$

Therefore,

$$x_{ks} = a_{ks}, \qquad k = w_1, w_2, \ldots, w_t, \quad s \text{ any ordinal,}$$

and with $x_{ks} = 0$ for $k > t + 1$ we have

$$f(I_{t+1}) = I_{t+1}A,$$

which completes the induction.

Let $\alpha_{w1}, \alpha_{w2}, \ldots, \alpha_{wi}$ be a finite set of arbitrary but fixed elements of R for any finite i, and define

$$N_i(\alpha_{w1}, \alpha_{w2}, \ldots, \alpha_{wi}) = \alpha_{w1}E_{w_1w_1} + \alpha_{w2}E_{w_2w_2} + \ldots + \alpha_{w_i}E_{w_iw_i},$$

where $w_1, w_2, \ldots, w_i$ is any finite set of ordinal numbers.

LEMMA 3. *For f, $f(N_i) = N_iA$, for $i = 1, 2, \ldots, n$, with n finite.*

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Proof. $N_1 = \alpha_1 E_{w_1 w_1}$; hence for $I = 1$, $f(N_i) = N_i A$. Suppose the lemma is valid for $i = t$, that is

$$f(N_t) = N_t A,$$

and let

$$f(N_{t+1}) = (x_{ks}).$$

Now, $[f(N_{t+1}) - f(N_t)]_r = [N_{t+1} - N_t]_r$, so there exists a $T \in R'$ such that

$$(x_{ks}) - N_t A = (N_{t+1} - N_t)T.$$

Thus $x_{ks} = \alpha_k a_{ks}$, for $k = w_1, w_2, \ldots, w_t$, and s any ordinal. Also

$$[f(N_{t+1}) - f(\alpha_{wt+1} E_{wt+1wt+1})]_r = [N_{t+1} - \alpha_{wt+1} E_{wt+1wt+1}]_r.$$

Hence there exists a $T \in R'$ such that

$$(x_{ks}) - \alpha_{wt+1} E_{wt+1wt+1} = (N_t)T.$$

Thus, $x_{ks} = \alpha_k a_{ks}$ for $k = w_{t+1}$ and s any ordinal. This combined with $x_{ks} = 0$ for $k > t + 1$ gives

$$f(N_{t+1}) = N_{t+1} A,$$

which completes the induction. Hence $f(N_i) = N_i A$ for $i = 1, 2, \ldots, n$.

Again let $\alpha_{w1}, \alpha_{w2}, \ldots, \alpha_{wi}$ be an arbitrary but fixed finite set of elements of R and define for any ordinal w,

$$M_w = \alpha_{w1} E_{ww_1} + \alpha_{w2} E_{ww_2} + \ldots + \alpha_{wi} E_{ww_i},$$

where $w_1, w_2, \ldots, w_i$ is any finite set of ordinal numbers. Note that M_w has α_{wi} in the wth row and w_ith column while N_i has α_{wi} in the w_ith row and w_ith column. Thus, if we look at the sum of the non-zero elements (there are only a finite number) of each column of $M_w - N_i$, it is always zero. Also, the only columns with non-zero elements are $w_1, w_2, \ldots, w_i$.

LEMMA 4. *For f, $f(M_w) = M_w A$, for any ordinal w.*

Proof. Let $f(M_w) = (x_{ks})$; then

$$[(x_{ks}) - f(N_i)]_r = [M_w - N_i]_r,$$

so there exists a $T \in R'$ such that

$$(x_{ks}) - N_i A = (M_w - N_i)T.$$

But the sum of the non-zero elements of each column of $(M_w - N_i)T$ is zero; hence we have

$$x_{ks} = \sum_{j=1}^{i} \alpha_{w_j} a_{w_j s}, \qquad \text{for } k = w,\ s \text{ any ordinal.}$$

This, with the fact that $x_{ks} = 0$ for $k \neq w$, establishes that

$$f(M_w) = M_w A.$$

Let $S = (\alpha_{ij})$ be an arbitrary but fixed matrix in R', and denote by M_w the matrix whose wth row is identical with the wth row of S, the remaining rows consisting entirely of zeros.

LEMMA 5. *For* f, $f(S) = SA$.

Proof. Let $f(S) = (x_{ks})$. Since f is a motion we know that $f(S)$ is a right multiple of S. Thus, any particular row in $f(S)$ is obtained by multiplying the corresponding row vector of S by a row-finite matrix. Now

$$[f(S) - f(M_w)]_r = [S - M_w]_r.$$

Hence there exists a $T \in R'$ such that

$$(x_{ks}) - M_w A = (S - M_w)T.$$

The wth row of $(S - M_w)T$ has all elements zero; hence the wth row of (x_{ks}) is identical with the wth row of $M_w A$. Thus, the wth row of $f(S)$ is the wth row of S times A. Since this is true for any w, it follows that $f(S) = SA$.

The preceding lemmas establish that for any motion f of R' which sends zero into zero there exists a matrix $A \in R'$ such that

$$f(X) = SA, \qquad \text{for all } X \in R',$$

where A is determined as indicated from $f(E_{1j})$, $j = 1, 2, \ldots$. A matrix M is a unit in R' provided M has an inverse. The matrix A is unique and a unit, for assuming otherwise leads immediately to a contradiction of the fact that f is a motion. It is clear that for any unit matrix $A \in R'$, the mapping $X \to XA$ is a motion of R'.

Proof of Theorem 1. Let A, B be fixed elements of R' with A a unit, and consider the mapping F such that $F(X) = XA + B$, $X \in R'$. Suppose for X, $Y \in R'$ that $F(X) = F(Y)$; then

$$XA + B = YA + B \qquad \text{and} \qquad XA = YA,$$

which implies that $X = Y$. Thus, F is a one-to-one mapping.

Consider any $Y \in R'$, and let $X = (Y - B)A^{-1}$; then

$$F(X) = (Y - B)A^{-1}A + B = Y.$$

Hence F maps R' onto R'.

Let X, Y be arbitrary elements of R'.

$$[F(X) - F(Y)]_r = [XA + B - (YA + B)]_r = [XA - YA]_r = [(X - Y)A]_r;$$

therefore

$$[F(X) - F(Y)]_r \subset [X - Y]_r.$$

Now, let $Z \in [X - Y]_r$; that is, there exists some $T \in R'$ such that

$Z = (X - Y)T$. Also, $Z = (X - Y)AA^{-1}T$, which is clearly an element of $[(X - Y)A]_r = [F(X) - F(Y)]_r$; hence

$$[F(X) - F(Y)]_r \supset [X - Y]_r.$$

Therefore $[F(X) - F(Y)]_r = [X - Y]_r$ and we have established that F is a motion of R'.

Next, let F be any motion of R' and define a mapping f such that

$$f(X) = F(X) - F(0), \qquad X \in R'.$$

Clearly, f maps zero into zero, so there exists a unique unit matrix $A \in R'$ such that $F(X) = XA$. Hence

$$XA = F(X) - F(0) \qquad \text{or} \qquad F(X) = XA + F(0).$$

Let $B = F(0)$ and we have

$$F(X) = XA + B,$$

where A, B are fixed elements of R' and A is a unit.

It is worth noting that any motion of R' can be thought of as a rotation followed by a translation.

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