ARITHMETIC INVARIANTS OF SUBDIVISION OF COMPLEXES

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The following problem was raised by M. Brown. Let K be a finite simplicial complex, of dimension n, with $\alpha_i(K)$ simplexes of dimension i. Which of the linear combinations $\sum_0^n \lambda_i \alpha_i(K)$ have the property that they are unaltered by all stellar subdivisions of K? The most obvious invariant is the Euler characteristic; there are also some identities that hold for manifolds **(2)**, so, if K is a manifold, they remain true on subdivision. We shall see that no other expressions are ever invariant, but if K resembles a manifold in codimensions $\leqslant 2r$ (in a sense defined below) that r of the relations continue to hold.

From now on we make the convention that, for any K, $\alpha_{-1}(K) = 1$. Then $\sum_0^n \lambda_i \alpha_i(K') = \phi$ for all stellar subdivisions K' of K if and only if (putting $\lambda_{-1} = -\phi$). $\sum_{-1}^n \lambda_i \alpha_i(K') = 0$ for all K': we take this version as more convenient. Write $\chi_+(K) = \sum_{-1}^n (-1)^i \alpha_i(K)$ for the reduced Euler characteristic.

By an elementary (or simple) subdivision of K we mean the introduction of a point in some simplex as a new vertex, and consequent subdivisions (Alexander **1**); a stellar subdivision is a sequence of elementary subdivisions. We assume known the definition of the link (complement in **(1)**) of a simplex σ of K; this we write as $\mathrm{lk}(K, \sigma)$.

A simplex σ^{n-r} of K is called *good* if $\chi_+(\mathrm{lk}(K, \sigma)) = (-1)^{r-1}$, *bad* otherwise.

LEMMA 1. *Let K' be a stellar subdivision of K, τ^{n-s} a simplex of K', σ^{n-r} the least simplex of K containing it. Then τ is good or bad according as σ is.*

Proof. By induction, we can suppose that K' is an elementary subdivision. It is then easy to verify that $\mathrm{lk}(K', \tau) \cong \mathrm{lk}(\sigma', \tau) * \mathrm{lk}(K, \sigma)$, where $*$ denotes the join. But $\mathrm{lk}(\sigma', \tau) \cong S^{s-r-1}$, and $\chi_+(A * B) = -\chi_+(A)\chi_+(B)$. So

$$\chi_+(\mathrm{lk}(K', \tau)) = -1.(-1)^{s-r-1}.\chi_+(\mathrm{lk}(K, \sigma)),$$

and this equals $(-1)^{s-1}$ if and only if $\chi_+(\mathrm{lk}(K, \sigma)) = (-1)^{r-1}$.

We call K *good in codimension r* if every simplex of codimension $\leqslant r$ (i.e. dimension $\geqslant n - r$) is good. The invariance of this property under stellar subdivision follows from the lemma. In fact (although we do not need this for our main theorem) we have

PROPOSITION 1. *Being "good in codimension r" is a topologically invariant property.*

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Proof. K is good in codimension r if and only if the set of bad simplexes has dimension $<n - r$. A point is interior to a bad simplex if and only if, when we introduce that point as new vertex, it becomes a bad vertex. So it is enough to show that being a bad vertex is a topological property.

But if P is a vertex of K', and $\text{st}(K', P)$ its (open) star, we have isomorphisms

$$\begin{aligned} \tilde{H}_{i-1}(\text{lk}(K', P)) &\cong H_i(\overline{\text{st}}(K', P), \text{lk}(K', P)) && \text{as } \overline{\text{st}}(K', P) \text{ is contractible,} \\ &\cong H_i(K', K' - \text{st}(K', P)) && \text{by simplicial excision,} \\ &= H_i(K', K' - P) && \text{by a homotopy equivalence.} \end{aligned}$$

So $\chi_+(\text{lk}(K', P)) = -\chi(K, K - P)$, which is topologically invariant.

LEMMA 2. *If K is good in codimension r, then*

$$(1)_r \qquad (-1)^{r-1}\alpha_{n-r}(K) + \sum_{i=0}^{r} (-1)^i \binom{n-r+i+1}{n-r+1} \alpha_{n-r+i}(K) = 0.$$

Proof. For any simplex σ^{n-r}, write $L = \text{lk}(K, \sigma)$. Since σ is good,

$$(-1)^{r-1} + \sum_{i=0}^{r} (-1)^i \alpha_{i-1}(L) = 0.$$

We shall sum this over all $(n - r)$-simplexes of K. Note that an $(i - 1)$-simplex of L corresponds to an $(n - r + i)$-simplex of K, with σ as a face. Since each $(n - r + i)$-simplex of K has exactly $\binom{n-r+i+1}{n-r+1}$ faces of dimension $n - r$, we obtain (1).

The first term in the relation corresponding to $r = 2j - 1$ is $2\alpha_{n-2j+1}(K)$, so the relations (1) corresponding to odd values of r are linearly independent. We shall see that those for even values of r are dependent on them.

We say K has *type* r if r is the greatest integer $\leqslant \frac{1}{2}n$ such that K is good in codimension $2r - 1$.

THEOREM. *Let K be a finite simplicial complex of dimension n and type r. Then every set of numbers $(\lambda_{-1}, \lambda_0, \ldots, \lambda_n)$, such that $\sum_{-1}^{n} \lambda_i \alpha_i(K') = 0$ for all stellar subdivisions of K, is a linear combination of the $r + 1$ sets which appear in*

$$\sum_{-1}^{n} (-1)^i \alpha_i = \chi_+(K)\alpha_{-1}, \qquad (1)_1, (1)_3, \ldots, (1)_{2r-1}.$$

Proof. By Lemma 1, any subdivision of K also has type r, so the above $r + 1$ relations continue to hold.

We shall prove the result by induction on n, the induction step going from $n - 2$ to n. In the cases $n = 0, 1$, $r = 0$. If $n = 1$, subdividing an edge increases each of α_0 and α_1 by 1, so $\lambda_0 + \lambda_1 = 0$. The result is now immediate if $n = 0, 1$.

We now consider the general case. Suppose $(\lambda_{-1}, \lambda_0, \ldots, \lambda_n)$ has the stated property. Let L be the link of a 1-simplex σ^1 of K. Then the effect of subdividing σ^1 is to increase $\sigma_i(K)$ by $\alpha_{i-1}(L) + \alpha_{i-2}(L)$. Since

$$\sum_{-1}^{n} \lambda_i \alpha_i(K) = \sum_{-1}^{n} \lambda_i \alpha_i(K'),$$

we have, subtracting,

$$\sum_{-1}^{n} \lambda_i\{\alpha_{i-1}(L) + \alpha_{i-2}(L)\} = 0, \quad \text{or} \quad \sum_{-1}^{n-2} (\lambda_{i+1} + \lambda_{i+2})\alpha_i(L) = 0.$$

Now the effect on L of elementary subdivision of a simplex of K with σ^1 as face is to perform elementary subdivision of the corresponding simplex of L. Hence the above must hold for all stellar subdivisions of L.

Now K has type r. Since the link of a simplex of codimension i in L is also the link of a simplex of codimension i in K, L is good in codimension $2r - 1$, and has type r if $2r \leqslant n - 2$, and type $r - 1$, if $2r = n - 1$ or n. Hence the vector space of those $(\mu_{-1}, \mu_0, \ldots, \mu_{n-2})$ with $\sum_{-1}^{n-2} \mu_i \alpha_i(L') = 0$ for all stellar subdivisions L' of L has dimension $r + 1$ or r, by the induction hypothesis. We have the relation

$$\sum_{-1}^{n-2} (-1)^i \alpha_i(L) = \chi_+(L)\alpha_{-1}(L).$$

But if $2r \leqslant n - 2$, K is not good in codimension $n - 1$ and so, by Lemma 1, it has (after possible subdivision), both good and bad 1-simplexes. Hence, here $\chi_+(L)$ depends on σ^1, and this relation must be rejected.

There remain in each case at most r linearly independent sets $(\mu_{-1}, \mu_0, \ldots, \mu_{n-2})$ with $\sum_{-1}^{n-2} \mu_i \alpha_i(L) = 0$ for all links L of 1-simplexes in all stellar subdivisions of K. Thus $(\lambda_0 + \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_{n-1} + \lambda_n)$ lies in an r-dimensional vector space, and $(\lambda_0, \lambda_1, \ldots, \lambda_n)$ in an $(r + 1)$-dimensional space. In view of the relation $\lambda_{-1} = -\sum_0^n \lambda_i \alpha_i(K)$, the other λ_i determine λ_{-1}, so we have at most $r + 1$ linearly independent sets $(\lambda_{-1}, \lambda_0, \ldots, \lambda_n)$. Since we already possess $r + 1$ linearly independent sets, this is the complete number.

We note that it follows from the theorem that the relations $(1)_{2i}$ for $i \leqslant r-1$ follow from the $(1)_{2i-1}$ for $i \leqslant r$. However, we can prove more than this directly.

PROPOSITION 2. *The relations* $(1)_{2i-1}$ *for* $i \leqslant k$ *formally imply* $(1)_{2k}$.

Proof. We seek coefficients $x_1, \ldots, x_k$ which give a formal identity

$$-\alpha_{n-2k} + \sum_{i=0}^{2k} (-1)^i \binom{n - 2k + i + 1}{n - 2k + 1} \alpha_{n-2k+i}$$
$$= \sum_{j=1}^{k} x_j \left[\alpha_{n-2k+2j-1} + \sum_{i=0}^{2k-2j+1} (-1)^i \binom{n - 2k + 2j + i}{n - 2k + 2_j} \alpha_{n-2k+2j+i-1}\right].$$

We observe that there are $2k$ equations (equating coefficients of α_{n-r} for

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$0 \leqslant r < 2k$) for the k unknowns x_j: we shall simplify by some transformations. First let i, j run to ∞: if we can solve the extended system, we have (putting $\alpha_r = 0$ for $r > n$) the required identity. Next replace the α_r by formal powers α^r: these are linearly independent, so this makes no essential change. But we can sum the series, and the equation reduces to

$$\alpha^{n-2k}(-1 + (1+\alpha)^{-(n-2k+2)}) = \sum_{j=1}^{\infty} x_j \alpha^{n-2k+2j-1}(1 + (1+\alpha)^{-(n-2k+2j+1)}),$$

or

$$-1 + (1+\alpha)^{-(n-2k+2)} = \sum_{j=1}^{\infty} x_j \alpha^{2j-1}(1 + (1+\alpha)^{-(n-2k+2j+1)}).$$

Now substitute $1 + \alpha = e^{\beta}$: this gives an isomorphism between the formal power series rings in α and in β. Our equation becomes

$$-1 + e^{-\beta(n-2k+2)} = \sum_{j=1}^{\infty} x_j (e^{\beta} - 1)^{2j-1}(1 + e^{-\beta(n-2k+2j+1)})$$

or, multiplying by $e^{\frac{1}{2}\beta(n-2k+2)}$, and expressing by hyperbolic sines and cosines,

$$-2 \sinh \tfrac{1}{2}\beta(n - 2k + 2) = \sum_{j=1}^{\infty} x_j (2 \sinh \tfrac{1}{2}\beta)^{2j-1} \cosh \tfrac{1}{2}\beta(n - 2k + 2j + 1).$$

In this last equation, each term is an odd function of β. The coefficient of x_j is a power series with leading term $2\beta^{2j-1}$. Thus equating (in turn) coefficients of odd powers of β, we obtain a series of equations which provide an inductive definition of the desired coefficients x_i. (With the vanishing of coefficients of even powers of β, the number of equations is "reduced to the same" as the number of unknowns.)

COROLLARY 1. *Suppose the link of every even-dimensional simplex of K has Euler characteristic 2. Then K has characteristic 0.*

For K certainly has some odd dimension $2k - 1$; we see, as in Lemma 2, that $(1)_1, (1)_3, \ldots, (1)_{2k-1}$ hold, so by the Proposition, $(1)_{2k}$ holds, i.e. K has characteristic 0.

COROLLARY 2. *If every σ^{n-2i+1} in K^n is good for $1 \leqslant i \leqslant r$, so is each σ^{n-2i} for $1 \leqslant i \leqslant r$.*

We need only apply Corollary 1 to each $\operatorname{lk}(K^n, \sigma^{n-2i})$. It follows that if K is good in codimension $2r - 1$, it is also good in codimension $2r$.

We conclude with a few comments on manifolds and low dimensions (which suggested the problems treated above). Of course, any manifold of dimension $2r - 1$ or $2r$ is good in codimension $2r$. Conversely, if $n = 2$, K^2 is good in codimension 2 when each edge lies on just two triangles (we now see at once that the link of each vertex is a disjoint union of circles, with Euler characteristic 0), so K is a pseudo-2-manifold, obtained from an actual 2-manifold M by identifying some vertices. If the genus of M is g, and i identifications are made, then $\chi(K) = 2 - 2g - i$. So if $\chi(K) = 2$, $g = i = 0$, and K is a sphere S^2.

If $n = 3$, and K is good in codimension 3, then by the above, each vertex link is a sphere S^2, so K is a 3-manifold; in particular, $\chi(K) = 0$. In this case there is a well-known converse (**3**, p. 208): suppose K is good in codimension 2. Then if L is the link of any vertex, by the above, we have

$$\alpha_0(L) - \alpha_1(L) + \alpha_2(L) \leqslant 2.$$

Summing over vertices of K, this becomes

$$2\alpha_1(K) - 3\alpha_2(K) + 4\alpha_3(K) \leqslant 2\alpha_0(K),$$

and since $\alpha_2(K) = 2\alpha_3(K)$, this is equivalent to $\chi(K) \geqslant 0$. If we are given $\chi(K) = 0$, we must have equalities throughout, so each L is a sphere and again K is a manifold.

The relation with manifolds breaks down in higher dimensions: the suspension of any 3-manifold is good in codimension 4, but need not even be a homology 4-manifold.

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