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ON NILPOTENT AND POLYCYCLIC GROUPS

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A group G is torsion-free, finitely generated, and nilpotent if and only if G is a supersolvable R-group. An ordered polycyclic group G is nilpotent if and only if there exists an order on G with respect to which the number of convex subgroups is one more than the length of G. If the factors of the upper central series of a torsion-free nilpotent group G are locally cyclic, then consecutive terms of the series are jumps, and the terms are absolutely convex subgroups.

1. PRELIMINARIES

The definitions of a *partially ordered group, (fully) ordered group, 0-group, 0*-group, positive cone, convex subgroup, jump*, and *R-group* can be found in [**1**] or in [**4**]. A subgroup C of an 0-group G is *absolutely convex* if and only if C is convex with respect to each order on G. By the *length* of a polycyclic group G is meant the number of infinite cyclic factors in any cyclic normal series of G. A subgroup A of a group G is *isolated* if $a \in G$, n a positive integer, and $a^n \in A$ imply $a \in A$. Finally, if S is a nonempty subset of a group G, then the *isolator* of S is the intersection of all isolated subgroups of G containing S.

2. RESULTS

THEOREM 1. *If G is a torsion-free, nilpotent group with upper central series $\{1\} = Z_0 \subset Z_1 \subset \cdots \subset Z_n = G$ and Z_{i+1}/Z_i is locally cyclic for $i = 0, 1, \cdots, s$, then $\{1\}, Z_1, \cdots, Z_s, Z_{s+1}$ are absolutely convex subgroups of G and $\{1\} \prec Z_1 \prec \cdots \prec Z_s \prec Z_{s+1}$ are jumps in the family of convex subgroups of any order on G.*

PROOF: If G is abelian, then $\{\{1\}, G\}$ is the family of convex subgroups of G with respect to any order on G; otherwise, there exists a subgroup C of G convex with respect to some order $\leqslant$ on G such that $\{1\} \neq C \subset G$. Let $1 \neq c \in C$ and $g \in G - C$. Then, as G is locally cyclic, $\langle g, c\rangle = \langle g_0\rangle$ for some $g_0 \in G$. Thus, there exists an integer k such that $g_0^k \in C$. Without loss of generality, we may assume $1 \leqslant g_0$. Then $1 \leqslant g_0 \leqslant g_0^k$, whence, $g_0 \in C$, a contradiction.

Suppose now that G is nonabelian and let there be given an order with positive cone $P(G)$. Suppose $Z_1 = Z(G)$ is locally cyclic and let $\{1\} \neq C$ be a convex subgroup of G

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(if no convex subgroup C of G exists such that $\{1\} \subset C \subset G$, then G is known to be 0-isomorphic to a subgroup of the additive group of real numbers, whence abelian). As G satisfies the maximal condition for subgroups locally, all convex subgroups are normal in G (see [**1**, p.54]) and, thus, $\{1\} \neq Z_1 \cap C$ since G is nilpotent. Suppose $Z_1 \not\subseteq C$, so there exists $z \in Z_1 - C$. Let $1 \neq x \in Z_1 \cap C$ and consider $\langle z, x\rangle$. Now $\langle z, x\rangle \subseteq Z_1$ and Z_1 is locally cyclic, so for some integers m and n, $z = a^m$ and $x = a^n$. Thus, $Z^n = a^{mn} = x^m$ and, hence, $z^n = x^m \in C$. Since C is convex and, thus, isolated, we have that $z \in C$, a contradiction. Therefore, $Z_1 \subseteq C$, where C denotes any nontrivial subgroup of G convex with respect to $P(G)$. If A is the intersection of all nontrivial subgroups of G which are convex with respect to $P(G)$, then A is convex with respect to $P(G)$ and $Z_1 \subseteq A$. Moreover, A is the unique minimal convex subgroup of G; that is, $\{1\} \prec A$ is a jump in the family of subgroups of G which are convex with respect to $P(G)$. Thus, by Lemma 1 of [**2**], $A \subseteq Z(G) = Z_1$. Therefore, $A = Z_1$, Z_1 is convex with respect to $P(G)$, and $\{1\} \prec A = Z_1$ is a jump in the family of convex subgroups of G with respect to $P(G)$.

If $G = Z_2$, we are finished. If not, we consider the torsion-free, nilpotent group G/Z_1. Since Z_1 is convex, the given order $P(G)$ induces an order on G/Z_1. Now $Z(G/Z_1) = Z_2/Z_1$ and Z_2/Z_1 is locally cyclic, so by applying the above argument for G and Z_1 to G/Z_1 and Z_2/Z_1, we have that Z_2/Z_1 is a convex subgroup of G/Z_1 and that $\bar{1} \prec Z_2/Z_1$ is a jump in the family of convex subgroups of G/Z_1. But there is a one-to-one correspondence between the convex subgroups of G/Z_1 and the convex subgroups of G containing Z_1. Thus, both Z_1 and Z_2 are convex with respect to $P(G)$ and $\{1\} \prec Z_1$, $Z_1 \prec Z_2$ are jumps. Repeated applications of the argument to $G/Z_2, \ldots, G/Z_s$ complete the proof. ∎

THEOREM 2. *An ordered, polycyclic group G is nilpotent if and only if there exists an order on G with respect to which the number of convex subgroups is $L(G)+1$, where $L(G)$ denotes the length of G.*

PROOF: First, let us assume that for some order on the polycyclic group G the number of convex subgroups is $r+1$, where $r = L(G)$. Then for each jump $D \prec C$ in the chain of convex subgroups, C/D is an infinite cyclic group. Thus, $\operatorname{Aut}(C/D)$ is cyclic of order two. Note, as G satisfies the maximal condition for subgroups, that C/D is a normal subgroup of the ordered group G/D and that conjugation of C/D by an element of G/D is an 0-automorphism of C/D. Since there are only two automorphisms of C/D and only one–the identity–is order-preserving, we have $Dc^g = Dc$ for each $c \in C$ and each $g \in G$. Thus, $C/D \subseteq Z(G/D)$ for each jump $D \prec C$, so, as the chain of convex subgroups is finite, G is nilpotent.

Next, let G be a torsion-free, finitely generated, nilpotent group. We shall induct on the length of G. Let $\{1\} = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$ be the upper central series for

G. If $L(G) = 1$ then $G = Z_1$ is an infinite cyclic group, and $\{\{1\}, G\}$ is the family of convex subgroups of G with respect to the order on G given by $1 \leqslant g$, where $G = \langle g \rangle$. Let us assume the theorem true for all torsion-free, finitely generated, nilpotent groups G such that $L(G) < k$; let G be such a group and suppose $L(G) = k$. Let P_1 denote an arbitrary, but fixed, order on G. Choose $1 \neq z \in Z(G)$, and let $C_1 = I(z)$ be the isolator of z in G. By [**3**, p.246] C_1 is a torsion-free, locally cyclic group. As G is polycyclic, C_1 is finitely generated, whence C_1 is an infinite cyclic, normal, isolated subgroup of G. Thus, $L(G/C_1) = k - 1$ and, thus, by inductive assumption, there exists an order P_2 on G/C_1 such that the number of convex subgroups is k. Therefore, $P(G) = (P_1 \cap C_1) \cup \{x \mid x \in G - C_1 \ \& \ xC_1 \in P_2\}$ is an order on G with respect to which C is convex and with respect to which the number of convex subgroups is $k + 1$. ∎

THEOREM 3. *A group G is a supersolvable R-group if and only if G is a torsion-free, finitely generated, nilpotent group.*

PROOF: If G is torsion-free, finitely generated, and nilpotent, then G is a supersolvable 0-group, whence a supersolvable R-group.

Suppose now that G is a supersolvable R-group. We first show $Z(G) \neq \{1\}$; Let $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ be a cyclic invariant series for G. Then G_1 is an infinite cyclic group, whence $\mathrm{Aut}(G_1)$ is of order two. Also, G_1 is normal in G, so $G/C_G(G_1) = N_G(G_1)/C_G(G_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(G_1)$. Therefore, $g \in G$ implies $g^2 \in C_G(G_1)$; that is, $[g^2, x] = 1$ for each $g \in G$ and each $x \in G_1$. But G is an R-group, so, by [**3**, p.244], $[g, x] = 1$. Thus $\{1\} \neq G_1 \subseteq Z(G)$.

Next, by [**3**, p.244], $G/Z(G)$ is a supersolvable R-group. As G_1 is an infinite cyclic subgroup of $Z(G)$, the length of $G/Z(G)$ is less than the length of G. By induction on the length of G, it follows that $G/Z(G)$ is nilpotent, whence G is nilpotent. ∎

COROLLARY. *A group G is a locally supersolvable, R-group if and only if G is torsion-free and locally nilpotent.*

Since a torsion-free, locally nilpotent group is an 0^*-group, we also have

COROLLARY. *A locally supersolvable group G is an 0^*-group if and only if G is an R-group.*

COROLLARY. *If G is a locally supersolvable, R-group, then G and each subgroup of G is an 0^*-group.*

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