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COMPACT LEFT MULTIPLIERS ON BANACH ALGEBRAS RELATED TO LOCALLY COMPACT GROUPS

M. J. MEHDIPOUR and R. NASR-ISFAHANI✉

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Abstract

We deal with the dual Banach algebras $L_0^\infty(G)^*$ for a locally compact group G. We investigate compact left multipliers on $L_0^\infty(G)^*$, and prove that the existence of a compact left multiplier on $L_0^\infty(G)^*$ is equivalent to compactness of G. We also describe some classes of left completely continuous elements in $L_0^\infty(G)^*$.

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1. Introduction and preliminaries

Let G be a locally compact group, and $L^\infty(G)$ be the usual Lebesgue space as defined in [6] equipped with the essential supremum norm $\|\cdot\|_\infty$. Let also $L_0^\infty(G)$ be the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ that vanish at infinity; that is, for each $\varepsilon > 0$, there is a compact subset K of G for which

$$\|f\chi_{G\setminus K}\|_\infty < \varepsilon,$$

where $\chi_{G\setminus K}$ denotes the characteristic function of $G \setminus K$ on G. For an extensive study of $L_0^\infty(G)$ see Lau and Pym [9]; see also Isik *et al.* [8] for the compact group case.

Let $L^1(G)$ be the group algebra of G defined as in [6] equipped with the convolution product $*$ and the norm $\|\cdot\|_1$. Remark that $L^\infty(G)$ is the continuous dual of $L^1(G)$ under the usual duality. For any $\phi \in L^1(G)$ and $g \in L_0^\infty(G)$ we have

$$\frac{1}{\Delta}\widetilde{\phi} * g \in L_0^\infty(G),$$

where Δ denotes the modular function of G and

$$\widetilde{\phi}(x) = \phi(x^{-1})$$

for all $x \in G$; see [9, Proposition 2.7]. So, for every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we may define the function $ng \in L^\infty(G)$ by

$$\langle ng, \phi \rangle := \left\langle n, \frac{1}{\Delta}\tilde{\phi} * g \right\rangle$$

for all $\phi \in L^1(G)$. It is also well known from [9] that the space $L_0^\infty(G)$ is left introverted in $L^\infty(G)$; that is, for each $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we have $ng \in L_0^\infty(G)$. This lets us endow $L_0^\infty(G)^*$ with the *first Arens product* '$\cdot$' defined by

$$\langle m \cdot n, g \rangle = \langle m, ng \rangle$$

for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$. Then $L_0^\infty(G)^*$ with this product is a Banach algebra; see [9].

Let $M(G)$ denote the measure algebra of G as defined in [6] endowed with the convolution product $*$ and the total variation norm $\|\cdot\|$. Then $M(G)$ is the continuous dual of $C_0(G)$, the space of all continuous functions on G vanishing at infinity. For any $\phi \in L^1(G)$ and $g \in L_0^\infty(G)$ we have

$$\frac{1}{\Delta}\tilde{\phi} * g \in C_0(G);$$

so, for every $\mu \in M(G)$ and $g \in L_0^\infty(G)$, we may define the function $\mu g \in L^\infty(G)$ by

$$\langle \mu g, \phi \rangle := \left\langle \mu, \frac{1}{\Delta}\tilde{\phi} * g \right\rangle$$

for all $\phi \in L^1(G)$. It follows that $\mu g \in L_0^\infty(G)$; in fact, $\mu g = ng$ for all extensions $n \in L_0^\infty(G)^*$ of $\mu \in C_0(G)^*$. This enables us to define $m \cdot \mu \in L_0^\infty(G)^*$ for all $m \in L_0^\infty(G)^*$ by

$$\langle m \cdot \mu, g \rangle = \langle m, \mu g \rangle$$

for all $g \in L_0^\infty(G)$; in fact, $m \cdot \mu = m \cdot n$ for all extensions $n \in L_0^\infty(G)^*$ of $\mu \in C_0(G)^*$.

For each $\phi \in L^1(G)$, we may consider ϕ as a linear functional in $L_0^\infty(G)^*$ defined by the usual way. So, there is a linear isometric embedding of $L^1(G)$ into $L_0^\infty(G)^*$. Also, observe that $\phi \cdot \psi = \phi * \psi$ for all $\phi, \psi \in L^1(G)$. It is well known that $L^1(G)$ is a closed ideal in $L_0^\infty(G)^*$; see [9]. Furthermore, an easy application of the Goldstine's theorem shows that $L^1(G)$ is weak* dense in $L_0^\infty(G)^*$. For any n in $L_0^\infty(G)^*$, the map $m \mapsto m \cdot n$ is weak*–weak* continuous on $L_0^\infty(G)^*$. For an element m in $L_0^\infty(G)^*$, the map $n \mapsto m \cdot n$ is not in general weak*–weak* continuous on $L_0^\infty(G)^*$ unless m is in $L^1(G)$; see Lau and Ülger [10] for details. For each $m, n \in L_0^\infty(G)^*$, there exist two nets (ϕ_α) and (ψ_β) in $L^1(G)$ such that $\phi_\alpha \to m$ and $\psi_\beta \to n$ in the weak* topology of $L_0^\infty(G)^*$, and therefore

$$m \cdot n = \text{weak}^* - \lim_\alpha \ \text{weak}^* - \lim_\beta \phi_\alpha * \psi_\beta.$$

This implies that the restriction map $\mathcal{P} : L_0^\infty(G)^* \to C_0(G)^*$ is a homomorphism. For any $m, n \in L_0^\infty(G)^*$ we have

$$m \cdot n = m \cdot \mathcal{P}(n).$$

Let us recall that an element $u \in L_0^\infty(G)^*$ is called a mixed identity if

$$\phi \cdot u = u \cdot \phi = \phi$$

for all $\phi \in L^1(G)$. Denote by $\Lambda_0(G)$ the nonempty set of all mixed identities u with norm one in $L_0^\infty(G)^*$, and note that $u \in \Lambda_0(G)$ if and only if it is a weak*-cluster point of an approximate identity in $L^1(G)$ bounded by one or, equivalently, an extension of δ_e from $C_0(G)$ to $L_0^\infty(G)$ with norm one, where $\delta_e \in M(G)$ denotes the Dirac measure at the identity element e of G; furthermore, $u \in \Lambda_0(G)$ if and only if $\|u\| = 1$ and

$$m \cdot u = m$$

for all $m \in L_0^\infty(G)^*$; that is, u is a right identity for $L_0^\infty(G)^*$ with norm one; see Ghahramani, Lau and Losert [5].

Let $\mathcal{A}$ be a Banach algebra; a bounded operator $T : \mathcal{A} \to \mathcal{A}$ is called a *left multiplier* if

$$T(ab) = T(a)b$$

for all $a, b \in \mathcal{A}$. For any $a \in \mathcal{A}$, the left multiplier $b \mapsto ab$ on $\mathcal{A}$ is denoted by λ_a; also, a is said to be a *left completely continuous element of* $\mathcal{A}$ if λ_a is a compact operator on $\mathcal{A}$. Right multipliers and right completely continuous elements are defined similarly.

Compact left or right multipliers on the second dual algebras $L^1(G)^{**}$ and $M(G)^{**}$ have been studied by Ghahramani and Lau in [2–4]. In the same papers, they have obtained some results on the question of existence of nonzero compact left or right multipliers on $L^1(G)^{**}$. Losert [11] has proved, among other things, that if G is noncompact, then there is no nonzero compact left multipliers on $L^1(G)^{**}$ or $M(G)^{**}$. The authors [12] have recently studied right completely continuous elements of $L_0^\infty(G)^*$; they proved that $L_0^\infty(G)^*$ has a certain right completely continuous element if and only if G is compact.

Our aim in this paper is to study compact left multipliers on $L_0^\infty(G)^*$. In Section 2 we prove that G is compact if and only if there is a nonzero compact left multiplier on $L_0^\infty(G)^*$. In Section 3 we study some classes of left completely continuous elements in $L_0^\infty(G)^*$. Finally, in Section 4 we investigate the relation between compact left multipliers on $L_0^\infty(G)^*$ and its right annihilator.

2. The existence of compact left multipliers on $L_0^\infty(G)^*$

We commence this section with the main result of the paper. First, let us recall that a linear functional $k \in L_0^\infty(G)^*$ is said to have *compact carrier* K if

$$k(g) = k(\chi_K g) \quad \text{for all } g \in L_0^\infty(G).$$

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THEOREM 2.1. *Let G be a locally compact group. Then the following assertions are equivalent:*

(a) *G is compact;*
(b) *there is a nonzero compact left multiplier on $L_0^\infty(G)^*$;*
(c) *$L_0^\infty(G)^*$ has a nonzero left completely continuous element.*

PROOF. (a) $\Rightarrow$ (b). Suppose that G is compact and m is the normalized left Haar measure on G. Then

$$m \cdot n = \langle n, 1\rangle m$$

for all $n \in L_0^\infty(G)^*$ and so m is a nonzero left completely continuous element of $L_0^\infty(G)^*$.

(b) $\Rightarrow$ (c). Suppose that there is a nonzero compact left multiplier T on $L_0^\infty(G)^*$. Choose $n \in L_0^\infty(G)^*$ with $T(n) \neq 0$. Then $\lambda_{T(n)} : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is compact. Now, choose an element u of $\Lambda_0(G)$. Since $m \cdot u = m$ for all $m \in L_0^\infty(G)^*$, it follows that

$$\lambda_{T(n)}(u) = T(n) \cdot u = T(n) \neq 0.$$

That is, $T(n)$ is a nonzero left completely continuous element of $L_0^\infty(G)^*$.

(c) $\Rightarrow$ (a). First, let us remark from Section 1 that for each $n \in L_0^\infty(G)^*$ and $x \in G$, the element $n \cdot \delta_x$ of $L_0^\infty(G)^*$ is defined by

$$\langle n \cdot \delta_x, g\rangle = \langle n, \delta_x g\rangle$$

for all $g \in L_0^\infty(G)$, where $\delta_x \in M(G)$ denotes the Dirac measure at $x \in G$ and $\delta_x g \in L_0^\infty(G)$ is defined by

$$\langle \delta_x g, \phi\rangle = \left(\frac{1}{\Delta}\tilde{\phi} * g\right)(x) \quad (x \in G).$$

Now, suppose that $L_0^\infty(G)^*$ has a nonzero left completely continuous element m and G is not compact. Let $\mathcal{C}$ be the family of all compact subsets of G directed by upward inclusion. For each $C \in \mathcal{C}$, there is an element x_C in G with $x_C \notin C$. Choose $u \in \Lambda_0(G)$ and note that $(u \cdot \delta_{x_C})_{C\in\mathcal{C}}$ is a bounded net in $L_0^\infty(G)^*$. Since m is a left completely continuous element of $L_0^\infty(G)^*$ there is a subnet $(x_{C(\gamma)})_{\gamma\in\Gamma}$ of the net $(x_C)_{C\in\mathcal{C}}$ such that

$$\|m \cdot (u \cdot \delta_{x_{C(\gamma)}}) - n\| \to 0$$

for some $n \in L_0^\infty(G)^*$. However, $m \cdot u = m$ and so

$$\begin{aligned} m \cdot (u \cdot \delta_{x_{C(\gamma)}}) &= (m \cdot u) \cdot \delta_{x_{C(\gamma)}} \\ &= m \cdot \delta_{x_{C(\gamma)}} \end{aligned}$$

for all $\gamma \in \Gamma$. This shows that

$$\|m \cdot \delta_{x_{C(\gamma)}} - n\| \to 0.$$

This together with $\|m \cdot \delta_x\| = \|m\|$ for all $x \in G$ imply that

$$\|m\| = \|n\|.$$

It follows that there exists $g \in L_0^\infty(G)$ with $\|g\|_\infty \leq 1$ such that

$$|\langle n, g\rangle| > \|m\|/2;$$

we may also assume that

$$\|g\chi_{G\setminus S}\|_\infty = 0$$

for some $S \in \mathcal{C}$. Furthermore, by [9, Proposition 2.6], there is $k \in L_0^\infty(G)^*$ with compact carrier K such that

$$\|k - m\| < \|m\|/4.$$

Now, $K^{-1}S \in \mathcal{C}$. Thus, there is $\gamma_0 \in \Gamma$ such that

$$K^{-1}S \subseteq D \quad \text{and} \quad \|m \cdot \delta_{x_D} - n\| \leq \|m\|/4,$$

where $D = C(\gamma_0)$. Then $\chi_K\,(\delta_{x_D} g) = 0$ in $L_0^\infty(G)$; indeed, for every $\phi \in L^1(G)$ we have

$$\begin{aligned}\langle \chi_K(\delta_{x_D} g), \phi\rangle &= \langle \delta_{x_D} g, \chi_K \phi\rangle \\ &= \left\langle \delta_{x_D}, \frac{1}{\Delta}\widetilde{(\chi_K \phi)} * g \right\rangle \\ &= 0;\end{aligned}$$

this is because $x_D \notin D$ and that $(1/\Delta)\widetilde{(\chi_K \phi)} * g$ is a continuous function with compact support contained in $K^{-1}S$. In particular,

$$\begin{aligned}\langle k \cdot \delta_{x_D}, g\rangle &= \langle k, \delta_{x_D} g\rangle \\ &= \langle k, \chi_K(\delta_{x_D} g)\rangle \\ &= 0.\end{aligned}$$

Consequently,

$$\begin{aligned}|\langle n, g\rangle| &\leq |\langle k \cdot \delta_{x_D} - n, g\rangle| \\ &\leq |\langle (k - m) \cdot \delta_{x_D}, g\rangle| + |\langle m \cdot \delta_{x_D} - n, g\rangle| \\ &\leq \|m\|/4 + \|m\|/4\end{aligned}$$

we have $|\langle n, g\rangle| \leq \|m\|/2$, a contradiction. □

For $I \subseteq L_0^\infty(G)^*$, the left annihilator of I is denoted by $\mathrm{lan}(I)$ and is defined by

$$\mathrm{lan}(I) = \{\iota \in I \mid \iota \cdot I = \{0\}\},$$

and the linear span of all products in I is denoted by I^2.

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COROLLARY 2.2. *Let I be a right ideal in $L_0^\infty(G)^*$ such that* $\text{lan}(I) = \{0\}$ *or* $\overline{I^2} = I$. *If G is not compact, then there is no nonzero compact left multiplier on I.*

PROOF. Suppose that $T : I \to I$ is a compact left multiplier. Fix $m, n \in I$. Then $T(m \cdot n)$ is a left completely continuous element of $L_0^\infty(G)^*$; indeed, for each $k \in L_0^\infty(G)^*$ with $\|k\| \le 1$ we have $n \cdot k \in I$, hence

$$\begin{aligned} T(m \cdot n) \cdot k &= T(m) \cdot n \cdot k \\ &= T(m \cdot n \cdot k) \\ &\in \{T(\iota) : \iota \in I,\ \|\iota\| \le \|m\|\,\|n\|\}. \end{aligned}$$

Since G is not compact, it follows from Theorem 2.1 that $T(m \cdot n) = 0$. So

$$T(m) \cdot I = \{0\},$$

and thus $T(m) \in \text{lan}(I)$. Therefore, $T = 0$ if $\text{lan}(I) = \{0\}$. Similarly, $T = 0$ if $\overline{I^2} = I$. □

Let us remark that Corollary 2.2 is applicable to any closed right ideal I of $L_0^\infty(G)^*$ with a bounded approximate identity; so, it improves the case $I = L^1(G)$ due to Sakai [13, Theorem 1].

3. Left completely continuous elements of $L_0^\infty(G)^*$

We commence this section with the following result which is needed in the following.

PROPOSITION 3.1. *Let G be a locally compact group. Then the following assertions are equivalent:*

(a) *$L_0^\infty(G)^*$ has a bounded approximate identity;*
(b) *$L_0^\infty(G)^*$ has an identity;*
(c) *$L_0^\infty(G)^* = L^1(G)$;*
(d) *G is discrete.*

PROOF. (a) ⇒ (b). Suppose that (a) holds. Let (u_γ) be a bounded approximate identity for $L_0^\infty(G)^*$, and let u be a weak* cluster point of (u_γ) in $L_0^\infty(G)^*$. Without loss of generality, we may assume that $u_\gamma \to u$ in the weak* topology. Let $n \in L_0^\infty(G)^*$. Then the weak* continuity of the map $m \mapsto m \cdot n$ on $L_0^\infty(G)^*$ shows that

$$u_\gamma \cdot n \to u \cdot n$$

in the weak* topology of $L_0^\infty(G)^*$. However,

$$u_\gamma \cdot n \to n$$

in the norm topology of $L_0^\infty(G)^*$. So $u \cdot n = n$.

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Now, for every $\phi \in L^1(G)$, using the weak* continuity of the map $k \mapsto \phi \cdot k$ on $L_0^\infty(G)^*$ we conclude that

$$\phi \cdot u_\gamma \to \phi \cdot u$$

in the weak* topology of $L_0^\infty(G)^*$. This together with that (u_γ) is a bounded approximate identity for $L_0^\infty(G)^*$ imply that

$$\phi \cdot u = \phi.$$

It follows from the weak* density of $L^1(G)$ in $L_0^\infty(G)^*$ that $n \cdot u = n$.

(b) ⇒ (c). It is well known from [9] that

$$L^1(G) = \bigcap \{u \cdot L_0^\infty(G)^* \mid u \in \Lambda_0(G)\}.$$

So, the result follows from the fact that $\Lambda_0(G) = \{u_0\}$, where u_0 is the identity element of $L_0^\infty(G)^*$; indeed, any $u \in \Lambda_0(G)$ is a right identity for $L_0^\infty(G)^*$, and so

$$u_0 = u_0 \cdot u = u.$$

(c) ⇒ (d). Let e denote the identity element of G and u be an extension of δ_e from $C_0(G)$ to an element u of $L_0^\infty(G)^*$. By assumption, there is $\phi \in L^1(G)$ such that

$$\langle u, g \rangle := \int_G \phi(x) g(x)\, dx \quad (g \in L_0^\infty(G)).$$

In particular, δ_e is absolutely continuous with respect to left Haar measure on G, and therefore G is discrete; see [6].

(d) ⇒ (a). This is clear. □

COROLLARY 3.2. *Let G be a locally compact group. Then G is discrete if and only if any left multiplier on $L_0^\infty(G)^*$ is of the form λ_m for some $m \in L_0^\infty(G)^*$.*

PROOF. Let e be the identity element of G and δ_e be the Dirac measure at e. If G is discrete and T is a left multiplier on $L_0^\infty(G)^*$, then $T = \lambda_{T(\delta_e)}$ trivially.

Conversely, suppose that any left multiplier on $L_0^\infty(G)^*$ is of the form λ_m for some $m \in L_0^\infty(G)^*$. Then there is $\delta \in L_0^\infty(G)^*$ such that λ_δ is the identity operator on $L_0^\infty(G)^*$. In particular, δ is a left identity for $L_0^\infty(G)^*$. Since $L_0^\infty(G)^*$ always has a right identity, it follows that $L_0^\infty(G)^*$ has an identity element. So, G is discrete by Proposition 3.1. □

It is obvious that $T|_{L^1(G)}$ is a compact left multiplier on $L^1(G)$ if T is a compact left multiplier on $L_0^\infty(G)^*$. Our next result shows that this is an 'if and only if' statement for certain left multipliers T on $L_0^\infty(G)^*$.

PROPOSITION 3.3. *Let G be a locally compact group and $\phi \in L^1(G)$. Then ϕ is a left completely continuous element of $L^1(G)$ if and only if ϕ is a left completely continuous element of $L_0^\infty(G)^*$.*

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PROOF. Suppose that $\lambda_\phi : L^1(G) \to L^1(G)$ is compact. Let $(e_\gamma)_{\gamma\in\Gamma}$ be an approximate identity for $L^1(G)$ bounded by one. Then for any $n \in L_0^\infty(G)^*$ with $\|n\| \le 1$, we have

$$\begin{aligned}\|\phi \cdot n - \phi * (e_\gamma \cdot n)\|_1 &= \|(\phi - \phi * e_\gamma) \cdot n\|_1 \\ &\le \|\phi - \phi * e_\gamma\|_1.\end{aligned}$$

Since $\phi \in L^1(G)$, it follows that

$$\phi * (e_\gamma \cdot n) \to \phi \cdot n.$$

Thus

$$\{\phi \cdot n : n \in L_0^\infty(G)^*,\ \|n\| \le 1\} \subseteq \{\phi * \psi : \psi \in L^1(G),\ \|\psi\|_1 \le 1\}^{-\|\cdot\|_1}.$$

This together with the fact that $\lambda_\phi : L^1(G) \to L^1(G)$ is compact show that

$$\{\phi \cdot n : n \in L_0^\infty(G)^*,\ \|n\| \le 1\}^{-\|\cdot\|_1}$$

is compact in $L^1(G)$. Consequently $\lambda_\phi : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is compact. □

REMARK 3.4. The 'only if' part of Proposition 3.3 does not remain valid for all left multipliers on $L_0^\infty(G)^*$; it is not true even for $m \in L_0^\infty(G)^*$ instead of $\phi \in L^1(G)$. In fact, let G be a locally compact group which is neither discrete nor compact, and choose $u \in \Lambda_0(G)$. On the one hand, since G is not discrete, it follows from Proposition 3.1 that there is $n \in L_0^\infty(G)^*$ such that $n \ne u \cdot n$. Set

$$m := n - u \cdot n,$$

then $\lambda_m|_{L^1(G)}$ is zero on $L^1(G)$ and, hence, compact. On the other hand, since G is not compact, it follows from Theorem 2.1 that $\lambda_m : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is not compact.

In the following, $P(G)$ denotes the set of all positive functions in $L^1(G)$.

COROLLARY 3.5. *Let G be a locally compact group. Then the following assertions are equivalent;*

(a) *G is compact;*
(b) *any $\phi \in L^1(G)$ is a left completely continuous element of $L_0^\infty(G)^*$;*
(c) *any $\phi \in P(G)$ is a left completely continuous element of $L_0^\infty(G)^*$;*
(d) *$L_0^\infty(G)^*$ has a nonzero left completely continuous element in $P(G)$;*
(e) *$L_0^\infty(G)^*$ has a nonzero left completely continuous element in $L^1(G)$.*

PROOF. Suppose that G is compact. Then any $\phi \in L^1(G)$ is a completely continuous element of $L^1(G)$; see Akemann [1, Theorem 4]. This together with Proposition 3.3 imply that ϕ is a completely continuous element of $L_0^\infty(G)^*$. That is (a) implies (b). Also, the implications (b) implies (c) implies (d) implies (e) are trivial. Finally, (e) implies (a) by Theorem 2.1. □

The right annihilator of $L_0^\infty(G)^*$ is denoted by $\mathrm{ran}(L_0^\infty(G)^*)$ and is defined by

$$\mathrm{ran}(L_0^\infty(G)^*) = \{r \in L_0^\infty(G)^* \mid L_0^\infty(G)^* \cdot r = \{0\}\}.$$

Let us remark that $\mathrm{ran}(L_0^\infty(G)^*)$ is the weak* closed ideal

$$\ker(\mathcal{P}) = \{n - u \cdot n \mid n \in L_0^\infty(G)^*\}$$

in $L_0^\infty(G)^*$ for all $u \in \Lambda_0(G)$; see Isik *et al.* [8, p. 139].

THEOREM 3.6. *Let G be a locally compact group. Then any left completely continuous element m of $L_0^\infty(G)^*$ has the form $m = \phi + r$ for some $\phi \in L^1(G)$ and $r \in \mathrm{ran}(L_0^\infty(G)^*)$.*

PROOF. Let m be a left completely continuous element of $L_0^\infty(G)^*$. Since $L^1(G)$ is an ideal in $L_0^\infty(G)^*$ and $\lambda_m : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is compact, it follows that $\lambda_m|_{L^1(G)}$ is a compact left multiplier on $L^1(G)$. Thus, there exists $\phi \in L^1(G)$ with $\lambda_m = \lambda_\phi$ on $L^1(G)$; see Akemann [1]. So, if we set

$$r := m - \phi,$$

then $r \cdot L^1(G) = \{0\}$ and, therefore,

$$\begin{aligned}\mathcal{P}(r) * L^1(G) &= \mathcal{P}(r) * \mathcal{P}(L^1(G)) \\ &= \mathcal{P}(r \cdot L^1(G)) = \{0\}.\end{aligned}$$

Since $\mathcal{P}(r) \in M(G)$, it follows that $\mathcal{P}(r) = 0$. That is, $r \in \mathrm{ran}(L_0^\infty(G)^*)$. □

As an immediate consequence of Proposition 3.3 and Theorem 3.6, we have the following corollary.

COROLLARY 3.7. *Let G be a locally compact group, and let m be a left completely continuous element of $L_0^\infty(G)^*$. Then the following statements hold:*

(i) $\mathcal{P}(m) \in L^1(G)$;
(ii) $m - \mathcal{P}(m) \in \mathrm{ran}(L_0^\infty(G)^*)$;
(iii) $u \cdot m = \mathcal{P}(m)$ *for all* $u \in \Lambda_0(G)$;
(iv) $\mathcal{P}(m)$ *is a left completely continuous element of* $L_0^\infty(G)^*$.

In the following, let $P_0(G)$ denote the set of all positive functionals on the C^*-algebra $L_0^\infty(G)$; also, let $\Delta_0(G)$ denote the set of all nonzero multiplicative linear functionals on $L_0^\infty(G)$, and note that $\Delta_0(G) \subseteq P_0(G)$. Before we present our next result, let us recall from Theorem 2.1 and its proof that G is compact if and only if $L_0^\infty(G)^*$ has a nonzero left completely continuous element in $P_0(G)$.

COROLLARY 3.8. *Let G be a locally compact group. Then the following assertions are equivalent:*

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(a) *G is finite;*
(b) *any $m \in L_0^\infty(G)^*$ is a left completely continuous element of $L_0^\infty(G)^*$;*
(c) *any $m \in P_0(G)$ is a left completely continuous element of $L_0^\infty(G)^*$;*
(d) *any $m \in \Delta_0(G)$ is a left completely continuous element of $L_0^\infty(G)^*$;*
(e) *$L_0^\infty(G)^*$ has a left completely continuous element in $\Delta_0(G)$.*

PROOF. The implications (a) implies (b) implies (c) implies (d) implies (e) are trivial. To complete the proof, suppose that there is $m \in \Delta_0(G)$ such that $\lambda_m : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is compact. Then $\mathcal{P}(m)$ is a nonzero multiplicative linear functional on the Banach algebra $C_0(G)$; indeed, $m \in P_0(G)$ and hence $\|\mathcal{P}(m)\| = \|m\| \neq 0$ by [9, Lemma 2.5]. So, there is an element $x \in G$ such that $\mathcal{P}(m)$ is a nonzero scalar multiple of the Dirac measure δ_x at x; see, for example, [7, Exercise 20.52]. By Corollary 3.7, $\mathcal{P}(m) \in L^1(G)$ and so $\delta_x \in L^1(G)$. This shows that G is discrete; see [6]. Now, we only need to recall from Theorem 2.1 that G is compact.

4. Compact left multipliers on $L_0^\infty(G)^*$ and ran$(L_0^\infty(G)^*)$

Before we state our next result, we need an elementary lemma.

LEMMA 4.1. *Let G be a locally compact group and $T : L_0^\infty(G)^* \to L_0^\infty(G)^*$ be a left multiplier. Then $T(L^1(G)) \subseteq L^1(G)$ and $T(\mathrm{ran}(L_0^\infty(G)^*)) \subseteq \mathrm{ran}(L_0^\infty(G)^*)$.*

PROOF. For each $\phi, \psi \in L^1(G)$, $T(\phi * \psi) = T(\phi) \cdot \psi$. Since $L^1(G)$ is an ideal in $L_0^\infty(G)^*$ and $L^1(G)^2 = L^1(G)$, we have $T(L^1(G)) \subseteq L^1(G)$.

Now, let $r \in \mathrm{ran}(L_0^\infty(G)^*)$. Then $T(r) \cdot \phi = T(r \cdot \phi) = 0$ for all $\phi \in L^1(G)$. Hence,

$$T(r) \cdot L^1(G) = \{0\}.$$

So $\mathcal{P}(T(r)) * L^1(G) = \{0\}$ and, hence, $\mathcal{P}(T(r)) = 0$; that is, $T(r) \in \mathrm{ran}(L_0^\infty(G)^*)$. □

For a subalgebra A of $L_0^\infty(G)^*$, we denote by $\mathcal{M}_{cl}(A)$ the set of all compact left multipliers on $L_0^\infty(G)^*$ with

$$T(L_0^\infty(G)^*) \subseteq A;$$

note that $T|_A$ is a compact left multiplier on A for all $T \in \mathcal{M}_{cl}(A)$.

PROPOSITION 4.2. *Let G be a locally compact group. Then the sets $\mathcal{M}_{cl}(L^1(G))$ and $\mathcal{M}_{cl}(\mathrm{ran}(L_0^\infty(G)^*))$ are closed ideals in $\mathcal{M}_{cl}(L_0^\infty(G)^*)$.*

PROOF. Clearly, $\mathcal{M}_{cl}(L^1(G))$ and $\mathcal{M}_{cl}(\mathrm{ran}(L_0^\infty(G)^*))$ are closed subspaces of $\mathcal{M}_{cl}(L_0^\infty(G)^*)$. Let $S \in \mathcal{M}_{cl}(L_0^\infty(G)^*)$ and $T \in \mathcal{M}_{cl}(L^1(G))$. Then $S \circ T$ is a compact left multiplier on $L_0^\infty(G)^*$. Now, if $n \in L_0^\infty(G)^*$, then $T(n) \in L^1(G)$ and, hence, $T(S(n)) \in L^1(G)$; moreover,

$$S(T(n)) \in L^1(G)$$

by Lemma 4.1. Therefore, $T \circ S$, $S \circ T \in \mathcal{M}_{cl}(L^1(G))$. The other case is similar.

We conclude the paper with the following result.

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THEOREM 4.3. *Let G be a locally compact group. Then $\mathcal{M}_{cl}(L_0^\infty(G)^*)$ is the Banach space direct sum of $\mathcal{M}_{cl}(L^1(G))$ and $\mathcal{M}_{cl}(\mathrm{ran}(L_0^\infty(G)^*))$.*

PROOF. Let $T \in \mathcal{M}_{cl}(L_0^\infty(G)^*)$ and choose $u \in \Lambda_0(G)$. Define the function $T_1 : L_0^\infty(G)^* \to L_0^\infty(G)^*$ by

$$T_1(n) := u \cdot T(n)$$

for all $n \in L_0^\infty(G)^*$, and set

$$T_2 := T - T_1.$$

Clearly T_1 and T_2 are compact left multipliers on $L_0^\infty(G)^*$.

Now, fix $n \in L_0^\infty(G)^*$, and note that $T(n)$ is a left completely continuous element of $L_0^\infty(G)^*$. Invoke Theorem 3.6 to conclude that $T(n) = \phi + r$ for some $\phi \in L^1(G)$ and $r \in \mathrm{ran}(L_0^\infty(G)^*)$. We therefore have

$$u \cdot T(n) = \phi \in L^1(G)$$

and

$$T(n) - u \cdot T(n) = r \in \mathrm{ran}(L_0^\infty(G)^*),$$

where $u \in \Lambda_0(G)$. That is $T_1 \in \mathcal{M}_{cl}(L^1(G))$ and $T_2 \in \mathcal{M}_{cl}(\mathrm{ran}(L_0^\infty(G)^*))$.

Finally, if $T \in \mathcal{M}_{cl}(L^1(G)) \cap \mathcal{M}_{cl}(\mathrm{ran}(L_0^\infty(G)^*))$, then for each $n \in L_0^\infty(G)^*$ we have $T(n) \in L^1(G) \cap \mathrm{ran}(L_0^\infty(G)^*)$ and, hence, $T(n) = 0$. □

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M. J. MEHDIPOUR, Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: mehdipour@sutech.ac.ir

R. NASR-ISFAHANI, Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran
e-mail: isfahani@cc.iut.ac.ir

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