

ON THE DISTANCE BETWEEN CONSECUTIVE DIVISORS OF AN INTEGER

BY

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ABSTRACT. Let $\omega(n)$ denote the number of distinct prime divisors of a positive integer n . Then we define $h: \mathbb{N} \rightarrow \mathbb{R}$ by $h(n) = 0$ if $\omega(n) \leq 1$ and $h(n) = \sum_{i=2}^{\omega(n)} 1/(q_i - q_{i-1})$ if $n = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_r^{\alpha_r}$, where $q_1 < q_2 < \dots < q_r$ are primes and $r \geq 2$. Similarly denote by $\tau(n)$ the number of divisors of n and let $H: \mathbb{N} \rightarrow \mathbb{R}$ be defined by $H(n) = \sum_{i=2}^{\tau(n)} 1/(d_i - d_{i-1})$, where $1 = d_1 < d_2 < \dots < d_{\tau(n)} = n$ are the divisors of n . We prove that there exists constants A and B such that $\sum_{n \leq x} h(n) = Ax + O(x(\log \log x)(\log x)^{-1})$ and $\sum_{n \leq x} H(n) = Bx + O(x(\log x)^{-1/3})$.

§1. Introduction. A natural way to estimate the average distance between the prime divisors of an integer $n = q_1^{\alpha_1} \dots q_r^{\alpha_r}$ ($r \geq 2$) is to study the arithmetical function

$$(1.1) \quad f(n) = \frac{1}{r-1} \sum_{i=2}^r (q_i - q_{i-1}) = \frac{P(n) - p(n)}{\omega(n) - 1}.$$

Here $q_1 < \dots < q_r$ are the prime divisors of n , and $P(n)$, $p(n)$, $\omega(n)$ denote respectively the largest prime divisor of n , the smallest prime divisor of n and the number of distinct prime divisors of n . Using a result of J.-M. De Koninck and A. Ivić [1], it follows easily that the average order of $f(n)$ is $cn/\log n$, where $c > 0$ is an absolute constant.

We now introduce another arithmetical function, which also provides information about the distance between the distinct prime divisors of n . We shall denote this function by $h(n)$ and define it as

$$(1.2) \quad h(n) = \begin{cases} 0 & \text{if } \omega(n) \leq 1, \\ \sum_{i=2}^{\omega(n)} \frac{1}{q_i - q_{i-1}}, & \text{if } n = q_1^{\alpha_1} \dots q_r^{\alpha_r}, r \geq 2, \\ & q_1 < \dots < q_r \text{ primes.} \end{cases}$$

In many ways this function is more complicated to estimate than $f(n)$, and in Theorem 1 below we shall show that there exists an absolute constant $A > 0$ such that

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$$\lim_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} h(n) = A.$$

An interesting, yet difficult problem is to determine the maximal order of $h(n)$. One has trivially

$$h(n) \leq \omega(n) \ll \log n / \log \log n.$$

Nevertheless $h(n)$ can be fairly large for some n , e.g.

$$(1.3) \quad \frac{\log N_k}{(\log \log N_k)^2} \ll h(N_k) \ll \frac{\log N_k \log \log \log N_k}{(\log \log N_k)^2},$$

where N_k is the product of the first k primes. This follows from the work of P. Erdős and A. Rényi [3], and the lower bound in (1.3) can be obtained simply as follows. If p_n denotes the n -th prime, then by the Cauchy-Schwarz inequality we obtain

$$(1.4) \quad \begin{aligned} k - 1 &= \sum_{2 \leq n \leq k} \left(\frac{p_n - p_{n-1}}{p_n - p_{n-1}} \right)^{1/2} \leq \left(\sum_{2 \leq n \leq k} (p_n - p_{n-1}) \right)^{1/2} \{h(N_k)\}^{1/2} \\ &= (p_k - 2)^{1/2} \{h(N_k)\}^{1/2}. \end{aligned}$$

But by the prime number theorem $p_k \sim k \log k$, $k \sim \log N_k / \log \log N_k$ as $k \rightarrow \infty$, hence (1.4) gives the lower bound in (1.3).

It seems equally interesting to study the analogues of (1.1) and (1.2) when one considers not only prime divisors of n , but all possible divisors of n . Thus if $1 = d_1 < d_2 < \dots < d_{\tau(n)}$ denote the consecutive divisors of n , where $\tau(n)$ is the number of divisors of n , then for $n \geq 2$ one may define

$$(1.5) \quad F(n) = \frac{1}{\tau(n) - 1} \sum_{i=2}^{\tau(n)} (d_i - d_{i-1}) = \frac{n - 1}{\tau(n) - 1}$$

as the average distance between the divisors of n . Using Theorem 1.2 of J.-M. De Koninck and A. Ivić [2], it follows readily by partial summation that the average order of $F(n)$ is $dn(\log n)^{-1/2}$ for some absolute $d > 0$.

Therefore it is perhaps more interesting to define the arithmetical function $H(n)$, the analogue of $h(n)$, as

$$(1.6) \quad H(n) = \begin{cases} 0 & \text{if } n = 1, \\ \sum_{i=2}^{\tau(n)} \frac{1}{d_i - d_{i-1}} & \text{if } n \geq 2, \end{cases}$$

where as before $1 = d_1 < \dots < d_{\tau(n)} = n$ are the consecutive divisors of n . Determining the maximal order of magnitude of $H(n)$ seems to be even more difficult than the corresponding problem for $h(n)$. Our main objective will be to prove that, similarly as $h(n)$, the function $H(n)$ has a finite mean value. The result is contained in Theorem 2 below, and provides information about the distribution of consecutive divisors of an integer. These types of problems have been investigated by several

authors, most notable by P. Erdős. A classical conjecture of his states that for almost all n

$$\min_{1 \leq i \leq \tau(n)-1} d_{i+1}/d_i \leq 2.$$

This conjecture, in an even stronger form, has been recently proved by H. Maier and G. Tenenbaum [4].

§2. Statement of results

THEOREM 1. *Let $h(n)$ be the arithmetical function defined by (1.2). Then*

$$(2.1) \quad \sum_{n \leq x} h(n) = Ax + O\left(\frac{x \log \log x}{\log x}\right),$$

where

$$(2.2) \quad A = \sum_{p_i < p_j} \frac{1}{(p_j - p_i)p_i p_j} \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) = 0.299 \dots$$

Here the sum is taken over all pairs of primes (p_i, p_j) such that $p_i < p_j$, while for a fixed pair (p_i, p_j) the product is over all primes p satisfying $p_i < p < p_j$.

For our second result we define (a, b) ($1 \leq a < b$ integers) to be a suitable pair if there exists a positive integer n such that a and b are two consecutive divisors of n . Clearly (a, b) is a suitable pair if and only if $a < d < b$ implies $d \nmid [a, b]$, where $[a, b]$ is the lowest common multiple of a and b . Further let $D_{a,b}$ consist of all integers of the form $(d/(d, [a, b]))$ ($a < d < b$), where $(d, [a, b])$ denotes the greatest common divisor of d and $[a, b]$ and where no element of $D_{a,b}$ is a multiple of another element of $D_{a,b}$. Finally, for a given suitable pair (a, b) , write $D_{a,b} = \{d_1, d_2, \dots, d_r\}$, then we shall denote by $R(a, b)$ the following expression

$$1 - \sum_{1 \leq i \leq r} \frac{1}{d_i} + \sum_{1 \leq i < j \leq r} \frac{1}{[d_i, d_j]} - \sum_{1 \leq i < j < k \leq r} \frac{1}{[d_i, d_j, d_k]} + \dots + (-1)^r \frac{1}{[d_1, d_2, \dots, d_r]}$$

where $[c_1, c_2, \dots, c_s]$ stands for the lowest common multiple of the integers c_1, c_2, \dots, c_s .

THEOREM 2. *Let $H(n)$ be the arithmetical function defined by (1.6). Then*

$$(2.3) \quad \sum_{n \leq x} H(n) = Bx + O(x(\log x)^{-1/3}),$$

where

$$B = \sum_{b=2}^{\infty} \sum_{a < b}^* \frac{1}{[a, b](b - a)} R(a, b)$$

and the star on the inner sum indicates that the summation runs through suitable pairs (a, b) .

§3. **The necessary lemmas.** In this section we shall formulate and prove two technical lemmas, which are necessary for the proof of Theorem 1 and Theorem 2.

LEMMA 1. *Let $p_i > p_j$ be any two fixed primes and let $p_{ij} = \prod_{p_i < p < p_j} p$, where p denotes primes. Then*

$$(3.1) \quad \sum_{n \leq y, (n, P_{ij})=1} 1 = y \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) + \theta 2^{j-i-1},$$

where $|\theta| \leq 1$.

PROOF OF LEMMA 1. The left-hand side of (3.1) is equal to

$$\begin{aligned} \sum_{n \leq y} \sum_{d|(n, P_{ij})} \mu(d) &= \sum_{d|P_{ij}} \mu(d)[y/d] = y \sum_{d|P_{ij}} \mu(d)/d \\ &+ \theta_1 \sum_{d|P_{ij}} 1 = y \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) + \theta 2^{j-i-1}, \end{aligned}$$

where $|\theta_1| \leq 1, |\theta| \leq 1$.

LEMMA 2. *Let (a, b) be a suitable pair, $g(d) = d/(d, [a, b])$ and $R(a, b)$ be as in §2. Then*

$$(3.2) \quad \sum_{m \leq y, a < d < b \Rightarrow g(d) \nmid m} 1 = yR(a, b) + O(2^{b-a}).$$

PROOF OF LEMMA 2. Set $D_{a,b} = \{d_1, \dots, d_r\}$. Then by the inclusion–exclusion principle, we have

$$\begin{aligned} \sum_{m \leq y, a < d < b \Rightarrow g(d) \nmid m} 1 &= [y] - \sum_{1 \leq i \leq r} \left[\frac{y}{d_i}\right] + \sum_{1 \leq i < j \leq r} \left[\frac{y}{[d_i, d_j]}\right] - \sum_{1 \leq i < j < k \leq r} \\ &\times \frac{y}{[d_i, d_j, d_k]} + \dots + (-1)^r \frac{y}{[d_1, d_2, \dots, d_r]} = yR(a, b) + O(2^r), \end{aligned}$$

and the result follows since $r \leq b - a - 1$.

§4. **Proof of Theorem 1.** Clearly we have

$$(4.1) \quad \begin{aligned} \sum_{n \leq x} h(n) &= \sum_{p_i < p_j \leq x} \frac{1}{p_j - p_i} \sum_{n \leq x, p_i p_j | n, (n, P_{ij})=1} 1, \\ &= \sum_{p_i < p_j \leq x, p_i p_j \leq x} \frac{1}{p_j - p_i} \sum_{m \leq x/p_i p_j, (m, P_{ij})=1} 1, \end{aligned}$$

where in the last sum m denotes a positive integer. Let $w(x)$ denote a function satisfying

$$(4.2) \quad \log \log x \ll w(x) \ll x^{1/3}$$

and which will be suitably chosen a little later. Since we have

$$\begin{aligned} \sum_{\substack{p_i < p_j \leq x \\ p_i p_j \leq x, p_j - p_i > w(x)}} \frac{1}{p_j - p_i} \sum_{\substack{m \leq x/p_i p_j \\ (m, p_{ij}) = 1}} 1 &\ll \frac{1}{w(x)} \sum_{p_i p_j \leq x} \left[\frac{x}{p_i p_j} \right] \\ &\leq \frac{1}{w(x)} \sum_{p_i p_j \leq x} \left(\frac{x}{p_i p_j} + 1 \right) \ll \frac{x}{w(x)} (\log \log x)^2, \end{aligned}$$

this means that (4.1) gives

$$(4.3) \quad \sum_{n \leq x} h(n) = \sum_{\substack{p_i < p_j \leq x \\ p_i p_j \leq x, p_j - p_i \leq w(x)}} \frac{1}{p_j - p_i} \sum_{\substack{m \leq x/p_i p_j \\ (m, p_{ij}) = 1}} 1 + O\left(\frac{x}{w(x)} (\log \log x)^2\right).$$

Next if $p_j > x^{1/2}$ in (4.3), then in view of (4.2) we have

$$p_i > p_j - w(x) > \frac{1}{2}x^{1/2}, \quad p_i p_j > \frac{1}{2}x,$$

hence

$$\begin{aligned} \sum_{\substack{p_i < p_j \leq x, p_i p_j \leq x \\ p_j - p_i \leq w(x), p_j > \sqrt{x}}} \frac{1}{p_j - p_i} \sum_{\substack{m \leq x/p_i p_j \\ (m, p_{ij}) = 1}} 1 &\ll \sum_{\substack{p_i p_j \leq x, p_j > \sqrt{x} \\ 0 < p_j - p_i \leq w(x)}} \left(\frac{x}{p_i p_j} + 1 \right) \\ &\ll \sum_{p_i p_j \leq x} 1 \ll x \frac{\log \log x}{\log x}. \end{aligned}$$

Therefore the contribution of p_j in (4.3) which satisfy $p_j > x^{1/2}$ is negligible. Hence using (4.3) and Lemma 1 we obtain

$$\begin{aligned} (4.4) \quad \sum_{n \leq x} h(n) &= x \sum_{\substack{p_i < p_j \leq x, p_i p_j \leq x \\ p_j - p_i \leq w(x), p_j \leq \sqrt{x}}} \frac{1}{(p_j - p_i) p_i p_j} \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right) \\ &+ O\left(\sum_{\substack{p_i < p_j \leq x, p_i p_j \leq x \\ p_j - p_i \leq w(x), p_j \leq \sqrt{x}}} \frac{2^{j-i}}{p_j - p_i}\right) + O\left(x \frac{\log \log x}{\log x}\right), \end{aligned}$$

if we choose

$$(4.5) \quad w(x) = \frac{1}{2} \log x \cdot \log \log x.$$

To estimate the first O -term in (4.4) we use the Brun–Titchmarsh inequality (see e.g. H. L. Montgomery [5], Theorem 4.4)

$$\pi(M + N) - \pi(M) \leq \frac{2N}{\log N}, \quad (M, N \geq 2)$$

This allows us to write

$$j - i = \pi(p_j) - \pi(p_i) \leq \frac{2(p_j - p_i)}{\log(p_j - p_i)} \leq \frac{1}{2} \frac{w(x)}{\log w(x)} \leq \frac{1}{2} \log x,$$

provided that $j - i \geq 2$. Estimating trivially the terms with $j = i + 1$ we obtain that the first error term in (4.4) is

$$\ll 2^{(1/2) \log x} \sum_{p_j \leq x^{1/2}} \sum_{n \leq p_j - 2} \frac{1}{p_j - n} \ll 2^{(1/2) \log x} \pi(x^{1/2}) \log x \ll x^{9/10}.$$

Thus to finish the proof of Theorem 1 it remains to show that the error made in replacing the first expression on the right-hand side of (4.4) by Ax is $\ll x(\log \log x)/(\log x)$. In view of the arguments which lead to (4.3) and (4.4) it will be sufficient to estimate the expression

$$R = \sum_{p_i < p_j, p_j > x} \frac{1}{(p_j - p_i)p_i p_j} \prod_{p_i < p < p_j} \left(1 - \frac{1}{p}\right).$$

Using partial summation and the prime number theorem we obtain

$$(4.6) \quad R \leq \sum_{p_i < p_j, p_j > x} \frac{1}{(p_j - p_i)p_i p_j} = \sum_{p_j > x} \frac{1}{p_j^2} \sum_{p_i < p_j} \left(\frac{1}{p_j - p_i} + \frac{1}{p_i}\right) \\ \ll \sum_{p_j > x} \frac{\log p_j}{p_j^2} \ll \frac{1}{x}.$$

This completes the proof of Theorem 1, and (4.6) shows that the value of the constant A in (2.1) is finite. A numerical calculation shows that $A = 0.299 \dots$

§5. Proof of Theorem 2. Let Σ^* denote summation over suitable pairs (a, b) , which were defined in §2. We have

$$(5.1) \quad \sum_{n \leq x} H(n) = \sum_{2 \leq n \leq x} \sum_{i=2}^{\tau(n)} \frac{1}{d_i - d_{i-1}} = \sum_{a < b, b \leq x-1}^* \frac{1}{b - a} \sum_{\substack{n \leq x, a|n, b|n \\ a < d < b = > d|n}} 1 \\ = \sum_{a < b, b \leq x-1}^* \frac{1}{b - a} \sum_{\substack{[a, b]m \leq x \\ a < d < b = > d \nmid [a, b]m}} 1 = \sum_{a < b, b \leq x-1}^* \frac{1}{b - a} \sum'_{m \leq x/[a, b]} 1,$$

where the dash in the last sum indicates that we restrict the sum to positive integers m for which the condition

$$a < d < b = > \frac{d}{(d, [a, b])} \nmid m$$

is satisfied. Let now $u(x), v(x)$ be two functions which satisfy

$$(5.2) \quad \log^\epsilon x \ll u(x) \leq \frac{1}{2} v(x) \ll \log^{1-\epsilon} x,$$

and which will be suitably determined a little later. We shall show first that the terms in (5.1) for which $b - a > u(x) \log x$ are small. This follows from

$$(5.3) \quad \sum_{2 \leq n \leq x} \sum_{i=2, d_i - d_{i-1} > u(x) \log x}^{\tau(n)} \frac{1}{d_i - d_{i-1}} \ll \frac{1}{u(x) \log x} \sum_{n \leq x} \tau(n) \ll \frac{x}{u(x)}.$$

The second step in the proof is to show that we can also neglect those terms in (5.1) for which $b > v(x) \log x$. Indeed in view of (5.2) and $ab = a, b$ we have

$$(5.4) \quad \sum_{\substack{v(x) \log x < b \leq x-1 \\ a < b, b-a \leq u(x) \log x}}^* \frac{1}{b-a} \sum'_{m \leq x/[a, b]} 1 \leq x \sum_{v(x) \log x < b \leq x} \sum_{b-u(x) \log x < a < b} \\ \times \frac{(a, b)}{(b-a)ab} \leq 2x \sum_{v(x) \log x < b \leq x} \sum_{b-u(x) \log x < b} \frac{(a, b)}{(b-a)b^2} \\ \ll xu(x) \log x \sum_{b > v(x) \log x} \frac{1}{b^2} \ll x \frac{u(x)}{v(x)},$$

since $(a, b) \leq b - a$ if $1 \leq a < b$. Thus from (5.1), (5.3) and (5.4) it follows that

$$(5.5) \quad \sum_{n \leq x} H(n) = \sum_{b \leq v(x) \log x} \sum_{a < b, b-a \leq u(x) \log x}^* \frac{1}{b-a} \sum'_{m \leq x/[a, b]} 1 \\ + O\left(\frac{x}{u(x)}\right) + O\left(x \frac{u(x)}{v(x)}\right).$$

Further we shall show that in the sum over a we may suppose that $b - a \leq \log x$, and not only $b - a \leq u(x) \log x$. Namely we have

$$(5.6) \quad \sum_{2 \leq n \leq x} \sum_{\substack{i=2, d_i - d_{i-1} > \log x \\ d_i \leq v(x) \log x}}^{\tau(n)} \frac{1}{d_i - d_{i-1}} \ll x \frac{v(x)}{\log x}$$

To obtain this estimate let $\alpha(n)$ for each n counted in (5.6) denote the number of i 's such that both $d_i - d_{i-1} > \log x$ and $d_i \leq v(x) \log x$ hold. Then obviously we must have $\alpha(n) \leq v(x)$, and the left-hand side of (5.6) is

$$\leq \sum_{2 \leq n \leq x} \frac{1}{\log x} \sum_{\substack{i=2, d_i - d_{i-1} > \log x \\ d_i \leq v(x) \log x}}^{\tau(n)} 1 \leq \frac{1}{\log x} \sum_{2 \leq n \leq x} \alpha(n) \ll x \frac{v(x)}{\log x}.$$

Thus we obtain

$$(5.7) \quad \sum_{n \leq x} H(n) = \sum_{b \leq v(x) \log x} \sum_{\substack{a < b \\ b-a \leq \log x}}^* \frac{1}{b-a} \sum'_{m \leq x/[a, b]} 1 \\ + O\left(\frac{x}{u(x)}\right) + O\left(\frac{xu(x)}{v(x)}\right) + O\left(\frac{xv(x)}{\log x}\right).$$

Now we shall simplify the triple sum in (5.7) by using Lemma 2. We have

$$(5.8) \quad \sum'_{m \leq x/[a,b]} 1 = \frac{x}{[a,b]} R(a,b) + O(2^{b-a}),$$

and

$$(5.9) \quad \sum_{b \leq v(x) \log x} \sum^*_{a < b, b-a \leq \log x} \frac{2^{b-a}}{b-a} \leq 2^{\log x} v(x) \log^2 x \ll x/u(x).$$

Hence we may write

$$(5.10) \quad \sum_{n \leq x} H(n) = x \sum_{b \leq v(x) \log x} \sum^*_{a < b, b-a \leq \log x} \frac{1}{(b-a)[a,b]} R(a,b) + O\left(\frac{x}{u(x)}\right) + O\left(x \frac{u(x)}{v(x)}\right) + O\left(x \frac{v(x)}{\log x}\right).$$

Noting that $R(a,b)$ denotes the density of integers with the constraints given in Lemma 2, we obtain that $0 \leq R(a,b) \leq 1$ for all a,b . Thus Theorem 2 will follow from (5.10) with a suitable choice of $u(x)$ and $v(x)$ if we can prove

$$(5.11) \quad \sum_{b > v(x) \log x} \sum^*_{a < b} \frac{1}{(b-a)[a,b]} + \sum_{b=2}^{\infty} \sum^*_{b-a > \log x, a < b} \frac{1}{(b-a)[a,b]} \ll \frac{(\log \log x)^2}{\log x} + \frac{1}{v(x)}.$$

Recalling that $a,b = ab$ it is seen that the first double sum in (5.11) does not exceed

$$\sum_{b > v(x) \log x} \frac{1}{b} \sum_{a < b, b-a \leq \log x} \frac{1}{a} + \sum_{b > v(x) \log x} \sum_{a < b, b-a > \log x} \frac{(a,b)}{(b-a)ab}.$$

Now we have

$$\begin{aligned} \sum_{b > v(x) \log x} \frac{1}{b} \sum_{a < b, b-a \leq \log x} \frac{1}{a} &\ll \sum_{b > v(x) \log x} \frac{1}{b} \int_{b-\log x}^b \frac{dt}{t} \\ &\ll \sum_{b > v(x) \log x} \frac{1}{b} \log \frac{1}{1 - \frac{\log x}{b}} \ll \log x \sum_{b > v(x) \log x} \frac{1}{b^2} \ll \frac{1}{v(x)}, \end{aligned}$$

so that (5.11) will follow from

$$(5.12) \quad \sum = \sum_{b=2}^{\infty} \sum_{a < b, b-a > \log x} \frac{(a,b)}{(b-a)ab} \ll \frac{(\log \log x)^2}{\log x}.$$

If we set $b - a = m$, then

$$\sum \ll \sum_{m > \log x} \frac{1}{m} \sum_{b > m} \frac{(b - m, b)}{(b - m)b} = \sum_{m > \log x} \frac{1}{m} S_m,$$

say. But for each $m \geq 1$

$$S_m = \sum_{d|m} \sum_{b > m, (b, m) = d} \frac{(b - m, b)}{(b - m)b} = \sum_{d|m} d \sum_{b > m, (b, m) = d} \frac{1}{(b - m)b}.$$

Hence setting $b = Bd, m = Md, (B, M) = 1$, we obtain

$$\begin{aligned} S_m &= \sum_{d|m} d \sum_{B > M, (B, M) = 1} \frac{1}{(Bd - Md)Bd} \\ &= \sum_{d|m} \frac{1}{Md} \sum_{B > M, (B, M) = 1} \left(\frac{1}{B - M} - \frac{1}{B} \right) \\ &= \frac{1}{m} \sum_{d|m} \sum_{B > M, (B, M) = 1} \frac{1}{B} \ll \frac{\tau(m) \log m}{m} \end{aligned}$$

Therefore using the elementary estimate

$$\sum_{n \leq x} \tau(n) = x \log x + O(x)$$

and partial summation, we obtain

$$\sum \ll \sum_{m > \log x} \frac{\tau(m) \log m}{m^2} \ll \frac{(\log \log x)^2}{\log x},$$

hence (5.11) follows. Collecting together all the estimates we finally obtain

$$\begin{aligned} \sum_{n \leq x} H(n) &= Bx + O\left(\frac{x}{u(x)}\right) + O\left(\frac{xu(x)}{v(x)}\right) + O\left(\frac{xv(x)}{\log x}\right) + O\left(\frac{x(\log \log x)^2}{\log x}\right) \\ &= Bx + O(x(\log x)^{-1/3}), \end{aligned}$$

if we choose

$$u(x) = \log^{1/3} x, v(x) = \log^{2/3} x.$$

This choice satisfies (5.2), hence we obtain (2.3) with B given by (2.4). A numerical calculation shows that $B = 1.77 \dots$

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