

1

Preliminaries

1.1 Notation

We denote by $\mathcal{L}^n$ the Lebesgue measure in the Euclidean n -space $\mathbb{R}^n$. In a metric space X , $d(A)$ stands for the diameter of A , $d(A, B)$ the minimal distance between the sets A and B , and $d(x, A)$ the distance from a point x to a set A . The closed ball with centre $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$ and the open ball by $U(x, r)$. In $\mathbb{R}^n$ we sometimes denote $B^n(x, r)$. The unit sphere in $\mathbb{R}^n$ is S^{n-1} . The Grassmannian manifold of linear m -dimensional subspaces of $\mathbb{R}^n$ is $G(n, m)$. It is equipped with an orthogonally invariant Borel probability measure $\gamma_{n,m}$. For $V \in G(n, m)$, we denote by P_V the orthogonal projection onto V .

For $A \subset X$, we denote by $\mathcal{M}(A)$ the set of non-zero finite Borel measures μ on X with support $\text{spt } \mu \subset A$. We shall denote by $f_{\#}\mu$ the push-forward of a measure μ under a map f : $f_{\#}\mu(A) = \mu(f^{-1}(A))$. The restriction of μ to a set A is defined by $\mu \lfloor A(B) = \mu(A \cap B)$. The notation $\ll$ stands for absolute continuity.

The characteristic function of a set A is χ_A . By the notation $M \lesssim N$, we mean that $M \leq CN$ for some constant C . The dependence of C should be clear from the context. The notation $M \sim N$ means that $M \lesssim N$ and $N \lesssim M$. By c and C , we mean positive constants with obvious dependence on the related parameters.

1.2 Hausdorff Measures

For $m \geq 0$, the m -dimensional Hausdorff measure $\mathcal{H}^m = \mathcal{H}_d^m$ in a metric space (X, d) is defined by

$$\mathcal{H}^m(A) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=1}^{\infty} \alpha(m) 2^{-m} d(E_i)^m : A \subset \bigcup_{i=1}^{\infty} E_i, d(E_i) < \delta \right\}.$$

Then $\mathcal{H}^0$ is the counting measure. Usually m will be a positive integer and then $\alpha(m) = \mathcal{L}^m(B^m(0, 1))$, from which it follows by the isodiametric inequality that $\mathcal{H}^m = \mathcal{L}^m$ in $\mathbb{R}^m$. The isodiametric inequality says that among the subsets of $\mathbb{R}^m$ with a given diameter, the ball has the largest volume; see, for example, [203, 2.10.33]. For non-integral values of m the choice of $\alpha(m)$ does not really matter. We denote by $\dim$ the Hausdorff dimension. The *spherical Hausdorff measure* S^m is defined in the same way but using only balls as covering sets.

The lower and upper m -densities of $A \subset X$ are defined by

$$\Theta_*^m(A, x) = \liminf_{r \rightarrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)),$$

$$\Theta^{*m}(A, x) = \limsup_{r \rightarrow 0} \alpha(m)^{-1} r^{-m} \mathcal{H}^m(A \cap B(x, r)).$$

The density $\Theta^m(A, x)$ is defined as their common value if they are equal.

We have

Theorem 1.1 *If A is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$2^{-m} \leq \Theta^{*m}(A, x) \leq 1 \text{ for } \mathcal{H}^m \text{ almost all } x \in A,$$

$$\Theta^{*m}(A, x) = 0 \text{ for } \mathcal{H}^m \text{ almost all } x \in X \setminus A.$$

When $m \leq 1$ the constant 2^{-m} is sharp; for $m > 1$ the best constant is not known.

We also have

Theorem 1.2 *If $A \subset X$ is $\mathcal{H}^m$ measurable and $\mathcal{H}^m(A) < \infty$, then*

$$\limsup_{\delta \rightarrow 0} \{d(B)^{-m} \mathcal{H}^m(A \cap B) : x \in B, d(B) < \delta\} = 1 \text{ for } \mathcal{H}^m \text{ almost all } x \in A.$$

For general measures, we have

Theorem 1.3 *Let $\mu \in \mathcal{M}(X)$, $A \subset X$, and $0 < \lambda < \infty$.*

- (1) *If $\Theta^{*m}(A, x) \leq \lambda$ for $x \in A$, then $\mu(A) \leq 2^m \lambda \mathcal{H}^m(A)$.*
- (2) *If $\Theta^{*m}(A, x) \geq \lambda$ for $x \in A$, then $\mu(A) \geq \lambda \mathcal{H}^m(A)$.*

For the above results, see [203, 2.10.17–19], [190, Section 2.2] or [321, Chapter 6].

We say that a closed set E is AD- m -regular (AD for Ahlfors and David) if there is a positive number C such that

$$r^m/C \leq \mathcal{H}^m(E \cap B(x, r)) \leq Cr^m \text{ for } x \in E, 0 < r < d(E).$$

A measure μ is said to be AD- m -regular if

$$r^m/C \leq \mu(B(x, r)) \leq Cr^m \text{ for } x \in \text{spt } \mu, 0 < r < d(\text{spt } \mu),$$

which means that $\text{spt } \mu$ is an AD- m -regular set.

1.3 Lipschitz Maps

Since Lipschitz maps are at the heart of rectifiability, we state here some basic well-known facts about them. We say that a map $f: X \rightarrow Y$ between metric spaces X and Y is *Lipschitz* if there is a positive number L such that

$$d(f(x), f(y)) \leq Ld(x, y) \text{ for } x, y \in X.$$

The smallest such L is the Lipschitz constant of f , which is denoted by $\text{Lip}(f)$.

Euclidean valued Lipschitz maps $f: A \rightarrow \mathbb{R}^k, A \subset X$, can be extended: there is a Lipschitz map $g: X \rightarrow \mathbb{R}^k$ such that $g|_A = f$, see [203, 2.10.43–44] or [321, Chapter 7].

Any Lipschitz map $g: \mathbb{R}^m \rightarrow \mathbb{R}^k$ is almost everywhere differentiable by Rademacher's theorem, see [203, 3.1.6] or [321, 7.3].

There is the Lusin type property: if $f: A \rightarrow \mathbb{R}^k, A \subset \mathbb{R}^m$ is Lipschitz, then for every $\varepsilon > 0$ there is a C^1 map $g: \mathbb{R}^m \rightarrow \mathbb{R}^k$ such that

$$\mathcal{L}^m(\{x \in A: g(x) \neq f(x)\}) < \varepsilon, \quad (1.1)$$

see [203, 3.1.16].