Canad. Math. Bull. Vol. 29 (4), 1986

NOTE ON CONTINUOUS AND PURELY FINITELY ADDITIVE SET FUNCTIONS

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ABSTRACT. The Sobczyk-Hammer respectively Yosida-Hewitt decomposition ([17], [19]) generates the class of continuous respectively purely finitely additive charges. In this paper, attention is limited to hereditable properties for these classes. It is proved that the property of continuity is preserved with respect to extensions and that if all extensions of a charge to a charge on a given field are continuous, then the original charge is continuous. An analogous heredity theorem for purely finite additivity holds true in the monogenic case.

1. **Preliminaries**. Let us now establish the setting for the work which follows. A *charge* on a field $\mathcal{U}$ of subsets of a set is a real-valued nonnegative finitely additive function defined on $\mathcal{U}$. A *measure* is a countably additive charge whose domain is a σ-field of subsets of a set. A charge μ on a field $\mathcal{U}$ (in a set Ω) will be called *continuous* if, and only if, given $\epsilon > 0$, there exists a partition $\{B_1, \ldots, B_n\}$ of Ω into a finite number of pairwise disjoint members of $\mathcal{U}$ such that $\mu(B_i) < \epsilon$ for every i. A charge μ is said to have the *Darboux property* if, and only if, for any $B \in \mathcal{U}$ and any α with $0 \leqq \alpha \leqq \mu(B)$ there exists a set $C \in \mathcal{U}$ such that $C \subset B$ and $\mu(C) = \alpha$. A set $A \in \mathcal{U}$ is an *atom* for μ if, and only if, $\mu(A) > 0$ and for any $E \in \mathcal{U}$, $E \subset A$, either $\mu(E) = 0$ or $\mu(E) = \mu(A)$. A charge μ is *nonatomic* if, and only if, there are no atoms for μ. If $\mathfrak{B}$ is a subfield of $\mathcal{U}$, then $\mu_{|\mathfrak{B}}$ will denote the restriction of μ on $\mathfrak{B}$. A charge ν on $\mathcal{U}$ is called *purely finitely additive* if, and only if, ν has no countably additive minorant (that is to say $\nu \geqq \kappa$ implies $\kappa = 0$ for any countably additive charge κ on $\mathcal{U}$ where $\leqq$ denotes the natural partial ordering on the set $ba\,(\Omega, \mathcal{U})$ of all bounded additive set functions on $\mathcal{U}$). Finally, we denote by $ba^+\,(\mathcal{U}, \nu, \mathcal{U}')$ respectively $ca^+\,(\Sigma, \lambda, \Sigma')$ the set of all extensions of a charge ν on a field $\mathcal{U}$ to a charge on a field $\mathcal{U}'$ where $\mathcal{U}$ is a subfield of $\mathcal{U}'$ respectively of a measure λ on a σ-field Σ to a measure on a σ-field Σ' where Σ is a sub-σ-field of Σ'.

2. **Extensions and restrictions**. A. Sobczyk and P. C. Hammer have proved that any charge on a field $\mathcal{U}$ can be decomposed uniquely into a continuous part and a part which can be written as a sum of at most two-valued charges on $\mathcal{U}$ ([17], [16]). We

Received by the editors August 6, 1984 and, in revised form, October 25, 1985.
AMS Subject Classification 28A10

https://doi.org/10.4153/CMB-1986-064-6 Published online by Cambridge University Press

show that the property of continuity is preserved with respect to extensions and that if all extensions of a charge to a charge on a given field are continuous, then the original charge is continuous. In the presence of extreme extensions the same holds true for measures — this may fail in the absence of extreme points.

THEOREM 1. *Let $\mathcal{U}$, $\mathcal{U}'$ be fields of subsets of a set Ω with $\mathcal{U} \subset \mathcal{U}'$ and let ν be a charge on $\mathcal{U}$. It holds that ν is continuous if, and only if, every $\mu \in b^+a\ (\mathcal{U}, \nu, \mathcal{U}')$ is continuous.*

PROOF. It suffices to prove that ν is continuous if every $\mu \in ba^+\ (\mathcal{U}, \nu, \mathcal{U}')$ is continuous. By theorem 1 in [14] the extreme points of $ba^+\ (\mathcal{U}, \nu, \mathcal{U}')$ can be characterized by the following approximation property (*): A charge μ on $\mathcal{U}'$ has the property (*) if, and only if, given $\epsilon > 0$ and $A' \in \mathcal{U}'$, there exists a set A in $\mathcal{U}$ with $\mu(A'\Delta A) < \epsilon$ for the symmetric difference $A'\Delta A$. Let μ be an extreme extension of ν, whose existence is guaranteed by a corollary to theorem 1 in [14], and which by assumption on $ba^+\ (\mathcal{U}, \nu, \mathcal{U}')$ is continuous. The technique in the proof for lemma 3.1 of [3] shows that already $\mu_{|\mathcal{U}}$ is continuous.

For nonatomic measures, a parallel theorem to Theorem 1 can be obtained. Note that for measures being continuous is the same as being nonatomic.

THEOREM 2. *Let $\mathcal{U}$, $\mathcal{U}'$ be σ-fields of subsets of a set Ω with $\mathcal{U} \subset \mathcal{U}'$ and ν a measure on $\mathcal{U}$. Let $ca_e^+\ (\mathcal{U}, \nu, \mathcal{U}')$ denote the subset of extreme points of $ca^+\ (\mathcal{U}, \nu, \mathcal{U}')$. Then, if $ca_e^+\ (\mathcal{U}, \nu, \mathcal{U}')$ is nonempty, it holds that ν is nonatomic if, and only if, every $\mu \in ca^+\ (\mathcal{U}, \nu\ \mathcal{U}')$ is nonatomic.*

Notice that $\mu \in ca_e^+\ (\mathcal{U}, \nu, \mathcal{U}')$ if, and only if, for every $A' \in \mathcal{U}'$ there is a set $A \in \mathcal{U}$ with $\mu(A'\Delta A) = 0$ (see [14]) and apply the same technique as in the proof for Theorem 1.

There are cases in which a nonatomic charge on a field can be extended to a charge which fails to be nonatomic. Thus, for nonatomic charges only a partial analogue of Theorem 1 holds true:

THEOREM 3. *Let $\mathcal{U}$, $\mathcal{U}'$ be fields of subsets of a set Ω with $\mathcal{U} \subset \mathcal{U}'$ and let ν be a charge on $\mathcal{U}$. Then ν is nonatomic, if every $\mu \in ba^+\ (\mathcal{U}, \nu, \mathcal{U}')$ is nonatomic.*

A theorem of D. Maharam ([10], Theorem 2) states that a charge μ on a σ-field is continuous if, and only if, μ has the Darboux property. Thus, the following partial analogue of Theorem 1 with respect to the Darboux property is true:

THEOREM 4. *Let $\mathcal{U}$, $\mathcal{U}'$ be σ-fields of subsets of a set Ω with $\mathcal{U} \subset \mathcal{U}'$ and let ν be a charge on $\mathcal{U}$. It holds that ν has the Darboux property if, and only if, every $\mu \in ba^+\ (\mathcal{U}, \nu, \mathcal{U}')$ has the Darboux property.*

It should be mentioned that in the theorem of D. Maharam just cited in general, the predicate *continuous* cannot be replaced by the predicate *nonatomic*: In [12] the

https://doi.org/10.4153/CMB-1986-064-6 Published online by Cambridge University Press

existence of a charge on a σ-field has been shown which is nonatomic and without the Darboux property.

It has been proven by K. Yosida and E. Hewitt ([19]) that any charge μ on a field of subsets of a set can be written uniquely as a sum $\mu = \lambda + \mu'$ of a countably additive part λ and a purely finitely additive part μ'. Translated to the case of purely finitely additive charges Theorem 1 fails as the following example shows: Let X be a set of cardinality $\aleph_1$, let $\mathcal{U}$ respectively $\mathcal{U}'$ denote the system $\{A \subset X \mid A$ countable or $X\backslash A$ countable$\}$ respectively the power set of X. Then, by a theorem of S. Ulam ([18]) the measure μ on $\mathcal{U}$ defined by $\mu(A) = 0$ respectively 1 if A respectively $X\backslash A$ is countable ($A \in \mathcal{U}$) only admits of purely finitely additive extensions to a charge on $\mathcal{U}'$. The monogenic case yields a positive result:

THEOREM 5. *Let $\mathcal{U}$, $\mathcal{U}'$ be fields of subsets of a set Ω with $\mathcal{U} \subset \mathcal{U}'$. Let ν be a charge on $\mathcal{U}$ and μ' a purely finitely additive charge on $\mathcal{U}'$ such that $ba^+(\mathcal{U}, \nu, \mathcal{U}') = \{\mu'\}$. Then ν also is purely finitely additive.*

PROOF. Let ν_1 be a countably additive charge on $\mathcal{U}$ with $\nu \geqq \nu_1$ whose extension to a measure on the σ-field $A^\sigma(\mathcal{U})$ generated by $\mathcal{U}$ will be denoted by $\bar{\nu}_1$. To prove $\nu_1 = 0$ notice that $\mathcal{U}' \subset (A^\sigma(\mathcal{U}))_{\bar{\nu}_1}$ holds for the completion $(A^\sigma(\mathcal{U}))_{\bar{\nu}_1}$ of $A^\sigma(\mathcal{U})$ with respect to $\bar{\nu}_1$ because for any $A' \in \mathcal{U}'$ and $\epsilon > 0$ there exist sets $A_1, A_2 \in \mathcal{U}$ such that $A_1 \subset A' \subset A_2$ and $\mu'(A_2\backslash A_1) < \epsilon$, implied by the chain $\nu_{*|\mathcal{U}'} = \mu' = \nu^*_{|\mathcal{U}'}$ which holds true for the interior respectively exterior charge ν_* respectively ν^* with respect to ν as the following argument will show (the interior respectively exterior charge ν_* respectively ν^* with respect to ν is defined by $\nu_*(T) = \sup_{\substack{M \subset T \\ M \in \mathcal{U}}} \nu(M)$ respectively $\nu^*(T) = \inf_{\substack{T \subset M \\ M \in \mathcal{U}}} \nu(M)$ for any $T \subset \Omega$): First, $\nu_*(E) \leqq \mu'(E) \leqq \nu^*(E)$ for all $E \in \mathcal{U}'$. Suppose now, there is a set $E_0 \in \mathcal{U}'$ with $\nu_*(E_0) < \nu^*(E_0)$. Then $\underline{\nu}$ respectively $\bar{\nu}$ defined by $\underline{\nu}(E) = \nu_*(E \cap E_0) + \nu^*(E \cap (\Omega\backslash E_0))$ respectively $\bar{\nu}(E) = \nu^*(E \cap E_0) + \nu_*(E \cap (\Omega\backslash E_0))$ for all sets E in the field $A(\mathcal{U} \cup \{E_0\})$ generated by $\mathcal{U} \cup \{E_0\}$ are two extensions of ν to a charge on $A(\mathcal{U} \cup \{E_0\})$ ([9], p. 269) with $\underline{\nu}(E_0) < \bar{\nu}(E_0)$ in contradiction to the monogenicity of μ' with respect to $\mathcal{U}$. Thus, the charge ν_1 admits of an extension to a countably additive charge ν_1' on $\mathcal{U}'$ with $\nu_1' \leqq \mu'$ and therefore $\nu_1 = 0$. The minorant property $\nu_1' \leqq \mu'$ can be shown by a more canonical indirect argument.

3. **Limits of sequences**. (1) A characterization of the predicate purely finitely additive by K. Yosida and E. Hewitt shows that this property is preserved with respect to limits of sequences of charges whose common domain is a σ-field: Let $(\mu_n)_{n \in \mathcal{N}}$ be a sequence of purely finitely additive charges on a σ-field $\mathcal{U}$ in a set Ω and μ_0 a charge on $\mathcal{U}$ with $\mu_n(A) \to \mu_0(A)$ for all $A \in \mathcal{U}$. Then μ_0 is also purely finitely additive.

PROOF. Let λ be a measure on $\mathcal{U}$ and $\epsilon > 0$. It suffices to show the existence of a set $A_0 \in \mathcal{U}$ such that $\mu_0(\Omega\backslash A_0) = 0$ and $\lambda(A_0) < \epsilon$ ([19], Theorem 1.18): Choose positive real numbers q_n ($n \in \mathcal{N}$) with $\Sigma_{k=1}^\infty q_k < \epsilon$. By Theorem 1.19 in [19] there are sets $A_n \in \mathcal{U}$ with $\mu_n(\Omega \backslash A_n) = 0$ and $\lambda(A_n) < q_n$ for all $n \in \mathcal{N}$. The set $\bigcup_{k=1}^{\infty} A_k \in \mathcal{U}$ fulfills the requirements.

(2) Whereas nonatomicity is preserved with respect to limits of sequences of measures under the topology of setwise convergence (notice that for nonatomic measures μ_n and a measure μ_0 on a σ-field $\mathcal{G}$ such that $\mu_n(A) \to \mu_0(A)$ for any $A \in \mathcal{G}$ the measure $\mu = \sum_{n=1}^{\infty} 1/2^n \mu_n$ is also nonatomic and μ_0 is absolutely continuous with respect to μ – then apply Theorem 2.4 of [7], thus μ_0 also is nonatomic) no analogous heredity results hold true in the case of charges concerning the predicates *nonatomic* and *continuous* as can be seen by direct arguments (for the nonatomic case, e.g., choose the field $\mathcal{U}$ and the charge μ' on $\mathcal{U}$ as in the second part of the example in [16], p. 450. Let μ_n be defined by $\mu_n(B) = \lambda(B \cap A_n)$ for any $B \in \mathcal{U}$ where λ denotes the Lebesgue measure and $A_n = [0, 1/4 + 1/(9 + n)] \cup (3/4 - 1/(9 + n), 1]$ for all $n \in \mathcal{N}$. Then μ_n is a nonatomic charge on $\mathcal{U} (n \in \mathcal{N})$ such that $\mu_n(B) \to \mu'(B)$ for any $B \in \mathcal{U}$).

4. **Remarks**. (1) Theorem 4 is no longer valid if it is translated to the case of fields: Define $S = [0,1)$, let $\mathcal{U}$ be the field which is generated by $\{[a, b) \mid a,b \in \mathbb{Q}$ with $0 \leqq a < b \leqq 1\}$, $\mathcal{U}'$ the σ-field of Borel sets in S, and ν the restriction of the Lebesgue measure λ on $\mathcal{U}$. Then every $\mu \in ba^+ (\mathcal{U}, \nu, \mathcal{U}')$ is continuous and thus has the Darboux property by the above theorem of D. Maharam in contrast to ν.

(2) The following example shows that Theorem 2 does not hold in the absence of extreme points in $ca^+(\mathcal{U}, \nu, \mathcal{U}')$: Let $\mathcal{U}'$ be the σ-field of Borel sets in the reals $\mathcal{R}$, define $\mathcal{U} = \{A \subset \mathcal{R} \mid A$ or $\mathcal{R} \backslash A$ is countable$\}$, and let ν be the measure on $\mathcal{U}$ defined by $\nu(A) = 0$ resp. 1 if A resp. $\mathcal{R} \backslash A$ is countable. Then $ca^+ (\mathcal{U}, \nu. \mathcal{U}')$ coincides with the family of all nonatomic probability measures on $\mathcal{U}'$. $ca_e^+ (\mathcal{U}, \nu, \mathcal{U}')$ is empty – this can be seen from the characterization of extreme extensions of a measure following Theorem 2.

There are structures under which $ca_e^+ (\mathcal{U}, \nu, \mathcal{U}')$ is nonempty: For example, let S be a compact Polish space (i.e. a compact space with a countable base), M a separable metric space, $\mathcal{U}'$ respectively $\mathfrak{B}$ the σ-field of Borel sets in S respectively M, f a continuous mapping of S into M, and ν a measure on $\mathcal{U}$ with $\mathcal{U} = f^{-1}(\mathfrak{B})$. Then, (because of the regularity of Borel measures on Polish spaces, see [1]) $ca^+ (\mathcal{U}, \nu, \mathcal{U}') \subset rca^+ (S)$, where $rca^+ (S)$ denotes the positive cone of the space $rca(S)$ of all real-valued countably additive regular set functions on the Borel sets in S. Let $rca(S)$ be provided with the weak* topology of the conjugate space of the Banach space $C(S)$ of all real-valued bounded continuous functions on S (under the supremum norm). Then $ca^+ (\mathcal{U}, \nu\, \mathcal{U}')$ is a compact subset of $rca(S)$: Choose a sequence $(\mu_n')_{n \in \mathcal{N}}$ in $ca^+ (\mathcal{U}, \nu, \mathcal{U}')$ converging to $\mu' \in rca^+(S)$ with respect to the relative weak* topology. Then, $\int (g \circ f)\, d\mu_n' \to \int (g \circ f)\, d\mu'$ for any $g \in C(M)$. Consequently, $\int g\, df(\nu) = \int g\, df(\mu')$ because of $\int (g \circ f) d\mu_n' = \int (g \circ f)\, d\nu$, where $h(\mu)$ denotes the image of a measure μ under a measurable mapping h. Thus, $\mu'_{|\mathcal{U}} = \nu$ by the coincidence of the measures $f(\nu)$ and $f(\mu')$ on ([4], Theorem 1.3 or [13], Theorem 5.9). It follows that ca$^+ (\mathcal{U}, \nu, \mathcal{U}')$

is closed in $rca(S)$ (notice the metrizability of the relative weak* topology on $rca^+(S)$ ([5], p. 1214) and the fact that $rca^+(S)$ is a closed subset of $rca(S)$). A corollary to the theorem of Alaoglu ([6], V.4.3) implies the compactness of ca^+ $(\mathcal{U}, \nu, \mathcal{U}')$. Because $\mathcal{U}$ is separable, the theorem of D. Landers and L. Rogge in [8] shows that ca^+ $(\mathcal{U}, \nu, \mathcal{U}')$ is nonempty. Thus ca^+ $(\mathcal{U}, \nu, \mathcal{U}')$ has extreme points ([6], V.8.2).

(3) The preservation result 3.(1) fails if the common domain $\mathcal{U}$ is a field but not a σ-field: Define $\Omega = [0, 1)$, $t_m = 1/2^m$ $(m \in \mathcal{N})$, let $\mathcal{U}$ $(t_m$: $m \in \mathcal{N})$ denote the field in Ω generated by $\{[\alpha, \beta) \mid \alpha, \beta \in \mathcal{R} : 0 \leqq \alpha < \beta \leqq 1\} \cup \{\{t_m\} : m \in \mathcal{N}\}$, and ν_s the restriction of the Dirac measure at s $(s \in \mathcal{R})$ on $\mathcal{U}$ $(t_m : m \in \mathcal{N})$. Then $(\mu_{t_n})_{n \in \mathcal{N}}$ defined by $\mu_{t_n}(A) = \lim_{\substack{t \to t_n \\ t > t_n}} \nu_t(A)$ for any $A \in \mathcal{U}$ $(t_m : m \in \mathcal{N})$ is a sequence of purely finitely additive charges on $\mathcal{U}$ $(t_m : m \in \mathcal{N})$ such that $\mu_{t_n}(A) \to \nu_0(A)$ for all $A \in \mathcal{U}$ $(t_m : m \in \mathcal{N})$. The convergence property of $(\mu_{t_n})_{n \in \mathcal{N}}$ follows from canonical, though longish calculations.

The preservation result 3.(1) cannot be translated to the case of nets $(\mu_\alpha)_{\alpha \in D}$: e.g., the restriction of the Lebesgue measure on the Borel σ-field in $[0, 1]$ admits of a representation as the limit of a net of purely finitely additive charges under the topology of setwise convergence ([11], 5.10).

I wish to thank Professor D. Plachky and the referee for valuable remarks.

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