Canad. Math. Bull. Vol. **20** (3), 1977

SOME RESULTS IN THE THEORY OF FIBRATIONS

BY
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§0. **Introduction.** I wish to present here some of the results of a research in the Theory of Fibrations initiated some time ago by Peter Booth, Philip Heath, and myself. The philosophy behind the work is to approach certain aspects of the Theory of Fibrations in a unified way through the systematic use of the sections of suitable fibrations; this yields general theorems, of which some well-known results are eventually particular cases.

In the first paragraph a useful functor—the *Section Functor*—is discussed; more precisely, if $p:E\rightarrow B$ is a Hurewicz fibration, there exists a covariant functor from a certain full subcategory of the fundamental groupoid of E into the category of sets. The construction of this functor is explained in detail in [7].

The second paragraph deals with some applications of the Section Functor; for example, if X, Y are based topological spaces with base points X and Y respectively, if $[X, x; Y, y]_*$ is the set of all base-homotopy classes of maps from X to Y taking x into y, if λ is a path from x to x' in X, μ is a path from y to y' in Y, then under suitable conditions on x and x', there is a bijection $[X, x; Y, y]_* \cong [X, x'; Y, y']_*$. This generalizes the non-relative version of [23, 7.3.3] which considers change of base point in just the second variable and, of course, change of base points in homotopy groups. The material of this paragraph is contained in [8].

The topic of the last part of the paper is the characterization of Universal F-fibrations in the sense of Allaud–Dold, where F is a fixed ground fibre; the results here presented will concern a forthcoming joint paper.

A few words about notation: I indicate with *Top*, *Top*$_*$, and *Set* respectively, the category of topological spaces and continuous functions (maps), the category of based topological spaces and based maps and, the category of sets and functions. Also, if K is any category, the family of objects of K will be denoted by obj K; if X, $Y \in$ Obj K, $K(X, Y)$ stands for the set of morphisms of K from X to Y.

No proofs are given; the definitions needed can be found either in the test or in a suggested reference.

Received by the editor Feb. 2, 1977.
(1) Research partly supported by NRC Grant A8777.

https://doi.org/10.4153/CMB-1977-051-9 Published online by Cambridge University Press

§1. **The section functor.** Let E and B be topological spaces. A map $p:E\to B$ is called a (Hurewicz) *fibration* if, for every $X\in\operatorname{Obj} Top$ and every commutative diagram

$$\begin{array}{ccc} X\times\{0\} & \xrightarrow{h} & E \\ \cap\big\downarrow & & \big\downarrow p \\ X\times I & \xrightarrow[H]{} & B \end{array}$$

(I is the unit interval $[0,1]$) there is a map $H':X\times I\to E$ such that $H'\mid X\times\{0\}=h$ and $pH'=H$. For example, given any two topological spaces X and Y, the projections $pr_1:X\times Y\to X$ and $pr_2:X\times Y\to X$ on the first and second factors respectively, are fibrations. Other simple examples are provided by pullbacks of a fibration and a map; more precisely, if $p:E\to B$ is a fibration and $f:A\to B$ is a map, the universal property of pullbacks shows that $p_f:A\sqcap E=\{(a,e)\in A\times E\mid f(a)=p(e)\}\to A$, $(a,e)\to a$, is a fibration.

A *section* of a fibration $p:E\to B$ is a map $s:B\to E$ such that $ps=1_B$. In what follows, sec p denotes the set of all sections of p (may be $\varnothing$); if the work is done within the category Top_* and $e\in E$, $p(e)=b\in B$ are the base points, $\sec_e p$ will stand for the set of all based sections of p. At a first glance, there are clearly two kinds of homotopies in sec p; the homotopies in the usual sense, which we call *free homotopies*, and the *vertical homotopies*, that is, those which at each stage are sections of p. Actually, as demonstrated by James and Thomas [16], these two coincide: $s, s'\in\sec p$ are vertically homotopic if, and only if, they are free-homotopic. With this in mind, call [sec p] the set of all free (equivalently, vertical) homotopy classes of sections of p. In the presence of base points, namely in $\sec_e p$, we can have four kinds of homotopies: *free, vertical, based* and *vertical-based* homotopies; precisely, $s, s'\in\sec_e p$ are *base-homotopic* if there exists a homotopy $H:s\simeq s'$ such that $H_t(b)=e$ for every $t\in I$; if, in addition, $H_t\in\sec_e p$ for every $t\in I$, s and s' are *vertically-based homotopic*. Of course, it would be interesting to know when two sections $s, s'\in\sec_e p$ are equivalent in the four senses described; we shall see later how to answer this question. For the moment we shall content ourselves with the following. Suppose that $b=p(e)$ is an *admissible* point of B, that is to say, such that $\{b\}$ is closed in B and the inclusion $\{b\}\subset B$ is a *cofibration*;(1) then, mimicking the corresponding result of [16], one shows that $s, s'\in\sec_e p$ are vertically-based homotopic if, and only if, they are based-homotopic. Thus, for

(1) A map $g:X\to Y$ is a *cofibration* if, given any $Z\in\operatorname{Obj} Top$ and maps $H:X\times I\to Z$, $h:Y\to Z$ such that $H|X\times\{0\}=h\circ g\times 1|X\times\{0\}$, there exists $H':Y\times I\to Z$ such that $H'|Y\times\{0\}=h$ and $H=H'\circ g\times 1$. A detailed account of the properties of fibrations and cofibrations can be found in [11].

https://doi.org/10.4153/CMB-1977-051-9 Published online by Cambridge University Press

any $e \in E$ with $b = p(e)$ admissible, let $[\sec_e p]$ be set of all based (equivalently, vertically-based) homotopy classes of sections of p which take b into e.

The reader is now reminded that a groupoid is a small category whose morphisms are invertible [10]. As an example, recall the fundamental groupoid πE of a topological space E. The objects of πE are the points of E; its morphisms are constructed as follows. If λ, λ' are paths from e to e' in E, that is to say, λ, λ' are maps from I into E such that $\lambda(0) = \lambda'(0) = e$, $\lambda(1) = \lambda'(1) = e'$, λ and λ' are said to be *homotopic rel e, e'* if there is a map $H: I \times I \to E$ such that $H(\ ,0) = \lambda$, $H(\ ,1) = \lambda'$, $H(0, t) = e$, $H(1, t) = e'$, for every $t \in I$. Then the morphisms from e to e' in πE are the homotopy classes rel e, e' of paths with end points e and e'.

In what follows, $\pi^{\cdot} E$ will denote the full subgroupoid of πE of classes (rel end-points) of paths with end-points over admissible points of B. Let me take the occasion to observe that admissible points are not so uncommon in nature; for instance, every point of a CW-complex is admissible [18]. Also notice that if B is locally equiconnected (for example, if B is a CW-complex [15, II.2]) then $\pi^{\cdot} E = \pi E$ [15, II.3 and II.8].

Let $[\lambda] \in \pi^{\cdot} E$ be a path class from e to e'; because the inclusion of $b' = p(e')$ in B is a cofibration, it is possible to construct a homotopy $G: B \times I \to B$ such that $G(\ ,0) = 1_B$ and $G(b', t) = p\lambda(1-t)$ for each $t \in I$. Now, for any given $s \in \sec_e p$, set up a commutative diagram

$$\begin{array}{ccc} B \times \{0\} \cup \{b'\} \times I & \xrightarrow{sG(\ ,1) \cup \lambda} & E \\ \cap\downarrow & & \downarrow p \\ B \times I & \xrightarrow[G^{-1}]{} & B \end{array}$$

where $G^{-1}(x, t) = G(x, 1-t)$ for all $x \in B$ and all $t \in I$. From [25, Theo. 4], we obtain a map $H: B \times I \to E$ which extends $sG(\ ,1) \cup \lambda$ and is such that $pH = G^{-1}$. Set $s' = H(\ ,1): B \to E$; it is easy to show that $s' \in \sec_{e'} p$. Now, in [7, Lemma] we have proved that the association $[s] \to [s']$—where $[s]$, $[s']$ are the base-homotopy classes of s, s'—is independent of λ within its path-class and of s within its base homotopy class in $[\sec_e p]$. In other words, if $[\lambda] \in \pi^{\cdot} E$ is a path-class from e to e', there is a function $\lambda_{\#}: [\sec_e p] \to [\sec_{e'} p]$ such that $\lambda_{\#}[s] = [s']$. Notice that for the construction of s', the hypothesis that $b = p(e)$ is admissible was not used; however, all the properties of $\pi^{\cdot} E$ will be needed to prove the following.

(1.1) THEOREM. *For any fibration $p: E \to B$ there is a covariant functor*

$$\mathcal{S}: \pi^{\cdot} E \to Set$$

https://doi.org/10.4153/CMB-1977-051-9 Published online by Cambridge University Press

—the Section Functor—which assigns to each object e of πE *the set* $[\sec_e p]$ *and to each morphism* $[\lambda]\in \pi^{\cdot}E$ *the function (bijection)* $\lambda_{\#}$ [7, *Theorem*].

For a fixed $e\in \operatorname{Obj} \pi^{\cdot}E$, let $F=\{x\in E \mid p(x)=p(e)\}$ be the *fibre* of p which contains e; if $\sec_e p\neq \emptyset$, the exact homotopy sequence of p shows that $\pi_1(F, e)$ can be viewed as a subgroup of $\pi_1(E, e)$. Hence, there is a well-defined function

$$\pi_1(F, e)\times[\sec_e p]\to[\sec_e p]$$

which takes every pair $([\lambda], [s])\in \pi_1(F, e)\times[\sec_e p]$ into $\lambda_{\#}[s]$. The following corollary is a consequence of (1.1).

(1.2) COROLLARY. *Given that* $e\in \operatorname{Obj} \pi^{\cdot}E$ *and* $\sec_e p\neq\emptyset$, *then:*

1) *the function* $\pi_1(F, e)\times[\sec_e p]\to[\sec_e p]$ *as defined before is a left-action;*

2) *if* $s, \bar{s}\in \sec_e p$, *s is homotopic to* $\bar{s}$ *in the vertical (or, equivalently, free) sense if, and only if, there exists* $[\lambda]\in \pi_1(F, e)$ *such that* $\lambda_{\#}[s]=[\bar{s}]$.

The proofs for the following group of results can be found in [8]; the reader is advised that in all of them, e is a fixed object of $\pi^{\cdot}E$ and F is the fibre of p which contains e. The first result gives the precise relationship between free and based homotopy classes of sections of p.

(1.3) PROPOSITION. *The following are equivalent:*

1) *the obvious function* $[\sec_e p]\to[\sec p]$ *is an injection;*

2) *if* $s, \bar{s}\in\sec_e p$, *the statements that s and* $\bar{s}$ *are homotopic in each of the four senses described earlier are equivalent;*

3) *the action of* $\pi_1(F, e)$ *on* $[\sec_e p]$ *is trivial.*

(1.4) PROPOSITION. *For every* $s\in\sec p$, *there is* $\bar{s}\in\sec_e p$ *which is free-homotopic to s if, and only if,* $s(b)$ *and e lie in the same path-component of F.*

(1.5) PROPOSITION. *If F is path-connected,* $[\sec_e p]\to[\sec p]$ *is a surjection; if furthermore, for every* $e'\in F$, $\sec_{e'} p\neq\emptyset$, *the condition is also sufficient.*

Combining (1.3) and (1.5) we obtain

(1.6) PROPOSITION. *If F is path-connected, there is a bijection between* $[\sec p]$ *and the set* $[\sec_e p]/\pi_1(F, e)$ *of the orbits of* $[\sec_e p]$ *under the action of* $\pi_1(F, e)$. *(Clearly, if F is simply-connected,* $[\sec p]\cong[\sec_e p]$*).*

§2. **Applications of the Section Functor.** Let Y be a topological space and let μ be a path in Y connecting the points y and y'. It is well-known that μ induces an isomorphism of homotopy groups $\mu *:\pi_n(Y, y)\cong\pi_n(Y, y')$, for every $n\geq 0$. Ignoring the algebraic structure involved, this result has a very easy generalization: let X be a topological space and let $x\in X$ be a *non-degenerate* base point of X (that is to say, the inclusion $\{x\}\subset X$ is a cofibration); then, μ induces a bijection $\mu_*:[X, x; Y, y]_*\cong[X, x; Y, y']_*$ (see [23, 7.3.3]). Notice that the base point of X was kept fixed; it would be interesting to know how the change of base point in both X and Y affects the set $[X, x; Y, y]_*$. The study of this problem fits nicely in the context of the work developed in §1; as we shall see, its solution is very simple.

Let us begin by observing that $[X, x; Y, y]_* \cong [\sec_{(x,y)} pr_2]$, where $pr_2 : X \times Y \to Y$ is the projection on the second factor. Next, the reader should notice that for the trivial fibration pr_2, Theorem 4 of [25], used in the construction of the Section Functor, can be proved under the hypothesis that the inclusion $A \subset X$ is a cofibration and without requiring A to be closed in X (Strøm's notation). This, together with the fact that $\pi(X \times Y) \cong \pi X \times \pi Y$, shows that the Section Functor associated to the fibration pr_2 becomes a functor from the category $\pi^* X \times \pi Y$ to *Set*, where $\pi^* X$ is the full subcategory of πX whose objects are the non-degenerate (not necessarily closed) points of X; for every $(x, y) \in \pi^* X \times \pi Y$, $\mathcal{T}(x, y) \cong [X, x; Y, y]_*$.

(2.1) Proposition. *Let $x, x' \in X$ and $y, y' \in Y$ be given with x and x' non-degenerate. Any pair of paths λ from x to x' in X and μ from y to y' in Y induces a bijection $(\lambda, \mu)_\# : [X, x; Y, y]_* \cong [X, x'; Y, y']_*$.*

Since the fibre of pr_2 which contains y is Y, one can expect the fundamental group of Y to act on $[X, x; Y, y]_*$, if x is non-degenerate. In fact,

(2.2) Proposition. *Let $x \in X$ and $y \in Y$ be given, with x non-degenerate. Then,*

1) *there is a left-action of $\pi_1(Y, y)$ on $[X, x; Y, y]_*$ which takes each pair $([\mu], [f]) \in \pi_1(Y, y) \times [X, x; Y, y]_*$ into $(\lambda_x, [\mu])_\#[f]$, where λ_x is the constant path at x in X;*

2) *two maps f. $\bar{f}$ from (X, x) into (Y, y) are homotopic if, and only if, there is $[\mu] \in \pi_1(Y, y)$ such that $(\lambda_x, [\mu])_\#[f] = [\bar{f}]$.*

The next result is obtained applying (1.3) to the fibration pr_2.

(2.3) Proposition. *If $x \in X$ is non-degenerate, the following statements are equivalent:*

1) *the function $[X, x; Y, y]_* \to [X, Y]$ (= set of all free homotopy classes of maps from X to Y) is an injection;*

2) *if $f, \bar{f} \in Top_*((X, x), (Y, y))$ the assertion that there is a homotopy from f to $\bar{f}$ is equivalent to the assertion that there is a base-homotopy from f to $\bar{f}$;*

3) *the action of $\pi_1(Y, y)$ on $[X, x; Y, y]_*$ is trivial.*

As in (1.5) we see that if Y is path connected and $x \in X$ is non-degenerate, $[X, x; Y, y]_* \to [X, Y]$ is a surjection; we also have, as a trivial particularization of (1.6), the following result of Dold [14, 4.10]:

(2.4) Proposition. *If Y is path-connected and $x \in X$ is non-degenerate, there is a bijection $[X, Y] \cong [X, x; Y, y]/\pi_1(Y, y)$.*

Recall that a *lifting* of a map $f:A\to B$ over a fibration $p:E\to B$ is a map $g:A\to E$ such that $pg=f$. Let $L(f, p)$ be the set of all liftings of f over p; it is easy to see that $L(f, p)\cong \sec p_f$, where $p_f: A\sqcap E\to A$ is the fibration induced from p by f. If $a\in A$, $e\in E$ are base-points and $L(f, a; p, e)$ are the liftings of f over p which take a into e, then $L(f, a; p, e)\cong \sec_{(a,e)} p_f$. Two liftings $g, \bar{g}\in L(f, p)$ are said to be *vertically homotopic* if there is a homotopy $H: g\cong \bar{g}$ such that $H_t\in L(f, p)$ for each $t\in I$. Denote the set of all vertical homotopy classes of liftings of f over p by $L[f, p]$ and the set of all vertically-based homotopy classes of liftings in $L(f, a; p, e)$ by $L[f, a; p, e]$; then, identify $L[f, p]$ with $[\sec p_f]$ and $L[f, a; p, e]$ with $[\sec_{(a,e)} p_f]$. It should be pointed out that in spite of the close relationship between liftings and sections of a convenient fibration, the generalizations:

1) free homotopic liftings are vertically homotopic, and

2) based homotopic liftings are vertically based homotopic, *do not seem to hold.* In the latter instance this applies even if a is admissible.

Theorem (1.1) applied to liftings shows easily that if (a, e), $(a', e')\in A\sqcap E$, with a, a' admissible, then for every path h from a to a' in A and every path k from e to e' in E such that $pk=fh$, there is a bijection $(h, k)_{\#}: L[f, a; p, e]\cong L[f, a'; p, e']$. It should also be noticed that for any pair (a, e) as before, if F is the fibre of p (or of p_f) over $f(a)$ (respectively, a) and if $\sec_{(a,e)} p_f\neq\emptyset$, the fundamental group $\pi_1(F, e)$ acts on $L[f, a; p, e]$; furthermore,

(2.5) Proposition. *The following conditions are equivalent:*

1) *the obvious function $L[f, a; p, e]\to L[f, p]$ is an injection:*

2) *if g, $\bar{g}$ $L[f, a; p, e]$ then g and $\bar{g}$ are vertically-homotopic if, and only if, they are vertically-based homotopic;*

3) *the action of $\pi_1(F, e)$ on $L[f, a; p, e]$ is trivial.*

(2.6) Proposition. *If F is path-connected, $L[f, a; p, e]\to L[f, p]$ is surjective; furthermore, there is a bijection $L[f, p]\cong L[f, a; p, e]/\pi_1(F, e)$.*

§3. **Universal grounded fibrations.** Let F be a fixed topological space; a *grounded fibration with fibre F*, or *F-fibration*, is a sequence of topological spaces and maps

$$F\xrightarrow[g]{} p^{-1}(*)\hookrightarrow E\xrightarrow[p]{} B$$

such that:

1) B is a based, path-connected CW-complex with base point $*$;

2) p is a fibration;

3) g is a homotopy-equivalence. Denote this fibration by $\xi=(p, g)$.

Let $\xi_1=(p_1, g_1)$ and $\xi_2=(p_2, g_2)$ be F-fibrations; a morphism $\xi_1\to\xi_2$ is a pair (α, β), $\alpha: E_1\to E_2$, $\beta: B_1\to B_2$, such that $\beta p_1=p_2\alpha$ and $\alpha/p_1^{-1}(*)\circ g_1\simeq g_2$. Clearly $\alpha\mid p_1^{-1}(*)$ is a homotopy equivalence; furthermore, if $B_1=B_2$ and β is the identity function, α will be a homotopy equivalence [13, 6.3]. Because α takes $p_1^{-1}(b)$

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into $p_2^{-1}(b)$—the *fibers* over b of p_1 and p_2, respectively—for every $b \in B$, α is called a *fiber homotopy equivalence*. Let $t_F B$ be the set[(1)] of all fiber homotopy equivalence classes of F-fibrations over B. Actually, t_F is a contravariant functor from the category C_* of based, path-connected CW-complexes and based maps to the category of sets; each $X \in \text{Obj } C_*$ is taken into $t_F X$ and each map $f: X \to Y$ defines a function $t_F(f): t_F Y \to t_F X$ which maps the class $\{\xi\}$ of an F-fibration ξ over Y into the class of the F-fibration over X obtained by taking the pullback of p and f. According to [14, 6.5], t_F is a half-exact functor; furthermore, t_F is *representable*, that is to say, there is an object B_∞ of C_* such that t_F and $[\ ,\ ; B_\infty, *]_*$ are naturally equivalent [14, 16.7 and 16.8]. Let $\theta: [\ ,\ ; B_\infty, *]_* \to t_F$ be the natural equivalence; a representative $\xi_\infty = (p_\infty, g_\infty)$ of $\theta(1_{B_\infty})$ is said to be *F-universal*. The obvious question one should investigate is the following: given that $\xi = (p, g)$ is an F-fibration, when is it F-universal? G. Allaud has taken up this problem in [1] and [2]; in [2] he came up with the following result. "Suppose that F is locally compact and let $E^{(F)}$ be the set $\{f: F \to E \mid \text{there is } b \in B \text{ such that imf} \subset p^{-1}(b) \text{ and } f: F \to p^{-1}(b) \text{ is a homotopy equivalence}\}$ with the compact-open topology. If $E^{(F)}$ is contractible then ξ is F-universal." Our objective is to obtain a theorem which is more general than Allaud's; furthermore, we would like to relate its proof to the study of the sections of some suitable fibration. Notice that because we are dealing in part with function spaces, it is advisable to work within the realm of a "Convenient Category". The study of these categories was pursued by several mathematicians (e.g. [5], [9], [12], [24], [26], and [27]); they are full subcategories of *Top* (or Top_*) satisfying a certain amount of "convenient" properties. In the words of the late N. Steenrod, "the demands which a convenient category should satisfy are first that it be large enough to contain all the particular spaces arising in practice. Second, it must be closed under standard operations; these are the formation of subspaces, product spaces $X \times Y$, function spaces $Y^X, \ldots$. Third, the category should be small enough so that certain reasonable propositions about the standard operations are true" e.g., an exponential law. An example of a convenient category can be obtained as follows. Let H be the category of all compact Hausdorff spaces and let $E: H \to Top$ be the imbedding functor. Next, take k to be the (left) Kan extension of E along itself[(2)] and define K as the image of *Top* by k; then, K is convenient [26]. If $X, Y \in \text{Obj } K$ define the product $X \times Y$ as the image by k of the usual product; also, define $K(X, Y)$ as $k(Y^X)$, where Y^X is the set of all maps from X into Y with the compact-open topology. One shows that for every X, Y, Z Obj K, $K(X \times Y, Z) \cong K(X, K(Y, Z))$ [26, 3.6]. One more comment is in order: the category K contains the compactly generated spaces (for their definition see [24]) and

(1) There is a set-theoretical problem here; for a discussion of this point and generalizations of the notion of F-fibration, see [20].

(2) For the definition and relevant properties of Kan extensions see [17], [19], and [21].

hence, all CW-complexes. From now on, all spaces and maps considered will be objects and morphisms of K.

(3.1) THEOREM. *Let ξ be the fibration*

$$F \xrightarrow[g]{} p^{-1}(*) \hookrightarrow E \xrightarrow[p]{} B$$

with B a based, path-connected CW-complex. If $\pi_j(E^{(F)}) = 0$ for every $j \geq 0$ (i.e., *$E^{(F)}$ is weakly-contractible*) *then ξ is F-universal.*

We have based the proof of the Theorem above on an ingenious construction due to Peter Booth; I refer to the *fibered mapping projection* of two maps $p: X \to B$ and $q: Y \to B$ first introduced in [3] for the category of quasitopological spaces (see [22]) and then, in [5] for K. Let me explain briefly such a construction and quote some of its properties. For a given $B \in \operatorname{Obj} K$, B. Hausdorff, let $(K \downarrow B)$ be the *comma-category* of objects of K over B [19], that is, the category whose objects are morphisms $p: X \to B$ of K; the morphisms from $p: X \to B$ to $q: Y \to B$ are morphisms $g: X \to Y$ in K such that $qg = p$. Given two objects p and q of $(K \downarrow B)$ as above, consider for each $b \in B$, the fibres $X_b = p^{-1}(b)$ and $Y_b = q^{-1}(b)$; then, take the set $(XY) = \bigcup_{b \in B} K(X_b, Y_b)$ and the set-theoretical function $(pq): (XY) \to B$ which maps any element $F: X_b \to Y_b$ into $b \in B$. It is possible to topologise (XY) so to make it an object of K and make (pq)—the *fibered mapping projection*—a morphism of that category [5], [6]. It is easily seen that $(K \downarrow B)$ is a category with finite products: the product of p and q is given by the pullback $p \sqcap q: X \sqcap Y \to B$. Clearly, for $p \in \operatorname{Obj}(K \downarrow B)$ fixed, $(p\_)$ and $- \sqcap p$ are endofunctors of $(K \downarrow B)$; moreover, $- \sqcap p$ is a left-adjoint ot $(p\_)$ [5]. Finally, if p and q are Hurewicz fibrations, so are (pq) and $q \sqcap p$ [5, 3.4].

With this in mind one can prove the following.

(3.2) PROPOSITION. *Let $p: X \to A$ and $q: Y \to B$ be fibrations, with A and B Hausdorff. Consider the fibration $p_* q: (B \times XY \times A) \to A$ constructed as the composite of the fibrations $(1_B \times pq \times 1_A): (B \times XY \times A) \to B \times A$ and $pr_2: B \times A \to A$. Then, there is a bijection between the set of all map pairs (h, k) with $h: X \to Y$, $k: A \to B$, $qh = kp$, and* $\sec p * q$.

Theorem (3.1) is then a consequence of (3.2) and

(3.3) PROPOSITION. *Let $\xi = (p, g)$ be a grounded fibration with weakly contractible ground fibre; then p has sections, any two sections are homotopic and any two based sections* (*same base*) *are based-homotopic.*

REMARK. Following [1, IV] one can show that if ξ is F-universal, with F of the homotopy type of a CW-complex, then $E^{(F)}$ is weakly contractible; moreover, if F is compact, $E^{(F)}$ is contractible. Thus [1] and (3.1) give rise to the following characterization of F-universal fibrations: "An F-fibration with ground fibre F of the homotopy type of a CW-complex is F-universal if, and only if, $E^{(F)}$ is weakly contractible." One can avoid entirely [1] and the condition that F be of the homotopy type of a CW-complex, by putting the notion of F-universality in a slightly more restrictive perspective. To wit, let HF_F be the homotopy category of grounded F-fibrations over C_* and corresponding morphisms. We then have that *ξ is a final object of HF_F if, and only if, $E^{(F)}$ is weakly contractible.* Here, the crucial point is that if $\xi = E(p, g)$ is an F-fibration over a sphere S^j, $\sec_* p \neq \emptyset$ and $[\sec_* p] \cong \pi_j(F)$, $j > 0$.

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