BISHOP'S CONDITION (β)†

by JON C. SNADER

(Received 8 September, 1983)

1. Introduction. In 1959, Bishop [**4**] published a seminal paper in which he studied various types of spectral decompositions or "duality theories" that an arbitrary bounded linear operator on a reflexive Banach space might have. In the course of his investigations, he isolated the following analytic property which he called condition (β).

DEFINITION 1.1. Let T be a bounded linear operator on a complex Banach space X, and suppose that for any open subset V of $\mathbf{C}$ and any sequence $f_n : V \to X$ of X-valued analytic functions such that $(\lambda I - T)f_n(\lambda) \to 0$ uniformly in norm on compact subsets of V, we also have that $f_n(\lambda) \to 0$ uniformly in norm on compact subsets of V. In this case we say that T has *condition* (β).

We remark that our definition of condition (β) is slightly different from Bishop's, his requiring only that $(f_n(\lambda))$ be uniformly bounded on compact subsets of V when $(\lambda I - T)f_n(\lambda) \to 0$ uniformly in norm on compact subsets of V.

In [**11**], Foiaş showed that every decomposable operator (and therefore spectral operators in the sense of Dunford, compact operators, and unitary, normal, and self-adjoint operators on a Hilbert space) enjoys condition (β). Indeed, in a sense, any operator that can be said to have a satisfactory spectral decomposition must have condition (β). More precisely, Erdelyi and Lange [**9**] defined axiomatically what they considered to be the minimum properties that an operator should possess in order to say it has a "spectral decomposition". Operators with these properties were said to have the *spectral decomposition property* (SDP). Albrecht [**1**], and independently Lange [**17**] and Nagy [**19**], in a beautiful and surprising result, proved the equivalence between SDP and decomposability. This is of particular interest to us not merely because it justifies our remark that any operator with a reasonable spectral decomposition must have condition (β), but also because the key to Albrecht's proof lies in establishing that any operator with the SDP enjoys condition (β).

The importance of condition (β) now begins to emerge; condition (β) is not merely a property that a certain large class of operators happens to possess, but rather is a useful tool which can be applied in proving important theorems. This importance is brought sharply into focus by the following remarkable characterization due to Lange [**16**].

THEOREM 1.2 (Lange's Theorem). *Let T be a bounded linear operator on a reflexive Banach space. Then T is decomposable if and only if T and its adjoint T^* have condition* (β).

† This material constitutes part of the Author's Ph.D. thesis written under the direction of R. G. Bartle at the University of Illinois.

There are other applications as well. In **[22]**, the author used condition (β) to study strongly analytic subspaces (see **[18]**) and to obtain a characterization of strongly decomposable operators **[2]** on a reflexive Banach space.

2. Bishop's condition (β). As we remarked previously, examples of operators enjoying condition (β) abound. On the other hand, the class of operators satisfying condition (β) is not too large.

EXAMPLE 2.1. This is an example of an operator without condition (β). Define $X=\ell_2(\mathbf{N})$, and let T be the left shift. Let $\{e_1, e_2, \ldots\}$ be the usual orthonormal basis in $\ell_2(\mathbf{N})$, and define $V=\{\lambda:|\lambda|<1\}$ and

$$f_n(\lambda)=\sum_{j=0}^{\infty} e_j\lambda^j \ (n=1,2,\ldots;\lambda\in V).$$

Now each f_n is analytic on V and is certainly not identically zero. However, as a routine calculation shows, $(\lambda I-T)f_n(\lambda)\equiv 0$ $(n=1,2,\ldots;\lambda\in V)$. Thus T does not have condition (β).

A useful criterion guaranteeing that an operator does *not* have condition (β) is provided by the following definition and theorem.

DEFINITION 2.2 **[7]**. A closed linear operator $T:D(\subset X)\to X$ is said to have the *single-valued extension property* (SVEP) if for every open set $V\subset \mathbf{C}$ and every analytic function $f:V\to D$, the condition $(\lambda I-T)f(\lambda)\equiv 0$ for λ in V implies that $f(\lambda)\equiv 0$ for λ in V.

The next result was known to Bishop. Its simple proof follows by taking $f_n(\lambda)=f(\lambda)$ $(n=1,2,\ldots)$.

THEOREM 2.3. *If a bounded linear operator on a Banach space has condition* (β), *then it also has the SVEP.*

The next Theorem is due to Finch **[10]**; its corollary provides a useful test for operators that fail to have condition (β).

THEOREM 2.4. *Let T be a closed linear operator on a Banach space X. If the range of T is all of X, but T is not one-one, then T does not have the SVEP.*

COROLLARY 2.5. *Let T be a bounded linear operator on a Banach space X. If the range of T is all of X, but T is not one-one, then T does not have condition* (β).

REMARKS 2.6. The following remarks are all easy consequences of Corollary 2.5. In all of the remarks, T is a bounded linear operator on a Banach space.

(1) If T is an isometry but is not onto, then T^* does not have condition (β). Indeed, as is well known, the fact that the range of T is closed implies that the range of T^* is also closed; further, if T is one-one, then the range of T^* is X^*. But now T^* is not one-one since the range of T is not X.

(2) If T has a right inverse but no left inverse, then T does not have condition (β).

https://doi.org/10.1017/S0017089500005759 Published online by Cambridge University Press

The hypothesis here is just another way of saying T is onto but not one-one. Notice that this remark provides another proof of the fact that the operator in Example 2.1 does not have condition (β).

(3) If T has condition (β), then $\lambda \in \rho(T)$, the resolvent set of T, if and only if the range of $(\lambda I - T)$ is all of X. As we shall see (Theorem 2.11), the operator $(\lambda I - T)$ enjoys condition (β) if T does. Now if $(\lambda I - T)$ is onto, it must also be one-one since it has condition (β). Therefore $(\lambda I - T)^{-1}$ exists, and $\lambda \in \rho(T)$. The reverse implication is trivial.

Now that we have established that the class of bounded operators enjoying condition (β) is a proper subset of the bounded operators, and have determined some necessary conditions for membership in that class, let us consider to what extent an operator having condition (β) implies that a related operator also has condition (β).

Perhaps the most natural related operator in the context of spectral theory is the restriction operator. The next theorem is immediate.

THEOREM 2.7. *Let T be a bounded linear operator on a Banach space X, and let $Y \subset X$ be a T-invariant subspace. If T has condition (β), then so does the restriction operator $(T \mid Y): Y \to Y$.*

If Y is a T-invariant subspace, an operator closely related to both T and $T \mid Y$ is the quotient operator T^Y on X/Y. Here the expected result does not hold. Before we can present the appropriate example, however, we need the following result which is interesting in its own right.

THEOREM 2.8. *If T is a bounded linear operator on a Banach space X, and if B is a linear isomorphism between the Banach spaces X and Y, then T has condition (β) if and only if BTB^{-1} does.*

Proof. Suppose that T has condition (β), that $f_n : V \to Y$, $(n = 1, 2, \ldots)$, are Y-valued analytic functions, and that $(\lambda I - BTB^{-1})f_n(\lambda) \to 0$ uniformly on compact subsets of V. Then for any compact $K \subset V$, we have that $B(\lambda I - T)B^{-1}f_n(\lambda) \to 0$ uniformly for λ in K; hence by the boundedness of B^{-1}, we see that $(\lambda I - T)B^{-1}f_n(\lambda) \to 0$ uniformly for λ in K. Since T has condition (β), we conclude that $B^{-1}f_n(\lambda) \to 0$ uniformly for λ in K. Finally, by the boundedness of B, we deduce that $f_n(\lambda) \to 0$ uniformly for λ in K. Since K was an arbitrary compact subset of V, it follows that BTB^{-1} has condition (β). The reverse implication follows by symmetry.

We are now ready to address the question of whether T^Y has condition (β).

EXAMPLE 2.9. Here we give an example of an operator T and a T-invariant subspace Y such that T has condition (β), but T^Y does not. Let $X = \ell_2(\mathbf{Z})$, let T be the left bilateral shift, and let $Y = \overline{\mathrm{sp}}\{e_{-1}, e_{-2}, e_{-3}, \ldots\}$, where $\{e_0, e_{\pm 1}, e_{\pm 2}, \ldots\}$ is the usual orthonormal basis for $\ell_2(\mathbf{Z})$. Now T is unitary and so certainly has condition (β), but T^Y is isomorphic to the left shift on $\ell_2(\mathbf{N})$, and hence does not have condition (β) by Example 2.1 and Theorem 2.8.

A characterization of those subspaces Y for which T^Y has condition (β) when T does, as well as some applications, can be found in [**22**].

The proof of the next theorem uses an idea due to Lange [**18**].

THEOREM 2.10. *Let T be a bounded linear operator on a Banach space X. If T has condition (β), and T^{-1} exists, then T^{-1} also has condition (β).*

Proof. Let $f_n : V \to X$ $(n=1,2,\dots)$ be X-valued analytic functions, let T have condition (β), and suppose that $(\lambda I - T^{-1})f_n(\lambda) \to 0$ uniformly on compact subsets of V. Let $K \subset V$ be compact. Now 0 is not in the spectrum $\sigma(T^{-1})$ of T^{-1} and so there exists an $r>0$ such that $\{\lambda \in \mathbf{C} : |\lambda| < r\}$ is disjoint from $\sigma(T^{-1})$. Define the compact sets K_1 and K_2 by

$$K_1 = \{\lambda \in \mathbf{C} : |\lambda| \leq r\} \cap K$$

$$K_2 = \{\lambda \in \mathbf{C} : |\lambda| \geq r\} \cap K$$

and notice that $K = K_1 \cup K_2$. Since $K_1 \subset \rho(T^{-1})$, it is clear that $f_n(\lambda) \to 0$ uniformly for λ in K_1. For $\lambda \in W = \{\lambda \in \mathbf{C} : |\lambda| > r/2\} \cap V$, we have $(\lambda I - T^{-1})f_n(\lambda) = -(\lambda^{-1}I - T)(T/\lambda)^{-1}f_n(\lambda)$ so that $(\lambda^{-1}I - T)(T/\lambda)^{-1}f_n(\lambda) \to 0$ uniformly on compact subsets of W. Define

$$g_n(\lambda) = (\lambda T)^{-1}f_n(1/\lambda)(n=1,2,\dots),$$

so that $g_n(1/\lambda)$ is analytic on W. Thus for λ in compact subsets of W, $(\lambda^{-1}I - T)g_n(1/\lambda) \to 0$ uniformly, and since T has condition (β), we see that $g_n(1/\lambda) \to 0$ uniformly for λ in $K_2 \subset W$. Now for λ in K_2, the operator T/λ is bounded, so we have that $f_n(\lambda) = (T/\lambda)g_n(1/\lambda) \to 0$ uniformly. Thus $f_n(\lambda) \to 0$ uniformly for λ in $K = K_1 \cup K_2$ proving the theorem.

Along the same lines we have the following result.

THEOREM 2.11. *If a bounded linear operator T on a Banach space X has condition (β), then*

(i) *the operator αT has condition (β) for any scalar α in $\mathbf{C}$,*
(ii) *the operator $(\alpha I - T)$ has condition (β) for any scalar α in $\mathbf{C}$,*
(iii) *the resolvent operator $R(\lambda; T) = (\lambda I - T)^{-1}$ has condition (β) for every $\lambda \in \rho(T)$.*

Proof. The proofs of (i) and (ii) are similar; we prove only (ii).

Let $f_n(\lambda) : V \to X$ $(n=1,2,\dots)$ be X-valued analytic functions, let α be in $\mathbf{C}$, and suppose that $\{\lambda I - (\alpha I - T)\}f_n(\lambda) \to 0$ uniformly on compact subsets of V. If we define

$$g_n(\lambda) = f_n(\lambda + \alpha) \quad (n=1,2,\dots),$$

then the g_n are analytic for $\lambda + \alpha$ in V, and $\{(\lambda - \alpha)I - T\}g_n(\lambda - \alpha) \to 0$ uniformly for λ in a compact subset of V. Since T has condition (β), we see that for any compact $K \subset V$, $g_n(\lambda - \alpha) \to 0$ uniformly for λ in K, and so $f_n(\lambda) = g_n(\lambda - \alpha) \to 0$ uniformly for λ in K.

Part (iii) is a corollary of (ii) above and Theorem 2.10.

As one would expect, condition (β) is preserved under the direct sum of operators.

THEOREM 2.12. *A bounded linear operator $T_1 \oplus T_2$ on the Banach space $X_1 \oplus X_2$ has condition (β) if and only if T_i has condition (β) on X_i $(i=1, 2)$.*

Proof. Suppose that $T_1 \oplus T_2$ has condition (β), that $f_n : V \to X_1$, $(n = 1, 2, \ldots)$ are X_1-valued analytic functions, and that $(\lambda I - T_1) f_n(\lambda) \to 0$ uniformly on compact subsets of V. Let I_i be the identity operator on X_i $(i = 1, 2)$ and define $g_n : V \to X_2$ for $n = 1, 2, \ldots$, by $g_n = 0 I_2$. Then we have that $(\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2))(f_n(\lambda) \oplus g_n(\lambda)) \to 0$ uniformly on compact subsets of V. Since $T_1 \oplus T_2$ has condition (β), then $f_n(\lambda) \oplus g_n(\lambda) \to 0$ uniformly on compact subsets of V. By the definition of convergence in $X_1 \oplus X_2$, we conclude that $f_n(\lambda) \to 0$ uniformly on compact subsets of V, and hence that T_1 has condition (β). The same argument, *mutatis mutandis*, shows that T_2 has condition (β).

On the other hand, if T_i on X_i $(i = 1, 2)$, have condition (β), if $f_n^1 \oplus f_n^2 : V \to X_1 \oplus X_2$ $(n = 1, 2, \ldots)$ are $X_1 \oplus X_2$-valued analytic functions with $f_n^i : V \to X_i$ vector valued analytic functions, and if

$$(\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2))(f_n^1(\lambda) \oplus f_n^2(\lambda)) \to 0$$

uniformly on compact subsets of V, then $(\lambda I_i - T_i) f_n^i(\lambda) \to 0$ uniformly on compact subsets of V for $i = 1, 2$. Since the T_i have condition (β), we see that $f_n^i(\lambda) \to 0$ uniformly on compact subsets of V for $i = 1, 2$, and therefore that $f_n^1(\lambda) \oplus f_n^2(\lambda) \to 0$ uniformly on compact subsets of V, completing the proof.

We have already seen, in Theorem 2.7, that if $Y \subset X$ is a T-invariant subspace, then $T \mid Y$ has condition (β) if T does. The converse, of course, is false.

EXAMPLE 2.13. This is an example of a T-invariant subspace Y, such that $T \mid Y$ has condition (β), but T does not.

Let $X = \ell_2(\mathbf{N}) \oplus \ell_2(\mathbf{N})$, let $R : \ell_2(\mathbf{N}) \to \ell_2(\mathbf{N})$ be the right shift, let $L : \ell_2(\mathbf{N}) \to \ell_2(\mathbf{N})$ be the left shift, and let $T : X \to X$ be $L \oplus R$. Then T does not have condition (β) by Example 2.1 and the last theorem. On the other hand, if we define $Y = (0) \oplus \ell_2(\mathbf{N})$, then Y is T invariant, and $T \mid Y$ is isomorphic to R. Since the right shift on $\ell_2(\mathbf{N})$ is a restriction of the unitary bilateral shift on $\ell_2(\mathbf{Z})$, it enjoys condition (β) by Theorem 2.7. Thus by Theorem 2.8, the operator $T \mid Y$ has condition (β).

In some cases we *can* conclude that T has condition (β) if we know that $T \mid Y$ does for certain T-invariant subspaces Y.

DEFINITION 2.14 [**8**, VII.3.17]. A subset of $\sigma(T)$ that is both open and closed in $\sigma(T)$ is called a *spectral set.*

DEFINITION 2.15. Let σ be a spectral set of $\sigma(T)$, and let f be a scalar-valued function, analytic on a neighborhood U of $\sigma(T)$, such that f is identically 1 on σ and vanishes on $\sigma(T) \setminus \sigma$. Define

$$E(\sigma) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, d\lambda,$$

where Γ is a finite union of rectifiable Jordan curves such that $\sigma(T)$ lies "inside" Γ, and such that $\Gamma \subset U$.

By [**8**, VII.3.20–1], each $E(\sigma)$ is a projection such that $E(\sigma)X$ reduces T and such that the map $\sigma \to E(\sigma)$ is an isomorphism of the Boolean algebra of spectral sets onto the Boolean algebra of all projections of the form $E(\sigma)$ with σ a spectral set.

THEOREM 2.16. *Let T be a bounded linear operator on a Banach space X. Suppose that $\sigma(T)$ is disconnected with spectral sets $\sigma_1, \ldots, \sigma_n$ such that $\sigma(T) = \bigcup_{j=1}^{n} \sigma_j$. Define*

$$Y_j = E(\sigma_j)X \; (j = 1, \ldots, n).$$

Then T has condition (β) if and only if each restriction $T \mid Y_j$ $(j = 1, \ldots, n)$ does.

Proof. From the discussion above, we have $X = \bigoplus_{j=1}^{n} Y_j$, and $T = \bigoplus_{j=1}^{n} (T \mid Y_j)$. By Theorem 2.12 it follows that T has condition (β) if and only if each $(T \mid Y_j)$, $j = 1, \ldots, n$, does.

For operators whose spectra exhibit an extreme behavior of this sort, we have a stronger result.

THEOREM 2.17. *Let T be a bounded linear operator on a Banach space X. If the spectrum of T is totally disconnected, then T is decomposable. In particular, T has condition (β).*

Proof. By [**1**] and [**20**], it suffices to show that if G_1 and G_2 are open sets in **C** and if $\sigma(T) \subset G_1 \cup G_2$, then there exist T-invariant subspaces Y_1 and Y_2 such that

(i) $X = Y_1 + Y_2$ and

(ii) $\sigma(T \mid Y_j) \subset G_j$ $(j = 1, 2)$.

Let us assume, therefore, that G_1 and G_2 are open sets that cover $\sigma(T)$. Each point of $\sigma(T)$ has a clopen neighborhood in either G_1 or G_2. Because $\sigma(T)$ is compact, a finite number of these clopen neighborhoods cover $\sigma(T)$. Let $N_1, \ldots, N_k$ be those clopen neighborhoods that are contained in G_1. For each point λ of $\sigma(T)$ such that λ is in $G_2 \setminus \left(\bigcup_{j=1}^{k} N_j \right)$ we may choose a clopen neighborhood $D_\lambda \subset G_2$ such that $D_\lambda \cap N_j = \varnothing$, $(j = 1, \ldots, k)$. Now $\{N_1, \ldots, N_k\} \cup \left\{ D_\lambda : \lambda \in \sigma(T) \cap \left(G_2 \setminus \left(\bigcup_{j=1}^{k} N_j \right) \right) \right\}$ is an open cover of $\sigma(T)$, so there exists a finite subcover. Let σ_1 be all members of the subcover from the set $\{N_1, \ldots, N_k\}$, and let σ_2 be the remaining neighborhoods in the subcover.

Now σ_1 and σ_2 are clopen and disjoint, so there exist scalar-valued analytic functions f_1 and f_2 such that f_1 is identically 1 on σ_1 and vanishes on σ_2 and such that f_2 is identically 1 on σ_2 and vanishes on σ_1. Define $E(\sigma_j)$ $(j = 1, 2)$, as in Definition 2.15. Now $Y_j = E(\sigma_j)$ $(j = 1, 2)$, are T-invariant subspaces such that $X = Y_1 + Y_2$, and it follows from [**8**, VII.3.20] that $\sigma(T \mid Y_j) \subset \sigma_j \subset G_j$ $(j = 1, 2)$. Thus T is decomposable, and in particular, T has condition (β).

In [**10**, Cor. 6], Finch showed that if a bounded linear operator T on a Banach space has the SVEP, then $\sigma(T^*) = \sigma_a(T^*)$, where $\sigma_a(T^*)$ is the approximate point spectrum of T^* [**6**, 1.15]. Similarly, if T^* has the SVEP, then $\sigma(T) = \sigma_a(T)$. In view of Theorem 2.3, these results hold if T or T^* have condition (β).

On the other hand, by the Duality Theorem for decomposable operators, [**12**], [**13**], T is decomposable if and only if T^* is. Thus for a decomposable operator, we have that $\sigma(T)=\sigma_a(T)$. These considerations allow us to verify the distinction between the class of operators with condition (β), and the class of decomposable operators.

EXAMPLE 2.18. This is an example of an operator that enjoys condition (β), but is not decomposable.

Let $X=\ell_2(\mathbf{N})$ and let T be the right shift. Now T is the restriction of a unitary operator, and so it has condition (β). As is well known, $\sigma(T)=\{\lambda\in\mathbf{C}:|\lambda|\leq 1\}$, while $\sigma_a(T)=\{\lambda\in\mathbf{C}:|\lambda|=1\}$. Thus $\sigma(T)\neq\sigma_a(T)$, and from the preceding remarks, it follows that T is not decomposable.

We remark also that the fact that $\sigma(T)\neq\sigma_a(T)$ shows that T^* cannot have condition (β). This is hardly surprising since T^* is our old friend the left shift on $\ell_2(\mathbf{N})$. Because this shows that condition (β) is not preserved under the adjoint operation, we record it as follows.

EXAMPLE 2.19. The right shift on $\ell_2(\mathbf{N})$ is an example of a bounded linear operator that has condition (β), but whose adjoint does not.

The next few examples show that condition (β) fails to be preserved under a variety of common operations. It seems entirely reasonable that if two operators have condition (β), then so does their sum. The next example shows that this is not the case.

EXAMPLE 2.20. In this example we have two operators that have condition (β), but whose sum does not.

Let T be any operator on a Hilbert space that does not have condition (β). Then $T=\frac{1}{2}(T+T^*)+\frac{1}{2}(T-T^*)$, and since the operators $\frac{1}{2}(T+T^*)$ and $\frac{1}{2}(T-T^*)$ are normal, they have condition (β).

Unlike the compact operators, for instance, the operators that have condition (β) do not form an ideal in the algebra of operators on a Banach space.

EXAMPLE 2.21. This example shows that the product of an operator with condition (β) and a bounded linear operator may not have condition (β), even if the two operators commute.

Let X be a Banach space such that not all its bounded linear operators have condition (β), and let T be a bounded linear operator that does not. Now it is trivial that the identity operator I on X has condition (β), and I certainly commutes with T, but $IT=TI=T$ does not have condition (β).

Another natural question is whether or not condition (β) is preserved under compact perturbations. The next example answers this in the negative. This example was used by Lange, [**18**], in another context.

EXAMPLE 2.22. This is an example of an operator that has condition (β), and a compact perturbation of it that does not.

In [**15**], Herrero gave an example of a compact operator K of $\ell_2(\mathbf{Z})$ such that $U+K$, where U is the right bilateral shift on $\ell_2(\mathbf{Z})$, has the following properties:

(i) $\sigma(U+K)=\{\lambda \in \mathbf{C}:|\lambda|=1\}$;

(ii) if $Y \neq (0)$ is an $(U+K)$-invariant subspace of $\ell_2(\mathbf{Z})$, then either $\sigma(U+K \mid Y)=\{\lambda \in \mathbf{C}:|\lambda|=1\}$ or $\sigma(U+K \mid Y)=\{\lambda \in \mathbf{C}:|\lambda| \leq 1\}$.

Note that by Schauder's theorem, U^*+K^* is also a compact perturbation of a unitary operator. Since U and U^* are unitary, they both have condition (β). On the other hand, if both $U+K$ and U^*+K^* have condition (β), then $U+K$ would be decomposable by Lange's Theorem. In order to show that this is not possible, let $G_1=\{\lambda \in \mathbf{C}: \operatorname{Re} \lambda < \frac{1}{2}\}$ and $G_2=\{\lambda \in \mathbf{C}: \operatorname{Re} \lambda > -\frac{1}{2}\}$. Now $\{G_1, G_2\}$ is an open cover for $\sigma(U+K)$; hence if $U+K$ is decomposable, there exist $(U+K)$-invariant subspaces Y_1 and Y_2 such that $\ell_2(\mathbf{Z})=Y_1+Y_2$, and $\sigma(U+K \mid Y_j) \subset G_j$ $(j=1,2)$. Now if one of the Y_j, say Y_1, is the zero subspace, then $Y_2=\ell_2(\mathbf{Z})$ and so $\sigma(U+K \mid Y_2)=\sigma(U+K)=\{\lambda \in \mathbf{C}:|\lambda|=1\} \not\subset G_2$. On the other hand, if neither Y_1 nor Y_2 is (0), then $\sigma(U+K \mid Y_1) \subset G_1$, and this contradicts condition (ii) above. Thus at least one of $U+K$ and U^*+K^* does not have condition (β).

Example 2.22 also provides us with another important verification.

Example 2.23. Here we exhibit an operator that has the SVEP, but does not have condition (β).

Let $U+K$ be as in the last example. Then $\sigma(U+K)=\sigma(U^*+K^*)=\{\lambda \in \mathbf{C}:|\lambda|=1\}$. But this implies that $U+K$ and U^*+K^* both have the SVEP, for if not there exists an open set $V \subset \mathbf{C}$, and a non-zero, $\ell_2(\mathbf{Z})$-valued analytic function $f: V \rightarrow \ell_2(\mathbf{Z})$ such that $(\lambda I-(U+K))f(\lambda)=0$ for $\lambda \in V$. That is, for λ in V, we have $(U+K)f(\lambda)=\lambda f(\lambda)$, so that $V \subset \sigma_p(U+K)$, the point spectrum of $U+K$. Thus if $U+K$ or U^*+K^* fails to have the SVEP, its spectrum $\{\lambda \in \mathbf{C}:|\lambda|=1\}$ would contain an open set, which is impossible. On the other hand, Example 2.22 shows that at least one of $U+K$ or U^*+K^* fails to have condition (β).

With this last example, we have shown that condition (β) is a genuine property in its own right, and that it fits strictly between the SVEP and decomposability in the sense that decomposability $\Rightarrow$ condition $(\beta) \Rightarrow$ SVEP.

Under stronger hypotheses we *can* conclude that condition (β) is preserved under certain types of perturbations. In **[3]** and **[21]** it is shown that if T is decomposable and S is an operator that commutes with T such that S has totally disconnected spectrum, then $T+S$ is decomposable. From this, the next theorem is immediate.

Theorem 2.24. *Let T be a decomposable operator on a Banach space X, and let S be a bounded linear operator on X that commutes with T. If*

(1) *S is a quasinilpotent operator, or*

(2) *S is a compact operator, or*

(3) *S has discrete spectrum,*

then $T+S$ is decomposable. In particular $T+S$ has condition (β).

The next theorem is almost immediate, but it is included for the sake of completeness.

Theorem 2.25. *Let T be an arbitrary bounded linear operator on a Banach space X, let*

Q be a quasinilpotent operator on X that commutes with T, and let K be a compact operator on X. Then QT, KT, and TK all have condition (β).

Suppose that T_n $(n=1,2,\ldots)$ are bounded linear operators on a Banach space, that each T_n enjoys condition (β), and that the T_n converge to an operator T in some topology. A natural question is whether or not T also has condition (β). This gains in interest when we notice that, in view of Lange's Theorem, it would imply that a convergent sequence of decomposable operators on a reflexive Banach space would converge to a decomposable operator. Unfortunately, this question has proved to be particularly intractable. Vasilescu **[23]** has shown that the uniform limit of commuting operators with the SVEP also has the SVEP, but we have been unable to extend the result to condition (β) even under hypotheses as strong as Vasilescu's. We can show, however, that convergence in the strong operator topology is not enough.

EXAMPLE 2.26. In this example we construct a sequence of operators with condition (β) that converges in the strong operator topology to an operator without condition (β).

Define $T_n:\ell_2(\mathbf{N})\to\ell_2(\mathbf{N})$ $(n=1,2,\ldots)$ by

$$\begin{aligned} T_ne_k &= 0 && \text{for} \quad k=1\\ &= e_{k-1} && \text{for} \quad 2\leqslant k\leqslant n\\ &= e_k && \text{for} \quad k>n, \end{aligned}$$

where $\{e_1, e_2, \ldots\}$ is the usual orthonormal basis, and extend T_n linearly and continuously to all of $\ell_2(\mathbf{N})$. Clearly the T_n are (uniformly) bounded linear operators. If we denote the identity operator on $\ell_2(\mathbf{N})$ by I and define $\hat{T}_n$ to be the left shift on $\mathbf{C}^n$, we may define the operator

$$\hat{T}_n\oplus I: C^n\oplus\ell_2(\mathbf{N})\to C^n\oplus\ell(\mathbf{N}).$$

Now $\sigma(\hat{T}_n\oplus I)=\sigma(\hat{T}_n)\cup\sigma(I)=\{0,1\}$, and so $\hat{T}_n\oplus I$ has condition (β) by Theorem 2.17. Since T_n is (topologically) isomorphic to $\hat{T}_n\oplus I$, we see, by Theorem 2.8, that each T_n has condition (β).

Let T be the left shift on $\ell_2(\mathbf{N})$. Since for any $x=(x_k)$ in $\ell_2(\mathbf{N})$ we have

$$\|(T-T_n)x\|^2\leqslant 2\sum_{k=n+1}^{\infty}|x_k|^2,$$

we see that the T_n converge to T in the strong operator topology. By Example 2.1, T does not have condition (β). This completes the example.

In many of the theorems and examples we have studied so far, the structure of the spectrum of the operator played a key role in determining whether or not the operator had condition (β). This leads to speculation as to whether or not those operators enjoying condition (β) can be characterized by their spectra. The next example shows that is is not the case—at least not in terms of the spectrum and its "fine structure".

EXAMPLE 2.27. We construct two operators whose spectra and fine structure are identical, but such that only one of them has condition (β).

Define the sets $D_0 = \{\lambda \in \mathbf{C} : |\lambda| < 1\}$ and $D = \{\lambda \in \mathbf{C} : |\lambda| \leq 1\}$. Let e_α be the function on D_0 such that

$$e_\alpha(\alpha) = 1, \quad \text{and}$$
$$e_\alpha(\beta) = 0 \quad \text{for} \quad \beta \neq \alpha.$$

Then $\{e_\alpha : \alpha \in D_0\}$ is the usual orthonormal basis for $\ell_2(D_0)$. Let M be the multiplication operator on $\ell_2(D_0)$ defined by $Me_\alpha = \alpha e_\alpha$, extended linearly and continuously to $\ell_2(D_0)$. Let L be the left shift on $\ell_2(\mathbf{N})$, and 0 the zero operator on $\ell_2(\mathbf{N})$ or $\ell_2(D_0)$ depending on context. Let X be the Banach space defined by $X = \ell_2(\mathbf{N}) \oplus \ell_2(D_0)$, and define the two operators S and T on X by $S = 0 \oplus M$ and $T = L \oplus 0$. By Theorem 2.12, T does not have condition (β) since L does not. On the other hand, M is normal and so has condition (β). Likewise, 0 has condition (β), and thus by Theorem 2.12 again, we infer that S does have condition (β).

A routine but tedious calculation shows that $\sigma(S) = \sigma_a(S) = D$, $\sigma_p(S) = D_0$, $\sigma_c(S) = D \setminus D_0$, and $\sigma_r(S) = \varnothing$. Here $\sigma_a(S)$, $\sigma_p(S)$, $\sigma_c(S)$, and $\sigma_r(S)$ are the approximate point, the point, the continuous, and the residual spectrum of S respectively. It is well known, [**14**, Prob. 6], that L, and therefore T, has the same spectrum and fine structure as S.

We turn now to the consideration of when a function of an operator has condition (β) if the operator itself does.

Let f be an analytic scalar-valued function defined on some neighborhood of the spectrum of an operator T. Then by $f(T)$ we mean

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1}\, d\lambda,$$

where Γ is the union of a finite number of rectifiable Jordan curves that do not intersect $\sigma(T)$. This, of course, is the celebrated Riesz–Dunford operational calculus; details can be found in [**8**, VII.3].

It is well known (see [**5**]) that if T is a bounded linear operator that has the SVEP (resp., is decomposable), and if f is an analytic scalar-valued function on some neighborhood of $\sigma(T)$, then $f(T)$ also has the SVEP (resp., is decomposable). Since condition (β) lies between the SVEP and decomposability in a certain sense (see the remarks following Example 2.23), it is natural to conjecture that an analogous result holds for condition (β). Indeed, the following is immediate.

THEOREM 2.28. *Let T be a bounded linear operator on a reflexive Banach space X. Let f be a scalar-valued analytic function defined on a neighborhood of $\sigma(T)$. Then if T and T^* have condition (β), so do $f(T)$ and $f(T^*)$.*

Proof. Since T and T^* have condition (β), they are decomposable by Lange's Theorem. Thus $f(T)$ and $f(T^*)$ are decomposable and therefore have condition (β).

Although the question of whether $f(T)$ has condition (β) when T does remains open in the general case, we do have the following somewhat more specialized result.

THEOREM 2.29. *Let T be a bounded linear operator on a Banach space X. Suppose that the spectrum of T is contained in the half-plane $\theta_0 < \arg \lambda < \theta_0 + \pi$. Then if T has condition (β), so does T^2.*

Proof. Let $\sqrt{\cdot}$ be the branch of the square root function having its branch cut at $\theta = \theta_0$. Suppose that $f_n : V \to X$ $(n = 1, 2, \ldots)$, are X-valued analytic functions. We may suppose that V lies in the same open half-plane as $\sigma(T)$.

Now if $(\lambda I - T^2)f_n(\lambda) \to 0$ uniformly on compact subsets of V, then we have $(\sqrt{\lambda}I - T)(\sqrt{\lambda}I + T)f_n(\lambda) \to 0$ uniformly on compact subsets of V also. Define $g_n(\lambda) := (\lambda I + T)f_n(\lambda^2)$ so that $g_n(\sqrt{\lambda})$ is analytic on V. Thus $(\sqrt{\lambda}I - T)g_n(\sqrt{\lambda}) \to 0$ uniformly on compact subsets of V, and since T has condition (β) by hypothesis, we infer that $g_n(\sqrt{\lambda}) \to 0$ uniformly on compact subsets of V. Therefore $(-\sqrt{\lambda}I - T)f_n(\lambda) = -g_n(\sqrt{\lambda}) \to 0$ uniformly on compact subsets of V. Now since $-\sqrt{\lambda}$ is in the resolvent set of T, we see that $(-\sqrt{\lambda}I - T)^{-1}$ is bounded so that $f_n(\lambda) = (-\sqrt{\lambda}I - T)^{-1}(-\sqrt{\lambda}I - T)f_n(\lambda) \to 0$ uniformly on compact subsets of V, proving that T^2 has condition (β).

COROLLARY 2.30. *Let T be a bounded linear operator on a Banach space X, and suppose that for some positive integer m, the spectrum of T^{2^m} does not surround the origin. Then if T has condition (β), so does T^{2^m}.*

REFERENCES

1. E. Albrecht, On decomposable operators, *Integral Equations and Operator Theory* **2** (1977), 1–10.

2. C. Apostol, Restrictions and quotients of decomposable operators in a Banach space, *Rev. Roumaine Math. Pures Appl.* **13** (1968), 147–150.

3. C. Apostol and R. G. Bartle, The sum and product of decomposable operators, preprint.

4. E. Bishop, A duality theorem for arbitrary operators, *Pacific J. Math.* **9** (1959), 379–397.

5. I. Colojoara and C. Foias, *Theory of generalized spectral operators*, (Gordon and Breach, New York, 1968).

6. H. R. Dowson, *Spectral theory of linear operators*, (Academic Press, 1978).

7. N. Dunford, Spectral operators, *Pacific J. Math.* **4** (1954), 321–354.

8. N. Dunford and J. T. Schwartz, *Linear operators*, (Wiley-Interscience, 1958).

9. I. Erdelyi and R. Lange, Operators with spectral decomposition properties, *J. Math. Anal. Appl.* **66** (1978), 1–19.

10. J. K. Finch, The single-valued extension property on a Banach space, *Pacific J. Math.* **58** (1975), 61–69.

11. C. Foiaş, On the maximal spectral spaces of a decomposable operator, *Rev. Roumaine Math. Pures Appl.* **15** (1970), 1599–1606.

12. S. Frunză, A complement to the duality theorem for decomposable operators, preprint.

13. S. Frunză, Spectral decomposition and duality, *Illinois J. Math.* **20** (1976), 314–321.

14. P. R. Halmos, *A Hilbert space problem book*, (Van Nostrand, 1967).

15. D. Herrero, Indecomposable compact perturbations of the bilateral shift, *Proc. Amer. Math. Soc.* **62** (1977), 254–258.

16. R. Lange, A purely analytic criterion for a decomposable operator, *Glasgow Math. J.* **21** (1980), 69–70.

17. R. Lange, On generalization of decomposability, *Glasgow Math. J.* **22** (1981), 77–81.

18. R. Lange, Strongly analytic subspaces, in *Operator theory and functional analysis*, edited by I. Erdelyi, Research Notes in Mathematics #38, (Pitman, San Francisco, 1979).

19. B. Nagy, Operators with the spectral decomposition property are decomposable, preprint.

20. M. Radjabalipour, Equivalence of decomposable and 2-decomposable operators, *Pacific J. Math.* **77** (1978), 243–247.

21. J. C. Snader, On T-strongly decomposable operators, to appear.

22. J. C. Snader, Strongly analytic subspaces and strongly decomposable operators, *Pacific J. Math.*, to appear.

23. F.-H. Vasilescu, On asymptotic behavior of operators, *Rev. Roumaine Math. Pures Appl.* **12** (1967), 353–358.

Department of Mathematics
University of South Florida
Tampa, Florida 33620
U.S.A.