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ABSOLUTE RIESZ SUMMABILITY OF A FOURIER RELATED SERIES, II

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This paper is an endeavour to improve upon the work begun in an earlier paper with the same title. We prove a general theorem on the summability $|R, \exp((\log w)^{\beta+1}), \gamma|$ of the series $\sum \{s_n(x)-s\}/n$, where $\{s_n(x)\}$ is the sequence of partial sums at a point x of the Fourier series of a Lebesgue integrable 2π-periodic function and s is a suitable constant. While the theorem improves upon the main result contained in the previous paper, corollaries to it include recent results due to Chandra and Yadava.

1. Definitions and notation

Let $e(w) = \exp((\log w)^{\beta+1})$, $\beta \geq 0$. A series $\sum u_n$ is said to be summable $|R, e(w), \gamma|$, $\gamma > 0$, and we write $\sum u_n \in |R, e(w), \gamma|$, if

$$\int_A^\infty e'(w)e^{-\gamma-1}(w) \left|\sum_{n<w} \{e(w)-e(n)\}^{\gamma-1}e(n)u_n\right| dw < \infty ,$$

where A is some constant.

Let $f \in L(-\pi, \pi)$ and be 2π-periodic and let

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$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) \equiv \sum_{0}^{\infty} A_n(t) .$$

Let the numbers x and s be fixed. We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\} - s ,$$

$$\chi(t) = \left(\log(k/t)\right)^{-1} \int_t^{\pi} \phi(u)(2 \sin \tfrac{1}{2}u)^{-1} du ,$$

$$G(n, t) = \int_t^{\pi} \left(\log(k/u)\right)^{b+1} \left(\log \log(k/u)\right)^{-\sigma} \frac{d}{du} \sin(n+\tfrac{1}{2})u du ,$$

$$Q(n, \alpha, a, c) = \{e(w)-e(n)\}^{\gamma-1} e(n) n^{\alpha-1} (\log n)^a (\log \log n)^c , \quad n < w ,$$

m will denote the integer determined by $m < w \le (m+1)$. Unless otherwise specified we use '$\sum$' to denote '$\sum_{n=3}^{\infty}$' and also write '$\sum_{n<w}$' to denote '$\sum_{n=3}^{m}$' . $K, K_1, K_2, \ldots$ denote absolute constants possibly different at different occurrences, and k denotes a suitable constant greater than or equal to $\pi \exp\left(e^2\right)$.

2. Theorem and remarks

2.1. We establish the following theorem.

THEOREM. *Let* $\beta, \gamma, \delta, \eta, \rho$ *and* σ *be real numbers such that* $\beta > 0$, $\gamma > 0$, $\eta \ge 1 + \delta$ *and* $\sigma \ge 1 + \rho$. *If*

$$\left(\log(k/t)\right)^{\eta} \left(\log \log(k/t)\right)^{\sigma} \chi(t) \in BV(0, \pi)$$

and

$$\left(\log(k/t)\right)^{\eta-1} \left(\log \log(k/t)\right)^{\sigma} t^{-1} \chi(t) \in L(0, \pi) ,$$

then

$$\sum \frac{s_n(x)-s}{n} (\log n)^{\delta} (\log \log n)^{\rho} \in |R, e(w), \gamma| .$$

2.2. REMARK 1. We note that the hypotheses on the function ϕ are independent of β and γ . Therefore in view of the consistency theorems

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for Riesz means (for the 'first theorem of consistency' refer to [2] and for a 'second theorem of consistency' refer to [4]), to obtain the best results we may choose $\gamma > 0$ as small as we please and similarly β may be taken any positive number however large.

REMARK 2. The case $\rho = 0$ and $\sigma = 1$ of the theorem (Corollary 1) gives an improvement on a previous result (see [3, Theorem 1]). Corollary 1 also extends a recent result due to Chandra and Yadava [1, Theorem 1]. A second corollary (Corollary 2) gives another result of Chandra and Yadava [1, Theorem 2].

3. Lemmas

We shall need the following lemmas for a proof of our theorem. These results are given in [3]. Lemmas 2, 3, 4 and 5 are given there for $c = 0$ and $\sigma = 0$. The modification in the proofs for other values of these parameters is rather routine.

LEMMA 1. *Let* b *and* η *be real numbers such that* $b + \eta > 0$ *and let* F *be a function defined over* $(0, \pi)$. *Then the following conditions*

$$\text{(i)} \quad F(t)\big(\log(k/t)\big)^{\eta} \in BV(0, \pi) ,$$

$$\text{(ii)} \quad F(t)\big(\log(k/t)\big)^{\eta-1} t^{-1} \in L(0, \pi) ,$$

are equivalent to the conditions

$$\text{(iii)} \quad \lim_{t\to 0+} F(t)\big(\log(k/t)\big)^{-b} = 0 , \text{ and}$$

$$\text{(iv)} \quad \int_0^{\pi} \big(\log(k/t)\big)^{b+\eta} \left| d\left\{F(t)\big(\log(k/t)\big)^{-b}\right\}\right| < \infty .$$

LEMMA 2. *Let* σ *and* b *be real numbers and* $b \geq 0$. *Then for* $0 < t < \pi$, *as* $n \to \infty$,

$$G(n, t) = O\Big((\log n)^{b}(\log\log n)^{-\sigma}\Big) + O\Big(nt\big(\log(k/t)\big)^{b+1}\big(\log\log(k/t)\big)^{-\sigma}\Big) .$$

LEMMA 3. *Let* $\beta > 0$, $0 < \gamma < 1$, $\alpha \geq 0$ *and* a *and* c *be real numbers. Then, as* $w \to \infty$,

$$\sum_{n<w} Q(n, \alpha, a, c) = O\Big(e^{\gamma}(w) w^{\alpha}(\log w)^{a-\beta}(\log\log w)^{c}\Big) + Q(m, \alpha, a, c) .$$

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LEMMA 4. *Let* γ *and* β *be positive and* δ *and* c *be any real numbers. Then the alternating series*

$$\sum (-1)^n n^{-1}(\log n)^{\delta}(\log\log n)^{c} \in |R, e(\omega), \gamma| .$$

LEMMA 5. *Let* $\beta > 0$, $0 < \gamma < 1$, $\alpha \ge 0$, δ *and* c *be real numbers,* $0 < t \le \pi$, $\omega \ge (2k/t)$ *and* θ *a constant independent of* n . *Then, as* $\omega \to \infty$,

$$\left|\sum_{n<\omega} Q(n, \alpha, \delta, c)\sin(nt+\theta)\right|$$

$$= O\left(t^{-\gamma}\omega^{\alpha-\gamma}(\log \omega)^{\delta+\beta(\gamma-1)}(\log\log \omega)^{c}e^{\gamma}(\omega)\right) + Q(m, \alpha, \delta, c) .$$

4. Proof of the theorem

In view of the 'first theorem of consistency' for Riesz means, it is sufficient to consider the case $0 < \gamma < 1$. Let $b \ge 0$ and be such that $b + \delta + 1 > 0$ and let us write $\chi^*(t) = \chi(t)\left(\log(k/t)\right)^{-b}\left(\log\log(k/t)\right)^{\sigma}$. Then using the Dirichlet integral and Lemma 1 we get

$$\frac{\pi}{2}\left\{s_n(x)-s\right\} = \int_0^{\pi} \frac{\sin(n+\frac{1}{2})u}{2\sin\frac{1}{2}u}\,\phi(u)du$$

$$= \left[-\sin(n+\tfrac{1}{2})t \int_t^{\pi} \frac{\phi(u)}{2\sin\frac{1}{2}u}\,du\right]_0^{\pi} + \int_0^{\pi} \chi(t)\left(\log(k/t)\right)\left(\sin(n+\tfrac{1}{2})t\right)'dt$$

$$= \left[-\chi^*(t)G(n, t)\right]_0^{\pi} + \int_0^{\pi} G(n, t)d\chi^*(t)$$

$$= \int_0^{\pi} G(n, t)d\chi^*(t) .$$

Therefore

$$\sum \frac{s_n(x)-s}{n}(\log n)^{\delta}(\log\log n)^{\rho} \in |R, e(\omega), \gamma|$$

if

$$\int_{e^2}^{\infty} e'(\omega)e^{-\gamma-1}(\omega)\left|\sum_{n<\omega} Q(n, 0, \delta, \rho)\int_0^{\pi} G(n, t)d\chi^*(t)\right|d\omega < \infty .$$

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Since, by Lemma 1,

$$\int_0^{\pi} \left(\log(k/t)\right)^{b+\eta} |d\chi^*(t)| < \infty ,$$

it is sufficient to show that, for $0 < t \le \pi$,

$$(1) \qquad I(t) = \int_{e^2}^{\pi} e'(\omega)e^{-\gamma-1}(\omega) \left| \sum_{n<\omega} Q(n,\ 0,\ \delta,\ \rho)G(n,\ t) \right| d\omega$$

$$= O\left(\left(\log(k/t)\right)^{b+\eta}\right) .$$

Let $\tau = 2(k/t)\left(\log(k/t)\right)^{\beta}$ and let

$$(2) \qquad I(t) = \int_{e^2}^{k/t} + \int_{k/t}^{\tau} + \int_{\tau}^{\infty} = I_1 + I_2 + I_3 \text{ , say.}$$

Write $L(t)$ for $\left(\log(k/t)^{b+1}\right)\left(\log \log(k/t)\right)^{-\sigma}$. Using Lemma 2 and Lemma 3 we obtain that

$$(3) \quad I_1 \le K_1 \int_{e^2}^{k/t} (\log \omega)^{\delta+b} (\log \log \omega)^{\rho-\sigma} \omega^{-1} d\omega$$

$$+ K_2 \int_{e^2}^{k/t} e'(\omega)e^{-\gamma-1}(\omega)Q(m,\ 0,\ \delta+b,\ \rho-\sigma)d\omega$$

$$+ K_3 tL(t) \int_{e^2}^{k/t} (\log \omega)^{\delta} (\log \log \omega)^{\rho} d\omega$$

$$+ K_4 tL(t) \int_{e^2}^{k/t} e'(\omega)e^{-\gamma-1}(\omega)Q(m,\ 1,\ \delta,\ \rho)d\omega$$

$$= O\left(\left(\log(k/t)\right)^{\delta}\left(\log \log(k/t)\right)^{\rho} L(t)\right)$$

$$+ K_1 \int_{e^2}^{k/t} e'(\omega)e^{-\gamma-1}(\omega)Q(m,\ 0,\ \delta+b,\ \rho-\sigma)d\omega$$

$$+ K_2 tL(t) \int_{e^2}^{k/t} e'(\omega)e^{-\gamma-1}(\omega)Q(m,\ 1,\ \delta,\ \rho)d\omega \text{ , for } 0 < t \le \pi .$$

Note that for $0 < \gamma < 1$ and for α, a, c and p and q real numbers such that $q > p \ge 3$,

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$$(4)\quad \int_p^q e'(w)e^{-\gamma-1}(w)Q(m,\ \alpha,\ a,\ c)dw$$

$$\le \sum_{m=[p]}^{[q]} \int_m^{m+1} e'(w)e^{-\gamma-1}(w)\{e(w)-e(m)\}^{\gamma-1}e(m)m^{\alpha-1}(\log m)^a(\log\log m)^c dw$$

$$\le K \sum_{[p]}^{[q]} m^{\alpha-1}(\log m)^a(\log\log m)^c\{1 - e(m)/e(m+1)\}^{\gamma}$$

$$\le K \sum_{[p]}^{[q]} m^{\alpha-\gamma-1}(\log m)^{a+\beta\gamma}(\log\log m)^c\ ,\ \text{by the mean value theorem.}$$

Therefore from (3) and (4) we get that, for $0 < t \le \pi$,

$$(5)\quad I_1 = O\left(\left(\log(k/t)\right)^{b+\delta+1}\left(\log\log(k/t)\right)^{\rho-\sigma}\right) + O\left(t^{\gamma}\left(\log(k/t)\right)^{b+\delta+1+\beta\gamma}\left(\log\log(k/t)\right)^{\rho-\sigma}\right) + K_3$$

$$= O\left(\left(\log(k/t)\right)^{b+\eta}\right)\ .$$

For I_2 , we first note that by the second mean value theorem

$$(6)\quad |G(n,\ t)| = \left|L(t)\left\{\sin(n+\tfrac{1}{2})t_1-\sin(n+\tfrac{1}{2})t\right\}\right|,\quad \text{for some } t_1 : 0 < t < t_1 < \pi,$$

$$\le 2L(t),\ \text{for}\ 0 < t \le \pi\ .$$

Therefore, using (6) and Lemma 3 we get

$$(7)\quad I_2(t) \le K_1L(t) \int_{k/t}^{\tau} (\log w)^{\delta}(\log\log w)^{\rho}w^{-1}dw + K_2L(t) \int_{k/t}^{\tau} e'(w)e^{-\gamma-1}(w)Q(m,\ 0,\ \delta,\ \rho)dw$$

$$\le K_1\left(\log(k/t)\right)^{b+\delta+1}\left(\log\log(k/t)\right)^{\rho-\sigma} \int_{k/t}^{\tau} w^{-1}dw + K_2L(t) \sum_{[k/t]}^{[\tau]} m^{-\gamma-1}(\log m)^{\delta+\beta\gamma}(\log\log m)^{\rho}\ ,\ \text{by (4)}$$

$$= O\left(\left(\log(k/t)\right)^{b+\eta}\right)\ ,\ \text{for}\ 0 < t \le \pi\ .$$

Next we note that

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$$(8)\qquad G(n,\ t) = [L(u)\sin(n+\tfrac{1}{2})u]_t^{\pi} - \int_t^{\pi} \sin(n+\tfrac{1}{2})u\,dL(u)$$

$$= L(\pi)(-1)^n - L(t)\sin(n+\tfrac{1}{2})t - \int_t^{\pi} \sin(n+\tfrac{1}{2})u\,dL(u)$$

and that for $r \geq 0$ and $b \geq 0$,

$$(9)\qquad \int_t^{\pi} u^{-r}dL(u) = O\left(t^{-r}L(t)\right) .$$

Therefore, after (8), (9) and (6), in view of Lemma 4, Lemma 5 and the result at (4) we obtain that

$$(10)\quad I_3 \leq K_1 \int_{e^2}^{\infty} e'(\omega)e^{-\gamma-1}(\omega)\left|\sum_{n<\omega} Q(n,\ 0,\ \delta,\ \rho)(-1)^n\right|d\omega$$

$$+ \left\{K_2t^{-\gamma}L(t)+K_3\left|\int_t^{\pi} u^{-\gamma}dL(u)\right|\right\} \int_{\tau}^{\infty} \omega^{-\gamma-1}(\log\ \omega)^{\delta+\beta\gamma}(\log\ \log\ \omega)^{\rho}d\omega$$

$$+ K_4L(t) \int_{\tau}^{\infty} e'(\omega)e^{-\gamma-1}(\omega)Q(m,\ 0,\ \delta,\ \rho)d\omega$$

$$\leq K_1 + K_2t^{-\gamma}L(t)\tau^{-\gamma}(\log\ \tau)^{\delta+\beta\gamma}(\log\ \log\ \tau)^{\rho}$$

$$+ K_3L(t) \sum_{[\tau]}^{\infty} m^{-\gamma-1}(\log\ m)^{\delta+\beta\gamma}(\log\ \log\ m)^{\rho}$$

$$= K_1 + O\left(\left(\log(k/t)\right)^{b+1+\delta}\left(\log\ \log(k/t)\right)^{\rho-\sigma}\right)$$

$$+ K_2\tau^{-\gamma}\left(\log(k/t)\right)^{b+1+\delta+\beta\gamma}\left(\log\ \log(k/t)\right)^{\rho-\sigma}$$

$$= O\left(\left(\log(k/t)\right)^{b+\eta}\right),\ \text{for}\quad 0 < t \leq \pi\ ,$$

and this completes the proof of the theorem.

5. Corollaries

We obtain the following results as special cases of our theorem.

COROLLARY 1. *Let* β, γ *and* δ *be real numbers such that* $\beta > 0$ *and* $\gamma > 0$. *If*

$$\left(\log(k/t)\right)^{\delta+1}\left(\log\ \log(k/t)\right)\chi(t) \in BV(0,\ \pi)$$

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and

$$\left(\log(k/t)\right)^{\delta}\left(\log\log(k/t)\right)t^{-1}\chi(t) \in L(0, \pi)$$

then

$$\sum \frac{s_n(x)-s}{n}(\log n)^{\delta} \in |R, e(w), \gamma| .$$

This corollary provides an improvement on a previous result [3, Theorem 1] and it also includes a theorem due to Chandra and Yadava [1, Theorem 1] - their result corresponds to the case $\delta = 1$.

The case $\delta = 0$ of Corollary 1 contains the following:

COROLLARY 2. *Let* $\beta > 0$ *and* $\gamma > 0$. *If*

$$\chi(0+) = 0 \quad \textit{and} \quad \int_0^{\pi} \left(\log(k/t)\right)\left(\log\log(k/t)\right)|d\chi(t)| < \infty$$

then

$$\sum \frac{s_n(x)-s}{n} \in |R, e(w), \gamma| .$$

Proof. Note that as $\chi(0+) = 0$,

$$\int_0^{\pi} |\left(\log(k/t)\log\log(k/t)\right)'\chi(t)|dt$$

$$= \int_0^{\pi} \left|\left(\log(k/t)\log\log(k/t)\right)' \int_0^t d\chi(u)\right|dt$$

$$\le \int_0^{\pi}\int_u^{\pi} |\left(\log(k/t)\log\log(k/t)\right)'|dt|d\chi(u)|$$

$$\le K\int_0^{\pi} \log(k/u)\log\log(k/u)|d\chi(u)| ,$$

and therefore

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$$(11) \quad \int_0^\pi |d\{\log(k/t)\log\log(k/t)\chi(t)\}|$$

$$\leq \int_0^\pi \left|\left(\log(k/t)\log\log(k/t)\right)'\chi(t)\right|dt + \int_0^\pi \log(k/t)\log\log(k/t)|d\chi(t)|$$

$$\leq K\int_0^\pi \log(k/t)\log\log(k/t)|d\chi(t)| \ ,$$

and then

$$(12) \quad \int_0^\pi t^{-1}\log\log(k/t)|\chi(t)|dt$$

$$\leq \int_0^\pi t^{-1}\left(\log\log(k/t)+1\right)|\chi(t)|dt$$

$$\leq \int_0^\pi |d\{\log(k/t)\log\log(k/t)\chi(t)\}| + \int_0^\pi \log(k/t)\log\log(k/t)|d\chi(t)|$$

$$\leq K\int_0^\pi \log(k/t)\log\log(k/t)|d\chi(t)| \ .$$

Thus from (11) and (12) we see that the hypotheses of Corollary 2 imply those of Corollary 1 in the case $\delta = 0$. Hence Corollary 2 follows from Corollary 1.

Corollary 2 is due to Chandra and Yadava ([*1*], Theorem 2).

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