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l'-ISOLATED MAPS AND LOCALIZATIONS

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0. Introduction. Let P be the set of primes, $l \subseteq P$ a subset and $l' = P - l$. Recall that an H_0-space is a space the rational cohomology of which is a free algebra.

Cassidy and Hilton defined and investigated l'-isolated homomorphisms between locally nilpotent groups. Zabrodsky [**8**] showed that if X and Y are simply connected H_0-spaces either with a finite number of homotopy groups or with a finite number of homology groups, then every rational equivalence $f : X \to Y$ can be decomposed into an l-equivalence and an l'-equivalence.

In this paper we define and investigate l'-isolated maps between pointed spaces, which are of the homotopy type of path-connected nilpotent CW-complexes. Our definition of an l'-isolated map is analogous to the definition of an l'-isolated homomorphism. As every homomorphism can be decomposed into an l-isomorphism and an l'-isolated homomorphism, every map can be decomposed into an l-equivalence and an l'-isolated map. This decomposition is unique, hence in case that X and Y satisfy the conditions of the first paragraph and $f : X \to Y$ is a rational equivalence, it coincides with Zabrodsky's decomposition. The construction of the decomposition is applied to study homotopy pull back diagrams and to study spaces and maps by means of their localizations.

Throughout this paper, commutative, pull back and pushout mean homotopy commutative, homotopy pull back and homotopy push out. Aside from Proposition 1.9 pullback should be understood as pullback in the category of path-connected spaces.

Among others we obtain the following propositions:

0.1 PROPOSITION (Theorem 3.5). *Let $f : X \to Y_1 \times Y_2$ be a rational equivalence. Suppose there exist subsets of the primes l_1 and l_2 satisfying:*

$$l_1 \cup l_2 = P,\ l_1 \cap l_2 = \emptyset,\ X_{l_i} \approx Z_i \times W_i \quad \textit{and}$$
$$f_{l_i} = g_i \times k_i : Z_i \times W_i \to (Y_1)_{l_i} \times (Y_2)_{l_i} \quad (i = 1, 2).$$

Then:

(a) *There exist spaces X_1 and X_2 so that $X \approx X_1 \times X_2$.*

(b) *There exist a homotopy equivalence $\epsilon : X \to X_1 \times X_2$ and maps*

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$g: X_1 \to Y_1$ and $k: X_2 \to Y_2$, so that

$$f \sim (g \times k)\epsilon: X \xrightarrow{\epsilon} X_1 \times X_2 \xrightarrow{g \times k} Y_1 \times Y_2.$$

0.2 PROPOSITION (Theorem 3.13). *Let X, μ and Y, ν be simply connected H-spaces and let $f: X \to Y$ be a map. Given subsets of the primes l_1, l_2, $l_1 \cup l_2 = P$ and rationalizations $\varphi_i: X \to X_0$, $\psi_i: Y \to Y_0$ $(i = 1, 2)$. If f_{l_i} $(i = 1, 2)$ is the localization of f at l_i for which the diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\varphi_i'"'] \arrow[dd, bend right=60, "\varphi_i"'] & Y \arrow[d, "\psi_i'"] \arrow[dd, bend left=60, "\psi_i"] \\
X_{l_i} \arrow[r, "f_{l_i}"] \arrow[d, "\varphi_i''"'] & Y_{l_i} \arrow[d, "\psi_i''"] \\
X_0 \arrow[r, "f_0"] & Y_0
\end{tikzcd}$$

commutes, then f is an H-map if and only if f_{l_1} and f_{l_2} are H-maps.

The paper is organized as follows: The first section deals with simple properties of l'-isolated maps. These properties are used in Section 2 to study pull backs, and in Section 3 to study properties of spaces and maps, especially of H_0-spaces and H-maps, by means of their localizations.

1. l'-isolated maps.

1.1 *Definition.* Let X, Y be nilpotent spaces, $f: X \to Y$ a map and $(\ )_l$- the l-localization operation. We say that f is *l'-isolated* if the square

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X_l & \xrightarrow{f_l} & Y
\end{array}$$

is a pull back.

1.2 COROLLARY. *A rational equivalence is l'-isolated if and only if it is an l'-equivalence.*

Proof. f is a rational equivalence and l'-isolated implies that the homotopy groups of the homotopy fiber of f are finite groups of order prime to l', hence f is an l'-equivalence.

Conversely, f is an l'-equivalence implies that f_l is an l'-equivalence, hence the fact that $X \to X_l$ and $Y \to Y_l$ are l-equivalences implies that the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X_l & \xrightarrow{f_l} & Y_l \end{array}$$

is a pull back and f is l'-isolated.

1.3 COROLLARY. *If a map is an l-equivalence and l'-isolated then it is a homotopy equivalence.*

1.4 LEMMA. (a) *If f: $X \to Y$ and g: $Y \to Z$ are l'-isolated, then $g \circ f$ is l'-isolated.*

(b) *If f: $X \to Y$ and g: $Y \to Z$ are such that g is l'-isolated and $g \circ f$ is l'-isolated, then f is l'-isolated.*

Proof. (a) Since f and g are l'-isolated the two squares in the diagram

(1.4.1)
$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow & & \downarrow & & \downarrow \\ X_l & \xrightarrow{f_l} & Y_l & \xrightarrow{g_l} & Z_l \end{array}$$

are pull back squares, hence [**6**] the rectangle is a pull back and $g \circ f$ is l'-isolated.

(b) Since g and $g \circ f$ are l'-isolated the right and the big squares in diagram (1.4.1) are pull back squares hence [**6**] the left square is a pull back and f is l'-isolated.

1.5 PROPOSITION. *Suppose X and Y are nilpotent spaces and f: $X \to Y$ is a map. Given a set of primes l, there exist a space $X(l, f)$, unique up to homotopy type, and maps f': $X \to X(l,f)$, f'': $X(l,f) \to Y$ so that $f'' \circ f' \sim f$, f' is an l-equivalence and f'' is l'-isolated. Furthermore, given a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \end{array}$$

and a set of primes l, there exists a map $g(l, f, \bar{f})$, so that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "g"] & X(l,f) \arrow[r, "f''"] \arrow[d, "{g(l,f,\bar{f})}"] & Y \arrow[d, "h"] \\ \bar{X} \arrow[r, "\bar{f}'"] & \bar{X}(l,\bar{f}) \arrow[r, "\bar{f}''"] & \bar{Y} \end{tikzcd}$$

If X, Y are of finite type, $f, \bar{f}$ are rational equivalences, and either X, Y have a finite number of homology groups or $\bar{X}$, $\bar{Y}$ have a finite number of homotopy groups, then $g(l, f, \bar{f})$ is unique up to homotopy.

Proof. $X(l, f)$, f' and f'' are constructed as follows: $X(l, f)$ is the pull back of

$$Y \rightarrow Y_l \xleftarrow{f_l} X_l,$$

f'' is the projection $f'': X(l, f) \rightarrow Y$ and $f': X \rightarrow X(l, f)$ is an l-equivalence that completes the following diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f'"] \arrow[ddr] & & Y \arrow[dd] \\ & X(l,f) \arrow[ur, "f''"] \arrow[d] & \\ & X_l \arrow[r, "f_l"] & Y_l \end{tikzcd}$$

The uniqueness of homotopy type of the component of the base point will follow from the second part of the proposition.

The existence of $g(l, f, \bar{f})$ follows from the following

1.5.1 LEMMA. *Given a pull back diagram*

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "f_2"] & X_1 \arrow[d, "f_{10}"] \\ X_2 \arrow[r, "f_{20}"] & X_0 \end{tikzcd}$$

if f_{20} is l'-isolated then f_1 is l'-isolated.

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Proof of 1.5.1. Consider the following cube:

$$\begin{tikzcd}
& X \arrow[rr, "f_1"] \arrow[dl, "f_{1l}"'] \arrow[dd, "f_2"] & & X_1 \arrow[dl] \arrow[dd, "f_{10}"] \\
X_l \arrow[rr] \arrow[dd] & & X_{1l} \arrow[dd] & \\
& X_2 \arrow[rr, "f_{20}"] \arrow[dl] & & X_0 \arrow[dl] \\
X_{2l} \arrow[rr, "f_{20l}"] & & X_{0l} &
\end{tikzcd}$$

Since the back face and the lower face are pull backs, X is the pull back of

$$X_{2l} \xrightarrow{f_{20l}} X_{01} \longleftarrow X_0 \xleftarrow{f_{10}} X_1 \quad [\mathbf{6}].$$

Therefore the commutativity of the right and left faces and the fact that the front face is a pull back imply [**6**] that the upper face is a pull back and f_1 is l'-isolated.

1.5.2 LEMMA. *Given a commutative diagram*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{g} & Y_1 \\ \Big\downarrow{\scriptstyle f_0} & & \Big\downarrow{\scriptstyle f_1} \\ \tilde{X} & \xrightarrow{h} & X_1 \end{array}$$

where g is an l-equivalence and h is l'-isolated. There exists $\tilde{f}$: $Y_1 \rightarrow \tilde{X}$ so that $\tilde{f} \circ g \backsim f_0$ and $h \circ \tilde{f} \backsim f_1$.

Proof of 1.5.2. Construct the pull back $W_{f_1,h}$ of f_1 and h and complete:

$$\begin{tikzcd}
\tilde{Y} \arrow[dr, "\hat{f}'"'] \arrow[drr, "\hat{f}"] \arrow[drrr, bend left, "g"] \arrow[ddrr, bend right, "f_0"'] & & & \\
& \tilde{Y}(l, \tilde{f}) \arrow[r, "\hat{f}''"'] & W_{f_1,h} \arrow[r, "\tilde{g}"'] \arrow[d, "\tilde{f}_1"] & Y_1 \arrow[d, "f_1"] \\
& & \tilde{X} \arrow[r, "h"] & X_1
\end{tikzcd}$$

As $\tilde{g}$ is l'-isolated (Lemma 1.5.1) and g is an l-equivalence, $\tilde{g}\hat{f}''$ is an

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l-equivalence and l'-isolated (Lemma 1.4(a)), hence $\tilde{g}\hat{f}''$ is a homotopy equivalence (Corollary 1.3). Let μ be the homotopy inverse of $\tilde{g}\hat{f}''$ and let $\alpha = \hat{f}''\mu$. Obviously $\tilde{g}\alpha \sim \tilde{g}\hat{f}''\mu \sim 1$. Consider the map $\tilde{f} = \tilde{f}_1\alpha$; since

$$\tilde{f}g \sim \tilde{f}_1\alpha g \sim \tilde{f}_1\hat{f}''\mu g \sim \tilde{f}_1\hat{f}''\mu(\tilde{g}\hat{f}'')\hat{f}' \sim \tilde{f}_1\hat{f}''\hat{f}' \sim \tilde{f}_1\hat{f} \sim f_0$$

and

$$h\tilde{f} = h\tilde{f}_1\alpha \sim f_1\tilde{g}\alpha \sim f_1,$$

$\tilde{f}$ is the desired map.

Now, apply 1.5.2 for $\tilde{Y} = X$, $Y_1 = X(l, f)$, $\tilde{X} = \bar{X}(l, \bar{f})$, $X_1 = \bar{Y}$, $g = f'$, $h = \bar{f}''$, $f_0 = \bar{f}' \circ g$, $f_1 = h \circ f''$ to obtain $g(l, f, \bar{f})$. The uniqueness of the homotopy type of $X(l, f)$ follows from the fact that the map $\tilde{f}$ which completes the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & X(l, f) \arrow[r] \arrow[d, "\tilde{f}"] & Y \arrow[d, equal] \\
X \arrow[r] & \bar{X}(l, f) \arrow[r] & Y
\end{tikzcd}$$

is both an l-equivalence and l'-isolated (Lemma 1.4(b)) and therefore (Corollary 1.3) it is a homotopy equivalence.

Finally, suppose X, Y are of finite type, f, $\bar{f}$ are rational equivalences and either X, Y are finite dimensional or $\bar{X}$, $\bar{Y}$ have a finite number of homotopy groups. If

$$f_1, f_2 \colon X(l, f) \to \bar{X}(l, \bar{f})$$

close the diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "g"] & X(l, f) \arrow[r, "f''"] \arrow[d, "f_1, f_2"] & Y \arrow[d, "h"] \\
\bar{X} \arrow[r, "\bar{f}'"] & \bar{X}(l, \bar{f}) \arrow[r, "\bar{f}''"] & Y
\end{tikzcd}$$

then $\bar{f}''f_1 \sim \bar{f}''f_2$ and $f_1f' \sim f_2f'$, hence the fact that $\bar{f}''$ is an l'-equivalence and f' is an l-equivalence implies [**4**, 5.3] that $f_1 \sim f_2$ and $g(l, f, \bar{f})$ is unique.

1.6 *Remark*. Let X and Y be spaces of finite type and let $f \colon X \to Y$ be a map. Denote by F the component of the base point of the fibre of f. Since the fibre of $f'' \colon X(l, f) \to Y$ is F_l, $X(l, f)$ is of finite type if and only if f is a rational equivalence.

1.7 Corollary. *If X, Y are H-spaces and $f \colon X \to Y$ is an H-map, then $X(l, f)$, f' and f'' admit H-structures.*

Proof. By the uniqueness of the 1.5 decomposition one has

$$X \times X(l, f \times f) \approx X(l, f) \times X(l, f)$$

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and if X, μ Y, ν are H-spaces and f is an H-map one has a commutative diagram

$$\begin{tikzcd} X \times X \arrow[r, "f' \times f'"] \arrow[d, "\mu"] & X(l,f) \times X(l,f) \arrow[r, "f'' \times f''"] \arrow[d, "\mu(l, f \times f, f)"] & Y \times Y \arrow[d, "\nu"] \\ X \arrow[r, "f'"] & X(l,f) \arrow[r, "f''"] & Y \end{tikzcd}$$

Let

$$i_\epsilon \colon X(l,f) \to X(l,f) \times X(l,f) \quad (\epsilon = 1, 2)$$

be the injections. Consider the maps $\mu(l, f \times f, f) \circ i_\epsilon$ $(\epsilon = 1, 2)$; since $\alpha_\epsilon f' \sim f'$ and $f''\alpha_\epsilon \sim f''$, α_ϵ is an l-equivalence and l'-isolated, hence a homotopy equivalence. Let γ_ϵ be the homotopy inverse of α_ϵ; $\gamma_\epsilon f' \sim f'$ and $f''\gamma_\epsilon \sim f''$. Then one can replace $\mu(l, f \times f, f)$ by $\mu(l, f \times f, f) \circ (\gamma_1 \times \gamma_2)$ which is an H-structure for $X(l, f)$ and

$$\mu(l, f \times f, f) \circ (\gamma_1 \times \gamma_2) \circ (f' \times f') \sim \mu(l, f \times f, f) \circ (f' \times f') \sim f' \circ \mu$$

$$f'' \circ \mu(l, f \times f, f) \circ (\gamma_1 \times \gamma_2) \sim \nu \circ (f'' \times f'') \circ (\gamma_1 \times \gamma_2) \sim \nu \circ (f'' \times f'')$$

so both f' and f'' are H-maps.

1.8 *Notation.* For every space X denote by X_n the n-stage of the Postnikov system of X and by $\mathscr{L}X$ the set of all maps $f \colon I \to X$ for which $f(0)$ = the base point of X.

1.9 PROPOSITION. *Given a map $f \colon X \to Y$ and a pull back diagram*

$$\begin{tikzcd} \hat{X}(l,f) \arrow[r, "f''"] \arrow[d] & Y \arrow[d] \\ X_l \arrow[r, "f_l"] & Y_l \end{tikzcd}$$

If f is a principal fibration then f'' is a principal fibration.

Proof. Suppose $f \colon X \to Y$ is induced by $g \colon Y \to Z$, then f_l is induced by g_l. Consequently the diagram

$$\begin{tikzcd} \hat{X}(l,f) \arrow[r] \arrow[d, "f''"] & X_l \arrow[r] \arrow[d, "f_l"] & \mathscr{L}Z_l \arrow[d] \\ Y \arrow[r] & Y_l \arrow[r, "g_l"] & Z_l \end{tikzcd}$$

is a composite of pull backs and f'' is a principal fibration.

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1.10 PROPOSITION. *Let $f: X \to Y$ be l'-isolated and let $\pi_n X$ be l'-torsion free. Then:*

(a) *$\pi_n Y$ is l'-torsion free.*

(b) *$f_n: X_n \to Y_n$ is l'-isolated.*

(c) *$f_\#: \pi_n X \to \pi_n Y$ is l'-isolated.*

Proof. (a) Since the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X_l \arrow[r, "f_l"] & Y_l \end{tikzcd}$$

is a pull back and $\pi_n X$ is l'-torsion free we have an exact sequence:

(1.10.1) $\quad 0 \to \pi_n X \xrightarrow{\alpha_n} \pi_n X_l \oplus \pi_n Y \xrightarrow{\beta_n} \pi_n Y_l \dots$

Consequently the fact that $\pi_n Y_l$ is l'-torsion free implies that the l'-torsion of $\pi_n Y$ belongs to the image of α_n and $\pi_n Y$ is l'-torsion free.

(b) Let W be the pull back of

$$X_{nl} \xrightarrow{f_{nl}} Y_{nl} \longleftarrow Y_n$$

and let $g: X_n \to W$ make the diagram

$$\begin{tikzcd} X_n \arrow[dr, "g"] \arrow[drr, "f_{nl}"] \arrow[ddr, bend right] & & \\ & W \arrow[r] \arrow[d] & Y_n \arrow[d] \\ & X_{nl} \arrow[r, "f_{nl}"] & Y_{nl} \end{tikzcd}$$

commutative. Consider $g_\#: \pi_m X_n \to \pi_m W$. For $m > n$,

$$\pi_m W = \pi_m X_m = 0,$$

hence $g_\#$ is an isomorphism. For $m < n$, one has a commutative diagram

(1.10.2)
$$\begin{tikzcd} \pi_{m+1} Y_{nl} \arrow[r] \arrow[d, equal] & \pi_m X_n \arrow[r] \arrow[d, "g"] & \pi_m X_n \oplus \pi_m Y_n \arrow[r] \arrow[d, equal] & \pi_m Y_{nl} \arrow[d, equal] \\ \pi_{m+1} Y_{nl} \arrow[r] & \pi_m W \arrow[r] & \pi_m X_n \oplus \pi_m Y_n \arrow[r] & \pi_m Y_{nl} \end{tikzcd}$$

hence, by the five lemma, $g_\#$ is an isomorphism. For $m = n$,

$$\pi_{m+1} Y_{n\,l} = 0,$$

hence (1.10.1) together with (1.10.2) imply that also in this case $g_\#$ is an isomorphism and therefore $X_n \approx W$.

(c) This follows from the fact that

$$\begin{tikzcd}
\pi_n X \arrow[r, "f_\#"] \arrow[d] & \pi_n Y \arrow[d] \\
\pi_n X_l \arrow[r, "(f_\#)_l"] & \pi_n Y_l
\end{tikzcd}$$

is a pull back if and only if the sequence

$$0 \to \pi_n X \to \pi_n X_l \oplus \pi_n Y \to \pi_n Y_l$$

is exact.

2. Pull back diagrams. In this section we study the relations between pull back diagrams and l'-isolated maps.

2.1 PROPOSITION. *Given a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "f_2"] & X_1 \arrow[d, "f_{10}"] \\
X_2 \arrow[r, "f_{20}"] & X_0
\end{tikzcd}$$

(a) *If f_{10} and f_2 are l-equivalences, f_{20} and f_1 are l'-isolated, then the square is a pull back.*

(b) *If* (a) *is satisfied and all the maps are rational equivalences, then the square is, also, a push out.*

Proof. (a) Let W be the pull back of

$$X_2 \xrightarrow{f_{20}} X_0 \xleftarrow{f_{10}} X_1.$$

There exists g: $X \to W$ so that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "f_1"] \arrow[dd, "f_2"'] \arrow[dr, dashed, "g"] & & X_1 \arrow[dd, "f_{10}"] \\
& W \arrow[ur] \arrow[dl] & \\
X_2 \arrow[rr, "f_{20}"] & & X_0
\end{tikzcd}$$

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Since $W \to X_1$ is l'-isolated (Lemma 1.5.1) and $W \to X_2$ is an l-equivalence, g is both l'-isolated and an l-equivalence, hence (Corollary 1.3) a homotopy equivalence.

(b) This is similar to (a).

2.2 PROPOSITION. *If $f: X \to Y$ is a rational equivalence then X is the pull back of $X(l, f) \to Y \leftarrow X(l', f)$.*

Proof. By Corollary 1.2 $X(l, f) \to Y$ is an l'-equivalence and $X(l', f) \to Y$ is an l-equivalence. Hence the square

$$\begin{array}{ccc} X & \longrightarrow & X(l,f) \\ \downarrow & & \downarrow \\ X(l',f) & \longrightarrow & Y \end{array}$$

satisfies the conditions of Proposition 2.1(a) and therefore it is a pull back square.

2.3 PROPOSITION. *Given a pull back diagram*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_{10}} \\ X_2 & \xrightarrow{f_{20}} & X_0 \end{array} \tag{2.3.1}$$

then the two squares in the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f_1'} & X(l, f_1) & \xrightarrow{f_1''} & X_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2(l, f_1, f_{20})} & & \downarrow{\scriptstyle f_{10}} \\ X_2 & \xrightarrow{f_{20}'} & X_2(l, f_{20}) & \xrightarrow{f_{20}''} & X_0 \end{array} \tag{2.3.2}$$

are pull back squares.

Proof. Let W be the pull back of

$$X_2(l, f_{20}) \xrightarrow{f_{20}''} X_0 \xleftarrow{f_{10}} X_{10}.$$

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Consider the following diagram

(2.3.3)
$$\begin{tikzcd}
X \arrow[r, "f_1'"] \arrow[d, "f_2"'] \arrow[rrd, dashed, "hf_1'"] & X(l,f) \arrow[rr, "f_1''"] \arrow[d, "f_2(l,f_1,f_{20})"'] \arrow[rd, dashed, "h"] & & X_1 \arrow[d, "f_{10}"] \\
 & & W \arrow[ru] \arrow[ld] & \\
X_2 \arrow[r, "f_{20}'"] & X_2(l,f_{20}) \arrow[rr, "f_{20}''"] & & X_0
\end{tikzcd}$$

Let h: $X(l, f) \rightarrow W$ close the diagram. Since (2.3.1) is a pull back, it follows from [**6**] that X is the pull back of

$$X_2 \xrightarrow{f_{20}'} X_2(l, f_{20}) \longleftarrow W.$$

Consequently, the fact that f_{20}' is an l-equivalence, and f_{20}'' is l-isolated imply that fh_1' is an l-equivalence and $W \rightarrow X_1$ is l'-isolated, hence (Corollary 1.3) h is a homotopy equivalence and the result follows.

2.4 Proposition. *Suppose all the spaces and the maps in diagram* (2.3.1) *are H-spaces and H-maps. Then*:

(a) *There exist H-structures on $X(l, f_1)$ and $X_2(l, f_2)$ so that all the maps in diagram* (2.3.2), *except possibly for f_1', are H-maps.*

(b) *If f_{20} is a rational equivalence and f_{10} is an l-equivalence, then the H-structures on $X(l, f_1)$ and $X_2(l, f_2)$ can be chosen so that f_1' is, also, an H map.*

Proof. (a) This follows from the fact that $X(l, f_1)$ is homotopy equivalent to the pull back of

$$X_2(l, f_{20}) \xrightarrow{f_{20}''} X_0 \xleftarrow{f_{10}} X_1$$

(Proposition 2.3) and $X_2(l, f_{20})$ admits an H-structure so that f_{10}' and f_{20}'' are H-maps (Corollary 1.7).

(b) Consider diagram (2.3.3) with $X(l, f_1) = W$. Since $f_1'' \circ f_1'$ and $f_2(l, f_1, f_{20}) \circ f_1'$ are H-maps and

$$(\Omega f_{10})_* \oplus (\Omega f_{20}'')_*: [X \times X, \Omega X_1] \oplus [X \times X, \Omega X_2(l, f_{20})] \rightarrow [X \times X, \Omega X_0]$$

is an epimorphism, it follows from [**1**, Proposition 10.3] that f_1' is an H-map.

2.5 Proposition. *Given a push out diagram*

(2.5.1)
$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \arrow[d, "f_{10}"] \\
X_2 \arrow[r, "f_{20}"] & X_0
\end{tikzcd}$$

If f_1: $X \to X_1$ is a rational equivalence, then the two squares in the diagram

$$\begin{tikzcd} X \arrow[r, "f_1'"] \arrow[d, "f_2"] & X(l,f_1) \arrow[r, "f_1''"] \arrow[d, "f_2(l,f_1,f_{20})"] & X_1 \arrow[d, "f_{10}"] \\ X_2 \arrow[r, "f_{20}'"] & X_2(l,f_{20}) \arrow[r, "f_{20}''"] & X_0 \end{tikzcd}$$

are push out squares.

Proof. Let W be the push out of $X_2 \xleftarrow{f_2} X \xrightarrow{f_2'} X(l,\ f_1)$ and let H: $W \to X_2(l,f_{20})$ close the following diagram

$$\begin{tikzcd} X \arrow[rr, dash, "f_1'"] \arrow[dd, "f_2"] & & X(l,f_1) \arrow[rr, dash, "f_1''"] \arrow[dl] \arrow[dd] & & X_1 \arrow[dd, "f_{10}"] \\ & W \arrow[dr, dashed, "h"] \arrow[rrrd, "f_{20}''h"] & & & \\ X_2 \arrow[ur] \arrow[rr, "f_{20}'"] & & X_2(l,f_{20}) \arrow[rr, "f_{20}''"] & & X_0 \end{tikzcd}$$

Since (2.5.1) is a push out it follows from [**6**] that X_0 is the push out of

$$W \longleftarrow X(l,f_1) \xrightarrow{f_1''} X_1.$$

Consequently, the fact that f_1' is an l-equivalence and f_1'' is an l'-equivalence imply that $X_2 \to W$ is an l-equivalence and $f_{20}''h$ is an l'-equivalence, hence h is a homotopy equivalence and the result follows.

2.6 PROPOSITION. *Let*

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "f_2"] & X_1 \arrow[d, "f_{10}"] \\ X_2 \arrow[r, "f_{20}"] & X_0 \end{tikzcd}$$

be a pull back square and let W be the pull back of

$$X_2(l,f_{20}) \xrightarrow{f_{20}''} X_0 \xleftarrow{f_{10}''} X_1(l,f_{10}).$$

If $h = f_1(l,f_2,f_{10})$, then

$$W = X(l,f_2)(l,\ h) = X(l,f_{10}f_1).$$

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Further, if $g = f_2(l, f_1, f_{20})$, *then* $W = X(l, f_1)(l, g)$ *and all the squares in the diagram*

$$\begin{tikzcd}
X \arrow[r, "f_1'"] \arrow[d, "f_2'"] & X(l, f_1) \arrow[r, "f_1''"] \arrow[d, "g'"] & X_1 \arrow[d, "f_{10}'"] \\
X(l, f_2) \arrow[r, "h'"] \arrow[d, "f_2''"] & W \arrow[r, "h''"] \arrow[d, "g''"] & X_1(l, f_{10}) \arrow[d, "f_{10}''"] \\
X_2 \arrow[r, "f_{20}'"] & X_2(l, f_{20}) \arrow[r, "f_{20}''"] & X_0
\end{tikzcd}$$

are pull back squares.

Proof. Since the lower right square is a pull back, h'' and g'' are l'-isolated and there exists h': $X(l, f_2) \to W$ so that $h''h' \sim h$ and $g''h' \sim f_{20}'f_2''$. Consequently the fact that the lower rectangle is a pull back (Proposition 2.3) implies [**6**] that the lower left square is a pull back. Hence h' is an l-equivalence and

$$W = X(l, f_2)(l, h) = X(l, f_{10}f_1)$$

(Proposition 1.5).

Let $g' = f_2'(l, f_1, h)$: $X(l, f_1) \to W$ and let $g = g''g'$. By Proposition 2.3 the right and left rectangles in the above diagram are pull backs. Consequently the fact that the two lower squares are pull backs implies [**6**] that the two upper squares are pull backs. Therefore g' is an l-equivalence and $W = X(l, f)(l, g)$.

2.7 Proposition. *Given a commutative diagram*

$$\begin{tikzcd}
& X \arrow[rr, "h_1"] \arrow[dl, "f"'] \arrow[dd, "h_2"] & & X_1 \arrow[dl, "f_1"] \arrow[dd, "h_{10}"] \\
Y \arrow[rr, "g_1"] \arrow[dd, "g_2"] & & Y_1 \arrow[dd, "g_{10}"] & \\
& X_2 \arrow[rr, "h_{20}"] \arrow[dl, "f_2"] & & X_0 \arrow[dl, "f_0"] \\
Y_2 \arrow[rr, "g_{20}"] & & Y_0 &
\end{tikzcd} \tag{2.7.1}$$

where:

(a) *X, Y are of finite type.*

(b) *f, f_0 are rational equivalences.*

(c) *Either X, Y are finite dimensional or X_0, Y_0 have a finite number of homotopy groups.*

Then the middle vertical square in the following diagram is commutative:

(2.7.2)
$$\begin{tikzcd}
 & & X \arrow[rrr, "h_1"] \arrow[dl, "f'"'] \arrow[ddd, "h_2"] & & & X_1 \arrow[dl] \arrow[ddd, "h_{10}"] \\
 & X(l,f) \arrow[rrr] \arrow[dl, "f''"'] \arrow[ddd] & & & X_1(l,f_1) \arrow[dl] \arrow[ddd] & \\
Y \arrow[rrr, "g_1"] \arrow[ddd, "g_2"'] & & & Y_1 \arrow[ddd, "g_{10}"] & & \\
 & & X_2 \arrow[rrr, "h_{20}"] \arrow[dl] & & & X_0 \arrow[dl, "f_0'"] \\
 & X_2(l,f_2) \arrow[rrr] \arrow[dl] & & & X_0(l,f_0) \arrow[dl, "f_0''"] & \\
Y_2 \arrow[rrr, "g_{20}"] & & & Y_0 & &
\end{tikzcd}$$

Proof. Since f' is an l-equivalence and f_0'' is an l'-equivalence, one obtains by chasing the diagram, that for every prime p, $h_{10}(l, f_1, f_0) \circ h_1(l, f, f_1)$ is mod-p homotopic to $h_{20}(l, f_2, f_0) \circ h_2(l, f, f_2)$ and the result follows from [**4**, 5.3].

2.8 PROPOSITION. *With the hypothesis of* 2.7, *if either*:

g_2: $Y \to Y_2$, g_{10}: $Y_1 \to Y_0$ *are l'-isolated and* h_1: $X \to X_1$, h_{20}: $X_2 \to X_0$ *are l-equivalences, or*

g_1: $Y \to Y_1$, g_{20}: $Y_2 \to Y_0$ *are l'-isolated and* h_2: $X \to X_2$, h_{10}: $X_1 \to X_0$ *are l-equivalences, then the middle vertical square is a pull back.*

Proof. The assumptions of the proposition imply that the middle vertical square is a commutative square, which satisfies the conditions of Proposition 2.1 and therefore it is a pull back square.

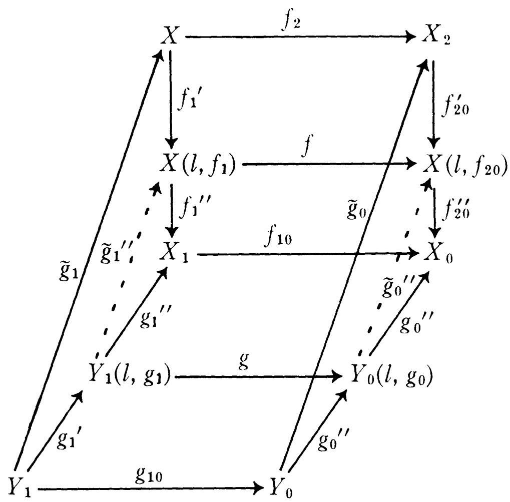
2.9 PROPOSITION. *If the base and cover in* (2.7.1) *are push outs and either the front and left faces or the left and back faces are pull backs, then the middle vertical square in* (2.7.2) *is a pull back.*

Proof. Suppose that the base and cover in (2.7.1) are push outs and that the left and front faces are pull backs, then the same is true for the front cube in (2.7.2). Thus, by Walker's Theorem 1.10 ([**7**]), the right and back faces of this front cube are pull backs; in particular, the middle vertical square is a pull back.

Similarly, suppose that the base and cover in (2.7.1) are push outs and that the left and back faces are pull backs, then the same is true for the back cube in (2.7.2). Thus by Mather's Theorem 18 ([**6**]), the right and front faces of this back cube are pull backs; in particular, the middle vertical square is a pull back.

The following two propositions deal with lifting problems:

2.10 PROPOSITION. *Suppose in the following diagram*



(a) *X is the pull back of* $X_1 \xrightarrow{f_{10}} X_0 \xleftarrow{f_{20}} X_2$.

(b) *gg_1' and f_{20} are rational equivalences.*

(c) $f_{10}g_1 \sim g_0 g_{10}$.

If all the spaces are CW-complexes of finite type, and either X_1 and Y_1 are finite dimensional or X_2 and X_0 have a finite number of homotopy groups, then the existence of a lifting, $\tilde{g}_0$: $Y_0 \to X_2$, of g_0 implies the existence of liftings

$$\tilde{g}_0'': Y_0(l, g_0) \to X(l, f_{20}),\ \tilde{g}_1: Y_1 \to X \quad \textit{and}$$
$$\tilde{g}_1'': Y_1(l, g_1) \to X(l, f_2)$$

of g_0'', g_1 and g_1'', respectively, so that the diagram commutes.

Proof. Denote $f = f_2(l, f_1, f_{20})$, $g = g_{10}(l, g_1, g_0)$. Since X is the pull back of

$$X_2 \xrightarrow{f_{20}} X_0 \xleftarrow{f_{10}} X_1$$

and $f_{20}\tilde{g}_0 g_{10} \sim f_{10}g_1$, there exists a lifting $\tilde{g}_1$ of g_1 which satisfies $f_2\tilde{g}_1 \sim \tilde{g}_0 g_{10}$. By Proposition 1.5 there exist liftings $\tilde{g}_0''$ and $\tilde{g}_1''$ of g_0 and g_1'', so that the left and right trapezoids and all the triangles are commutative. Hence (b) and the fact that

$$(f\tilde{g}_1'')g_1' \sim (\tilde{g}_0''g)g_1' \quad \text{and} \quad f_{20}''(f\tilde{g}_1'') \sim f_{20}''(\tilde{g}_0''g)$$

imply [**4**, 5.3] that $f\tilde{g}_1'' \sim \tilde{g}_0''g$ and the diagram commutes.

2.11 PROPOSITION. *Given a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "T"] & Y_0 \arrow[d, "S_0"] & Y \arrow[l, "g"'] \arrow[d, "S"] \\
X \arrow[r, "f"] & Y_0 & Y \arrow[l, "g"']
\end{tikzcd}$$

where:

(a) *X, Y and Y_0 are of finite type.*

(b) *f and g are rational equivalences.*

(c) *Either X and Y_0 are finite dimensional or Y and Y_0 have a finite number of homotopy groups.*

If either g is a principal fibration or Y, Y_0 are H-spaces and g is an H-map, then the existence of a lifting f': $X \rightarrow Y$, of f implies the existence of an integer m divisible only by primes dividing

$$\prod_{n=0}^{\infty} \exp H^n(x, \pi_n(\text{fiber } g))$$

and of liftings $\tilde{f}$: $X \rightarrow Y$ and $\tilde{f}''$: $X(l, f) \rightarrow Y(l, g)$ so that the following diagram commutes:

$$\begin{tikzcd}
& & Y \arrow[rrr, "S^m"] \arrow[d, "g'"] & & & Y \arrow[d, "g'"] \\
& & Y(l, g) \arrow[d, "g''"] & & & Y(l, g) \arrow[d, "g''"] \\
& & Y_0 \arrow[rrr, "S_0^m"] & & & Y_0 \\
& X(l, f) \arrow[ur, "f''"'] \arrow[uur, dotted, "\tilde{f}''"] & & & X(l, f) \arrow[ur, "f''"'] \arrow[uur, dotted, "\tilde{f}''"] & \\
X \arrow[rrr, "T^m"] \arrow[ur, "f'"'] \arrow[uuuurr, dotted, "\tilde{f}"] & & & X \arrow[ur, "f'"'] \arrow[uuuurr, dotted, "\tilde{f}"] & &
\end{tikzcd}$$

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Proof. By [**9**, 3.1] there exist an integer m divisible only by primes dividing

$$\prod_{n=0}^{\infty} \exp H^n(X, \pi_n(\text{fiber } g)),$$

and a lifting $\tilde{f}$: $X \to Y$ of f, so that $S^m\tilde{f} \sim \tilde{f}T^m$. Hence the considerations of the proof of Proposition 2.10 imply the result.

3. Localizations. In this section we use the decomposition of a map into an l-equivalence and an l'-isolated map, and the structure of the genus to study spaces and maps by means of their localizations.

3.1 LEMMA. *Let X, Y and W be nilpotent CW-complexes. Suppose Y is quasifinite and W is a connected H-space. Given a map f: $X \to Y$ satisfying: for every prime p, f_p^*: $[Y_p, W_p] \to [X_p, W_p]$ is onto, then f^* is onto.*

Proof. Let ϕ_n be the n-power map. For every map h: $X \to W$ and for every prime p there exists a map g_p': $Y_p \to W_p$ so that $g_p'f_p \sim h_p$. Since W is an H-space it follows from [**4**, 6.5] that there exists an integer n, $(n, p) = 1$, and a map g: $Y \to W$ so that $g_p \sim \phi_n g_p'$; as $g_pf_p \sim \phi_n h_p$ for all p, it follows from [**4**, 5.3] that $gf \sim \phi_n h$.

Suppose $n = p_1^{k_1} \cdot \ldots \cdot p_l^{k_l}$ where all the p_i are primes. The same considerations imply the existence of integers n_i, $(n_i, p_i) = 1$, and maps g_i: $Y \to W$ so that $g_if \sim \phi_{n_i}h$. Consequently if one defines k: $Y \to W$ by

$$k = g^a \cdot \prod_{i=1}^{l} g_i^{a_i}$$

where

$$an + \sum_{i=1}^{l} a_i n_i = 1$$

one obtains that $kf \sim h$ and f^* is onto.

3.2 PROPOSITION. *Let X and Y satisfy the conditions of* 3.1 *and let f: $X \to Y$ be a map. If for every prime p, Σf_p: $\Sigma X_p \to \Sigma Y_p$ has a left homotopy inverse, then Σf: $\Sigma X \to \Sigma Y$ has a left homotopy inverse.*

Proof. The result follows from Lemma 3.1 and from the fact that f^*: $[Y, W] \to [X, W]$ is onto for every H-space W if and only if Σf has a left homotopy inverse ([**8**], Corollary 1.1.4). The corollary follows from [**5**].

3.3 *Remark.* All spaces considered from now on, except in Proposition 3.13, are of the homotopy type of simply connected CW-complexes of finite type, which are either finite dimensional or have a finite number of non-zero homotopy groups.

3.4 PROPOSITION. *Let X, Y be H_0-spaces and let $l \subset P$ be a finite subset. Given localizations $X \to X_l$, $Y \to Y_l$ and a rational equivalence g_l: $X_l \to Y_l$, there exist a rational equivalence f: $X \to Y$ and a homotopy equivalence ϵ: $X_l \to X_l$ so that $f_l \sim g_l\epsilon$.*

Proof. Let X' be the pull back of $X_l \xrightarrow{g_l} Y_l \leftarrow Y$. Since X' is mod l-equivalent to X and l is finite there exists an l-equivalence g': $X \to X'$ [**10**]. Consequently it follows from the diagram

$$\begin{array}{ccccc} X & \xrightarrow{g'} & X' & \xrightarrow{g} & Y \\ \downarrow & & \downarrow & & \downarrow \\ X_l & \xrightarrow{g_l'} & X_l & \xrightarrow{g_l} & Y_l \end{array}$$

that $f = gg'$ and $\epsilon = g_l'$ are the desired maps.

3.5 THEOREM. *Let f: $X \to Y_1 \times Y_2$ be a rational equivalence. Suppose there exist subsets of the primes l_1 and l_2 satisfying:*

$$l_1 \cup l_2 = P,\ l_1 \cap l_2 = \phi,\ X_{l_i} \approx Z_i \times W_i \quad \textit{and}$$
$$f_{l_i} = g_i \times k_i\colon Z_i \times W_i \to (Y_1)_{l_i} \times (Y_2)_{l_i} \quad (i = 1, 2).$$

Then:

(a) *There exist spaces X_1 and X_2 so that $X \approx X_1 \times X_2$.*

(b) *There exist a homotopy equivalence ϵ: $X \to X_1 \times X_2$ and maps g: $X_1 \to Y_1$ and k: $X_2 \to Y_2$, so that*

$$f \sim (g \times k)\epsilon\colon X \xrightarrow{\epsilon} X_1 \times X_2 \xrightarrow{g \times k} Y_1 \times Y_2.$$

Proof. Let Z^i and W^i be the pull backs of

$$Z_i \xrightarrow{g_i} Y_{1_{l_i}} \leftarrow Y_1 \quad \text{and} \quad W_i \xrightarrow{k_i} Y_{2_i} \leftarrow Y_2 \quad (i = 1, 2),$$

respectively, and let f_i: $X \to Z^i \times W^i$ close the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "f_i"] \arrow[rr, bend left, "f"] \arrow[dr] & Z^i \times W^i \arrow[r, "g^i \times k^i"] \arrow[d] & Y_1 \times Y_2 \arrow[d] \\ & Z_i \times W_i \arrow[r, "g_i \times k_i"] & (Y_1)_{l_i} \times (Y_2)_{l_i} \end{tikzcd}$$

Obviously f_i is an l_i-equivalence and $g^i \times k^i$ is a $P - l_i$ equivalence. Consequently if $X_1 \times X_2$ is the pull back of

$$Z^1 \times W^1 \xrightarrow{g^1 \times k^1} Y_1 \times Y_2 \xleftarrow{g^2 \times k^2} Z^2 \times W^2$$

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the map $\epsilon: X \to X_1 \times X_2$ which closes the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "\epsilon"] \arrow[drr, "f_1"] \arrow[ddr, "f_2"'] & & \\
& X_1 \times X_2 \arrow[r] \arrow[d] & Z' \times W' \arrow[d, "g^1 \times k^1"] \\
& Z^2 \times W^2 \arrow[r, "g^2 \times k^2"] & Y_1 \times Y_2
\end{tikzcd}$$

is a homotopy equivalence [**4**, 5.3] and the result follows.

In case X is an H_0-space we obtain a stronger result:

3.6. THEOREM. *Let X be an H_0-space and let $l \subseteq P$ be a subset. Suppose there exist spaces Y_i, Z_i $(i = 1, 2)$ satisfying $X_l \approx Y_1 \times Y_2, X_{l'} \approx Z_1 \times Z_2$ $(Y_1)_0 \approx (Z_1)_0$, $(Y_2)_0 \approx (Z_2)_0$. Then there exist spaces U_1 and U_2 so that $X \approx U_1 \times U_2$.*

In order to prove this theorem and Propositions 3.9 and 3.10, we need the following definition and notations:

3.7 *Definition.* Let X be an H_0-space and let $f: X \to X$ be a map. Suppose $QH^{n_i}(X, Q) \neq 0$ for $i = 1, \ldots, k$ and $d = (d_1, \ldots, d_k) \in \mathbf{Z}^k$. We say that f *realizes* d if for every i,

$$\det\,(QH^{n_i}(f, Z)/\text{torsion}) = d_i.$$

3.8 *Notations.* Let t be an integer and let X be an H_0-space. Denote by $Z_t^*(Z_t = Z/tZ)$ the units in Z_t, by $G(X)$ the genus of X, by $K(X)$ the space $K(QH^*(X, Z)/\text{torsion})$ of [**10**], by $l(X)$ the number of integers n satisfying $QH^n(X, Q) \neq 0$, by $[X, X]_t$ the set of homotopy classes of t-equivalences $f: X \to X$ ([**10**]), and by $\alpha_X: [X, X]_t \to (Z_t^*/\pm 1)^{l(X)}$ the composition

$$[X, X]_t \to \text{Aut}\,(QH^*(X, Z)/\text{torsion} \otimes Z_t) \xrightarrow{|\det|} (Z_t^*/\pm 1)^{l(X)}.$$

Proof of Theorem 3.6. Let φ_i: $Y_i \to (Y_i)_0$ and ψ_i: $Z_i \to (Z_i)_0$ $(i = 1, 2)$ be rationalizations and let $W_1 \times W_2$ be the pull back of

$$Y_1 \times Y_2 \xrightarrow{\varphi_1 \times \varphi_2} (Y_1)_0 \times (Y_2)_0 \xleftarrow{\psi_1 \times \psi_2} Z_1 \times Z_2.$$

Obviously $W_1 \times W_2 \in G(X)$.

Suppose $QH^{n_i}(W_1 \times W_2, Q) \neq 0$ for $1 \leqq i \leqq l(W_1 \times W_2)$ and η_i: $W_i \to K(W_i)$ $(i = 1, 2)$ is any map yielding an isomorphism on

$QH^*(W_i, Z)$/torsion. Since the map $SL(n, Z) \to SL(n, Z_t)$ is an epimorphism it follows from [**10**, Propositions 2.5 and 2.6] that there exists

$$d = (d_{n1}, \dots, d_{n\,l(W_1\times W_2)}) \in Z^{l(W_1\times W_2)}$$

so that X is the pull back of

$$W_1 \times W_2 \xrightarrow{\eta_1 \times \eta_2} K(W_1) \times K(W_2) \xleftarrow{f} K(W_1) \times K(W_2),$$

where f is any map which realizes d. Obviously one can choose f_i: $K(W_i) \to K(W_i)$ $(i = 1, 2)$, so that $f_1 \times f_2$ realizes d. Hence $X \approx U_1 \times U_2$ where U_i is the pull back of

$$W_i \xrightarrow{\eta_i} k(W_i) \xleftarrow{f_i} k(W_i).$$

3.9 Proposition. *Let X be an H_0-space and let $l \subseteq P$ be a subset. Suppose φ: $X \to K(X)$ is a rational equivalence which induces an isomorphism on $QH^*(X, Z)$/torsion. If $H^*(X, Z)$ is torsion free then there exists an epimorphism $G(X) \to G(X(l, \varphi))$.*

Proof. Suppose $QH^{m_i}(X, Q) \neq 0$ for $i = 1, \dots, l(x)$. Let f: $X \to X$ realize $d = (d_1, \dots, d_{l(X)})$. Since $H^*(X, Z)$ is torsion free there exists f': $K(X) \to K(X)$ so that $f'\varphi \sim \varphi f$. Consequently (Proposition 1.5) there exists a unique map f'': $X(l, \varphi) \to X(l, \varphi)$ which closes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi'"] & X \arrow[d, "\varphi'"] \\ X(l, \varphi) \arrow[r, "f''"] \arrow[d, "\varphi''"] & X(l, \varphi) \arrow[d, "\varphi''"] \\ K(X) \arrow[r, "f'"] & K(X) \end{tikzcd} \tag{3.9.1}$$

Define a map g: $[X, X]_t \to [X(l, \varphi), X(l, \varphi)]$ by $g(f) = f''$ if and only if f'' closes diagram (3.9.1). Since f'' is unique and it realizes d, g is well defined and its image is contained in $[X(l, \varphi), X(l, \varphi)]_t$.

Consider the following diagram:

$$\begin{tikzcd} {[X, X]_t} \arrow[r, "\alpha_X"] \arrow[d, "g"] & (Z_t^*/\pm 1)^{l(X)} \arrow[r] \arrow[d, equal] & G(X) \arrow[r] & 0 \\ {[X(l, \varphi), X(l, \varphi)]_t} \arrow[r, "\alpha_{X(l,\varphi)}"] & (Z_t^*/\pm 1)^{l(X \cdot \varphi)} \arrow[r] & G(X(l, \varphi)) \arrow[r] & 0 \end{tikzcd}$$

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Since the diagram is commutative and its rows are exact (main theorem in [**10**]) there exists h: $G(X) \to G(X(l, \varphi))$ and h is an epimorphism.

3.10 PROPOSITION. *Let X be an H_0-space and let φ: $X \to K(X)$ be a rational equivalence. Then $Y \in G(X)$ if and only if for every subset $l \subseteq P$ there exist spaces $Y_1 \in G(X(l, \varphi))$, $Y_2 \in G(X(l', \varphi))$ so that Y is the pull back of*

$$Y_1 \xrightarrow{f} K(X) \xleftarrow{g} Y_2,$$

where f is an l'-equivalence and g is an l-equivalence.

Proof. Suppose $Y \in G(X)$. By Proposition 4.6.4 in [**8**] there exist an integer t, a t-equivalence f_0: $K(X) \to K(X)$ and maps f: $Y \to X$, η: $Y \to K(X)$ so that the square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\varphi} \\ K(X) & \xrightarrow{f_0} & k(X) \end{array}$$

is a pull back and φ is a t'-equivalence. By [**8**, Lemma 4.2.1] f and η are, also, a t-equivalence and a t'-equivalence, respectively. Hence $f(l, \eta, \varphi)$ and $f(l', \eta, \varphi)$ are t-equivalences (Proposition 2.3) and therefore $Y(l, \eta) \in G(X(l, \varphi))$ and $Y(l', \eta) \in G(X(l', \varphi))$. Consequently the fact that Y is the pull back of $Y(l', \eta) \to K(X) \leftarrow Y(l, \eta)$ (Proposition 2.2) implies that $Y_1 = Y(l, \eta)$ and $Y_2 = Y(l', \eta)$ are the desired spaces.

The converse is obvious.

3.11 *Definition.* Define the genus $G(f)$ of f as follows: Consider first of all (homotopy classes of) maps f': $X' \to Y'$ such that for every prime p there exist homotopy equivalences h_p: $X_p' \to X_p$ and k_p: $Y_p' \to Y_p$ so that $f_p h_p \sim k_p f_p'$. Call two such (homotopy classes of) maps f': $X' \to Y'$, f'': $X'' \to Y''$ *equivalent* if there exist homotopy equivalences h: $X' \to X''$ and k: $Y' \to Y''$ so that $f''h \sim kf'$ and let $G(f)$ be the set of equivalence classes.

3.12 PROPOSITION. *Let X, Y be H-spaces with primitively generated rational cohomology and let f: $X \to Y$ be an H-rational equivalence. For every $l \leqq P$ there exist an epimorphism $G(f) \to G(f'')$.*

Proof. By [**3**, Lemma 4.2] there exists an integer t so that each map $\bar{f}$: $\bar{X} \to \bar{Y}$ in $G(f)$ is the pull back of $X \xrightarrow{f} Y \xleftarrow{k} \bar{Y}$, where k is a t-equivalence.

Let $\bar{f} \in G(f)$ ($\bar{f}$: $\bar{X} \to \bar{Y}$) be the pull back of $X \xrightarrow{f} Y \xleftarrow{k} Y$ and let

$k''\colon \bar{X}(l, \bar{f}) \to X(l, f)$ close the diagram

$$\begin{tikzcd} \bar{X} \arrow[r, "k'"] \arrow[d, "\bar{f}'"] & X \arrow[d, "f'"] \\ \bar{X}(l, \bar{f}) \arrow[r, "k''"] \arrow[d, "\bar{f}''"] & X(l, f) \arrow[d, "f''"] \\ \bar{Y} \arrow[r, "k"] & Y \end{tikzcd}$$

By Proposition 2.3 the lower square is a pull back, hence $\bar{f}'' \in G(f'')$ [**3**, Lemma 4.2]. Define a map $\eta\colon G(f) \to G(f'')$ by $\eta(\bar{f}) = \bar{f}''$. The definition of the multiplications in $G(f)$ and $G(f'')$ implies that η is a homomorphism. To show that η is an epimorphism, suppose $g'' \in G(f'')$ is the pull back of $\bar{Y} \xrightarrow{k} Y \xleftarrow{f''} X(l, f)$, where k is a t-equivalence.

$$\begin{tikzcd} \hat{X} \arrow[r, "k''"] \arrow[d, "g''"] & X(l, f) \arrow[d, "f''"] \\ \bar{Y} \arrow[r, "k"] & Y \end{tikzcd}$$

Let $g'\colon \bar{X} \to \hat{X}$ be the pull back of $\hat{X} \xrightarrow{k''} X(l, f) \xleftarrow{f'} X$. Obviously [**3**, Lemma 4.2] $g''g'\colon \bar{X} \to \bar{Y}$ belongs to $G(f)$ and η is an epimorphism.

The last two propositions apply the decomposition of a rationalization into an l-equivalence and an l'-equivalence to deal with the question: When is a map between H-spaces an H-map?

3.13 THEOREM. *Let X, μ and Y, ν be simply connected H-spaces and let $f\colon X \to Y$ be a map. Given subsets of the primes l_1, l_2, $l_1 \cup l_2 = P$, and rationalizations $\varphi_i\colon X \to X_0$, $\psi_i\colon Y \to Y_0$ $(i = 1, 2)$. If f_{l_i} $(i = 1, 2)$ is the localization of f at l_i for which the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi_i'"'] & Y \arrow[d, "\psi_i'"] \\ X_l \arrow[r, "f_{l_i}"] \arrow[d, "\varphi_i''"'] & Y_{l_1} \arrow[d, "\psi_i''"] \\ X_0 \arrow[r, "f_0"] & Y_0 \end{tikzcd}$$

commutes, then f is an H-map if and only if f_{l_1} and f_{l_2} are H-maps.

Proof. Let μ_0 and ν_0 be the multiplications on X_0 and Y_0 induced by φ_1 and ψ_1, respectively, and let $\epsilon: X_0 \to X_0$, $\eta: Y_0 \to Y_0$ be homotopy equivalences satisfying $\epsilon\varphi_1 \sim \varphi_2$, $\eta\psi_1 \sim \psi_2$. Then $\epsilon\mu_0(\epsilon^{-1} \times \epsilon^{-1})$ and $\eta\mu_0(\eta^{-1} \times \eta^{-1})$ (ϵ^{-1} and η^{-1} are the homotopy inverses of ϵ and η) are multiplications on X_0 and Y_0 induced by φ_2 and ψ_2, respectively and ϵ and η are H-maps.

Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\varphi_1'"] \arrow[dd, "\varphi_2'"'] \arrow[drr, "f"] & X_{l_1} \arrow[d, "\varphi_1''"] \arrow[drr, "f_{l_1}"] & & \\
& X_0 \arrow[d, "\epsilon"] \arrow[drr, "f_0"] & Y \arrow[r, "\psi_1'"] \arrow[dd, "\psi_2'"] & Y_{l_1} \arrow[d, "\psi_1''"] \\
X_{l_2} \arrow[r, "\varphi_2''"] \arrow[drr, "f_{l_2}"'] & X_0 \arrow[drr, "\eta f_0 \epsilon^{-1}"] & & Y_0 \arrow[d, "\eta"] \\
& & Y_{l_2} \arrow[r, "\psi_2''"] & Y_0
\end{tikzcd}$$

Since all spaces and maps in the diagram, except possibly for f, are H-spaces and H-maps and Y is the pull back of

$$Y_{l_2} \xrightarrow{\psi_2''} Y_0 \xleftarrow{\eta} Y_0 \xleftarrow{\psi_1''} Y_{l_1},$$

it follows from [**1**, 10.3] that f is an H-map.

The converse is obvious.

3.14 PROPOSITION. *Let X, μ, Y, ν, X_i, μ_i, Y_i, ν_i $(i = 1, 2)$ be H-spaces and let $f: X \to Y$ be a map. Suppose $H^*(X, Q)$, $H^*(Y, Q)$ are primitively generated and $H^*(f, Q)$ is either a monomorphism, an epimorphism or zero. Suppose, also, that there exist subsets of the primes l_1, l_2, $l_1 \cup l_2 = P$, and l_i-equivalences $g_i: X_i \to X$, $h_i: Y \to Y_i$, so that the map $h_i f g_i: X_i, \mu_i \to Y_i, \nu_i$ $(i = 1, 2)$ is an H-map. If either*

(a) *$H^*(X_i, Q)$ and $H^*(Y_i, Q)$ $(i = 1, 2)$ are primitively generated, or*

(b) *$(g_i)_0$ and $(h_i)_0$ $(i = 1, 2)$ are H-maps, then there exist H-structures on X and Y so that f is an H-map.*

Proof. (a) Let μ_0 and ν_0 be the standard multiplications on

$$X_0 = \prod_{\text{finite}} K(Q, n_i) \quad \text{and} \quad Y_0 = \prod_{\text{finite}} K(Q, m_i),$$

respectively. Since $H^*(X_i, Q)$ and $H^*(Y_i, Q)$ $(i = 1, 2)$ are primitively

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generated, one can choose H-rational equivalences, φ_i: $X_i, \mu_i \to X_0, \mu_0$ and ψ_i: $Y_i, \nu_i \to Y_0, \nu_0$ $(i = 1, 2)$, so that $PH^*(h_{10}f_0g_{10}, Q)$ and PH^*-$(h_{20}f_0g_{20}, Q)$ are represented by diagonal matrices. Moreover, since h_{i0} and g_{i0} $(i = 1, 2)$ are homotopy equivalences, one can choose φ_i and ψ_i so that the matrices of $PH^*(h_{10}f_0g_{10}, Q)$ and $PH^*(h_{20}f_0g_{20}, Q)$ have nonzero entries in the same rows. Consequently, there exists an H-homotopy equivalence

$$\epsilon: Y_0, \nu_0 \to Y_0, \nu_0$$

so that

$$\epsilon(h_{10}f_0g_{10}) \sim h_{20}f_0g_{20}.$$

Let μ_{il_i} and ν_{il_i} $(i = 1, 2)$ be the multiplication on $(X_i)_{l_i}$ and $(Y_i)_{l_i}$. Identify $(X_i)_{l_i}$ and $(Y_i)_{l_i}$ with X_{l_i} and Y_{l_i}, by means of the l_i-equivalences g_i: $X_i \to X$ and h_i: $Y \to Y_i$, and let Y', ν' and X', μ' be the pull backs of

$$Y_{l_1}, \nu_{1l_1} \xrightarrow{\psi_1''} Y_0, \nu_0 \xleftarrow{\epsilon} Y_0, \nu_0 \xleftarrow{\psi_2''} Y_{l_2}, \nu_{2l_2}$$

and

$$X_{l_1}, \mu_{1l_1} \xrightarrow{\varphi_1''} X_0, \mu_0 \xleftarrow{\varphi_2''} X_{l_2}, \mu_{2l_2},$$

respectively. Suppose f: $X' \to Y'$ closes the diagram

$$\begin{tikzcd}
X', \mu' \arrow[r] \arrow[drr, "f'"] \arrow[dddd] & X_{l_1}, \mu_{1l_1} \arrow[drr, "h_{1l_1}f_{l_1}g_{1l_1}"] \arrow[dd, "\varphi_1''"] & & \\
& & Y', \nu' \arrow[r] \arrow[dddd] & Y_{l_1}, \nu_{1l_1} \arrow[dd, "\psi_1''"] \\
& X_0, \mu_0 \arrow[drr, "h_{10}f_0g_{10}"] \arrow[dd, "1"] & & \\
& & & Y_0, \nu_0 \arrow[dd, "\epsilon"] \\
X_{l_2}, \mu_{2l_2} \arrow[r, "\varphi_2''"] \arrow[drr, "h_{2l_2}f_{l_2}g_{2l_2}"'] & X_0, \mu_0 \arrow[drr, "h_{20}f_0g_{20}"] & & \\
& & Y_{l_2}, \nu_{l_2} \arrow[r, "\psi_2''"] & Y_0, \nu_0
\end{tikzcd}$$

Since all spaces and maps in the diagram, except possibly for f', are H-spaces and H-maps, it follows from [**1**, 10.3] that f' is an H-map. Therefore the fact that $f' \in G(f)$ and the structure of the genus of f [**3**, Proposition 3.16] imply that f is an H-map.

(b) Since $(g_i)_0$ and $(h_i)_0$ $(i = 1, 2)$ are H-equivalences, the fact that $H^*(X, Q)$ and $H^*(Y, Q)$ $(i = 1, 2)$ are primitively generated implies that $H^*(X_i, Q)$ and $H^*(Y_i, Q)$ $(i = 1, 2)$ are, also, primitively generated and therefore (b) follows from (a).

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