

INJECTIVITY AND RELATED CONCEPTS
IN MODULAR VARIETIES
II. THE CONGRUENCE EXTENSION PROPERTY

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Varieties with enough injectives satisfy the congruence extension property (CEP). We investigate the CEP in modular varieties by using the techniques developed in the first part of the present paper. As corollaries, we obtain the results of B. Davey and J. Kollár for the congruence distributive case as well as the description of all varieties of groups and rings with CEP, given by B. Biró, E.W. Kiss and P.P. Pálfy.

1. Preliminaries

Since a complete background and bibliography of the concepts related to injectivity is found in Kiss, Márki, Pröhle and Tholen [6], here we give only those results, which we need explicitly. The reader is referred to the first part (Kiss [5]) for terminology.

DEFINITION 1.1. An algebra A has CEP if all the congruences of its subalgebras are restrictions of congruences of A . A class of algebras has CEP if so does each of its members.

PROPOSITION 1.2. *The class of CEP algebras of a variety V is*

Received 3 December 1984.

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\$A2.00 + 0.00.

closed under subalgebras, direct limits (Biró, Kiss and Pálfy [1]), homomorphic images if V is CP (Kiss [4]), and finite direct products if V is CD. \square

We summarise a few basic, elementary methods, some of which have been used in Davey [2]. For algebras $B \leq A$ and $\beta \in \text{Con } B$, we denote by β^A the smallest congruence of A collapsing the classes of β , that is, $\beta^A = \text{Cg}_A(\beta)$.

LEMMA 1.3. Let $B \leq A$ be algebras, $\beta \in \text{Con } B$.

- (1) β can be extended to A if and only if $\beta = \beta^A/B$.
- (2) If $B \leq C \leq A$ have CEP, then $\beta^A = (\beta^C)^A$ and $\beta^A/C = \beta^C$.
- (3) If there is an $\alpha \in \text{Con } A$ such that $\alpha/B \leq \beta$ and β/α can be extended to A/α from its subalgebra B/α , then β can be extended to A .
- (4) A has CEP if and only if for each $B \leq A$, all the congruences β of B , with $B/\beta \text{ Si}$, can be extended to A .
- (5) A has CEP if and only if for each $a, b, c, d \in A$, $(c, d) \in \text{Cg}_A(a, b)$ implies that $(c, d) \in \text{Cg}_{\langle a, b, c, d \rangle}(a, b)$ (Day [3]).

Proof. (1), (2), (3) are easy, (4) follows from the Birkhoff theorem, since the restriction of the meet of some congruences is the meet of their restrictions. \square

Finally we present two theorems which we generalize, and a problem which we solve.

THEOREM 1.4 (Davey [2]). Let V be a CD variety. If $\text{Si}(V)$ is axiomatic, then V has CEP if and only if $\text{Si}(V)$ has CEP.

THEOREM 1.5 (Kollár [7]). Let V be a CD variety. If $F_V(4)$ is finite, then V has CEP if and only if $\text{Si}(V)$ has CEP.

PROBLEM 1.6 (Herrmann [9]). Is every CP + CEP variety disconnected?

From now on, we work in a fixed congruence modular variety.

2. General results

First of all, we try to understand the relationship between CEP and finite direct products. On this basis, we prove theorems that, having information on particular subclasses, decide whether a variety satisfies CEP.

PROPOSITION 2.1. *If an algebra A satisfies $CEP + C2 + S$, then so do its subalgebras.*

Proof. Let $B \leq A$, $\alpha, \beta \in \text{Con } B$. Then $[\alpha, \beta] = [\alpha^A, \beta^A]/B$ by CEP and S , so $C2$ holds in B . Therefore S also holds in B by [5], Proposition 3.4.

THEOREM 2.2. *If $A \times A$ has CEP, then A has $C2 + S + CEP$. Consequently, every CEP variety has $C2 + S$. Conversely, the direct product of finitely many algebras with $C2 + S + CEP$ also has CEP.*

Proof. If $A \times A$ has CEP, then its diagonal, which is isomorphic to A , has CEP also, as a subalgebra. We show that A satisfies $C2$. Let $\alpha, \beta \in \text{Con } A$. As $(\Delta_{\alpha, \beta})^{A \times A} = \Delta_{1, \beta}$ holds by Lemma 1.3 (2), Lemma 1.3 (1) gives $\Delta_{\alpha, \beta} = (\Delta_{1, \beta})/\alpha$, hence $\alpha[1, \beta] = [\alpha, \beta]$, that is, A has $C2$.

In order to show S in A , we use the notation $A^{\Delta_{\alpha, \beta}}$ to emphasise that we work in the algebra A . Let $B \leq A$ and $\beta = (1_B)^A$. Applying (2) of Lemma 1.3 twice, we obtain that $(A^{\Delta_{1_A, \beta}})^{B \times B} = B^{\Delta_{1_B, 1_B}}$, hence

$[1_B, 1_B] = [1_A, \beta]/B$. By $C2$ we have

$$[1_A, \beta]/B = ([1_A, 1_A]/B)(\beta/B) = [1_A, 1_A]/B,$$

since $\beta/B = 1_B$. Thus A satisfies S .

Suppose now that A_1 and A_2 have $C2 + S + CEP$, let $B \leq A_1 \times A_2$, and $\beta \in \text{Con } B$. We want to extend β to $A_1 \times A_2$. By (4) of Lemma 1.3 we may assume that B/β is Si. As B has $C2$ ([5] Prop. 3.1, 3.4 and Proposition 2.1), B/β is either affine or prime ([5], Proposition 4.1).

We use (3) of Lemma 1.3. If B/α is affine, we may choose

$\alpha = [1_A, 1_A]$, since affine algebras have CEP, and $A_1 \times A_2$ has S by [5] Proposition 3.4. If B/β is prime, then [5], Theorem 1.8 shows that the kernel of a projection of $A_1 \times A_2$ is below β . Choosing it to be α , Lemma 1.3 (3) works again. \square

Now we generalize Davey's result. Recall that $P^2(K) = \{A \times A : A \in K\}$.

THEOREM 2.3. *A variety V has CEP if and only if*

- (i) $P^2\text{Si}(V)$ has CEP and
- (ii) $P_U\text{Si}(V)$ has CEP.

Proof. First we show that V has $C2 + S$. For $C2$, this is clear (by Theorem 2.2, [5], Propositions 4.1 and 4.2). To establish S it suffices to show that the prime algebras of V satisfy S by [5], Corollary 4.6. Applying [5], Theorem 1.8 (the Generalized Jónsson's Theorem) with $\alpha = 0$ we obtain that the prime algebras are always in $\text{SP}_U\text{Si}(V)$, thus they satisfy CEP. So let $B \leq A$ be CEP algebras of V , and suppose that there is a pair $(a, b) \in [1_A, 1_A]/B - [1_B, 1_B]$. Let α be an extension of $[1_B, 1_B]$ to A , and let θ be maximal among the congruences of A containing α but not containing (a, b) . Then A/α is a Si factor of A not satisfying S , a contradiction. Thus V satisfies S .

Now we copy the proof of the second statement of Theorem 2.2. Let $B \leq A \in V$, and $\beta \in \text{Con } B$ such that B/β is Si. If this factor is affine, then $\alpha = [1_A, 1_A]$ works in Lemma 1.3 (3), as A has property S . Otherwise B/β is prime, in this case consider a subdirect representation of A with Si algebras, and choose α to be an ultra-filtral congruence below β ([5], Theorem 1.8). This works in Lemma 1.3 (3) by condition (ii). \square

PROPOSITION 2.4. *Let K be a class of algebras, $V = V(K)$. If $P^2\text{HP}_U(K)$ has CEP, and $F_V(2)$ is finite or $C2$, then V has CEP. If V is CP, then the operator H can be omitted from the first condition.*

Proof. V has $C2 + S$ by [5], Proposition 5.3. Thus each Si member

S is either affine or prime. In the second case, S is in $\text{HSP}_{\mathbf{U}}(K)$. Since this class is closed under H , S and $P_{\mathbf{U}}$, Theorem 2.3 applies. \square

Starting in the direction of finiteness conditions, we generalize Kollár's result (Theorem 1.5).

PROPOSITION 2.5. *If $F_V(4)$ is finite, then the variety V has CEP if and only if $P^2\text{Si}(V)$ has CEP.*

Proof. We apply Lemma 1.3 (5) (Day's observation). Let $a, b, c, d \in A \in V$, and decompose A into a subdirect product of Si algebras $A \cong \prod\{A_i : i \in I\}$. Then $B = \langle a, b, c, d \rangle$ is finite by our assumption. Hence there is a finite $X \subseteq I$ such that the kernel $\alpha = \text{Cg}_A(X)$ of the projection of A into $A' = \prod\{A_i : i \in X\}$ separates the elements of B . But A' has CEP by the assumptions and Theorem 2.2. So Lemma 1.3 (3) shows, with α above, that every congruence of B can be extended to A . \square

PROPOSITION 2.6. *Let K be a finite set of finite algebras. Then $V(K)$ has CEP if and only if $P^2H(K)$ has CEP. If $V(K)$ is CP, then H can be omitted.*

Proof. Though this is a formal consequence of Proposition 2.4, the reader will easily find an elementary argument using [5], Proposition 5.4, Theorem 2.2 and $V = \text{DP}_{\text{sf}}\text{Si}(V)$. \square

PROPOSITION 2.7. *If V_1 and V_2 are two CEP subvarieties of a modular variety, then $V_1 \vee V_2$ has CEP.*

This statement follows immediately from [5], Proposition 6.1 and Theorem 2.2. Finally we solve Herrmann's problem (Problem 1.6).

PROPOSITION 2.8. *There exists a finitely generated CP + CEP variety, which is not disconnected.*

Proof. Let A be the symmetric group on five letters, with its elements as nullary operations. Then A has no subalgebras, and its only nontrivial congruence (given by the alternating group) is perfect, and equals $[1, 1]$. Thus A has $C2 + S + \text{CEP}$. So $V(A)$ has CEP by

Proposition 2.6 and Theorem 2.2. However, it contains no neutral algebra by the Generalized Jónsson's Theorem. \square

3. Applications

We give the description of all varieties of groups and associative rings over commutative rings K with identity (named K -rings) that have CEP. Though most of these results have been obtained in Biró, Kiss and Pálffy [1], our proofs are much simpler.

An easy computation shows that the commutator of two normal subgroups M and N of a group G is $[M, N] = \langle n^{-1}m^{-1}nm : n \in N, m \in M \rangle$, and the commutator of the ideals I, J of a K -ring R is the K -ideal generated by $IJ + JI$, even if R is non-associative. The affine K -rings are the zerorings.

PROPOSITION 3.1. *A group (Lie-algebra) with property S is commutative. The varieties of groups (Lie-algebras) with CEP are exactly the commutative ones.*

Proof. Let H be an Abelian subgroup of the commutator subgroup G' of a group G . Then S yields $H' = H \cap G'$, that is, H has one element. The proof is the same for Lie-algebras, since their subalgebras generated by one element are also commutative. \square

PROPOSITION 3.2. *K -rings with $C2 + S$ are disconnected.*

Proof. Suppose that R has $C2 + S$, and let $T \leq R$. S gives that $T^2 = R^2 \cap T$ and $T^3 = R^3 \cap T$, so as $C2$ yields $R^3 = R^2$, we have $T^3 = T^2$. Let A be the two-sided annihilator of R . Since nilpotent ideals are clearly zerorings, factorising by a maximal one, we have to show that $A \neq 0$ provided $R^2 \neq R$.

Let $u \in R - R^2$, $T = \langle u \rangle$. Then $T^3 = T^2$ yields a polynomial p such that $u^2 = u^3p(u)$. Let $s = u^2p(u) - u$, then $s \neq 0 = s^2$. We show $s \in A$. Since $t \in R^2$ and $t^2 = 0$ imply that $t = 0$ (by S), for any $r \in R$, $(srs)^2 = 0$ gives $srs = 0$, so $(sr)^2 = 0$, hence $sr = 0$, and similarly $rs = 0$. \square

COROLLARY 3.3. *A CEP variety of K-rings is disconnected.*

This corollary follows from [5], Theorem 7.2. To get a description of these varieties, we have to recall the characterization of congruence distributive K-algebra varieties. This has been done by McKenzie [8]. Actually, his paper contains the description of C2 varieties of K-rings as well, the previous argument has been included for its relative simplicity.

THEOREM 3.4 (McKenzie [8]). *A variety V of K-rings (with or without identity) is neutral if and only if it is generated by division K-rings of bounded finite cardinality if and only if the equation $x^n = x$ holds in V for some $n > 1$. Such varieties are commutative. V has C2 if and only if it is disconnected.*

If we speak about varieties of K-rings with identity, we assume that the subalgebras contain the identity element. Let us summarize now the results on K-rings.

THEOREM 3.5. *The following are equivalent for a variety V of K-rings without identity:*

- (1) V has CEP;
- (2) V has C2 + S ;
- (3) V has C2 ;
- (4) V is disconnected (see the description in Theorem 3.4);
- (5) V satisfies the equation $(xy)^n = xy$ for some $n > 1$.

The following are equivalent for a variety V of K-rings with identity:

- (1') V has CEP;
- (2') V is neutral.

Proof. (5) \Rightarrow (4). It turns out from the proof of Theorem 3.3 of [1] that if a variety V satisfies (5), then the Si elements of the Z-reduct of V are either zero-rings or fields of size at most n. So (4) holds by [5], Theorem 7.2.

(4) \Rightarrow (1). By Proposition 2.7 we may assume that V is neutral. The Si algebras of V are just the division-K-rings by Theorem 3.4 and Jónsson's Theorem, so Theorem 1.4 applies. The remaining statements of the

theorem are clear from the previous assertions. \square

We mention that the description of all semigroup varieties with CEP is found in Biró, Kiss and Pálffy [1].

PROBLEM 3.6. Can one apply the methods of the paper to describe all varieties of quasigroups and alternative rings having the CEP?

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