BULL. AUSTRAL. MATH. SOC. 17B56
VOL. 54 (1996) [517–527]

ON THE COHOMOLOGY OF A CLASS OF NILPOTENT LIE ALGEBRAS

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Let $\mathfrak{g}$ denote a finite dimensional nilpotent Lie algebra over $\mathbb{C}$ containing an Abelian ideal $\mathfrak{a}$ of codimension 1, with $z \in \mathfrak{g}\backslash\mathfrak{a}$. We give a combinatorial description of the Betti numbers of $\mathfrak{g}$ in terms of the Jordan decomposition $\mathfrak{a} = \bigoplus_{l=1}^{t} \mathfrak{a}_l$ induced by $ad(z)|_{\mathfrak{a}}$. As an application we prove that the filiform-nilpotent Lie algebras arising in the case $t = 1$ have unimodal Betti numbers.

INTRODUCTION

Let $\mathfrak{g}$ denote a finite dimensional complex nilpotent Lie algebra containing an Abelian ideal $\mathfrak{a}$ of codimension one, with $z \in \mathfrak{g}\backslash\mathfrak{a}$. We compute the Betti numbers

$$b_i(\mathfrak{g}) = \dim\left(H^i(\mathfrak{g}, \mathbb{C})\right)$$

for the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in $\mathbb{C}$. Choose a basis for $\mathfrak{a}$ with respect to which the matrix representation of $ad(z)|_{\mathfrak{a}}$ is in lower triangular Jordan canonical form. Denote the corresponding decomposition of $\mathfrak{a}$ by $\mathfrak{a} = \bigoplus_{l=1}^{t} \mathfrak{a}_l$, where each $\dim(\mathfrak{a}_l) = n_l$. Then our main result is the following:

THEOREM 1. *The i^{th} Betti number $b_i(\mathfrak{g})$ is given by*

$$b_i(\mathfrak{g}) = \kappa_i(\mathfrak{g}) + \kappa_{i-1}(\mathfrak{g})$$

for each $1 \leqslant i \leqslant \dim(\mathfrak{g})$, where $\kappa_i(\mathfrak{g})$ denotes the sum

$$\sum_{\substack{k_1+\cdots+k_t=i \\ 0\leqslant k_l\leqslant n_l}} \#\left\{ \left((\alpha_{1,1},\ldots,\alpha_{k_1,1}),\ldots,(\alpha_{1,t},\ldots,\alpha_{k_t,t}) \right) \in \mathbb{Z}^i \;\middle|\; \begin{array}{c} 1 \leqslant \alpha_{1,l} < \cdots < \alpha_{k_l,l} \leqslant n_l \\ \sum \alpha_{k,l} = \left\lceil \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1) \right\rceil \end{array} \right\}$$

Received 30th January, 1996
The first author would like to thank Grant Cairns for his advice and encouragement, and Sally Davison, Barry Jessup and Louis Santharoubane for helpful suggestions and discussions.

for each $1 \leqslant i \leqslant \dim(\mathfrak{g})$, *and* $\kappa_0(\mathfrak{g}) = 1$. *Here* $\lceil x \rceil$ *denotes the least integer not smaller than* x, *and* $\#$ *indicates cardinality.*

One may identify two extreme cases in the applications of Theorem 1. Firstly the case of an arbitrary number of Jordan blocks t for $ad(z)|_{\mathfrak{a}}$ all of equal length $n_l = 2$, and secondly the case of a single Jordan block of arbitrary length $n_1 = n$. In the first case one is interested in the family $\{\mathfrak{p}_t \colon t \in \mathbb{N}\}$ of $(2t+1)$-dimensional 2-step nilpotent Lie algebras, where each $\mathfrak{p}_t$ has basis $\{x_1, \dots, x_t, x_{t+1}, \dots, x_{2t}, z\}$ and non-zero relations $[z, x_j] = x_{t+j}$ for each $1 \leqslant j \leqslant t$. Explicit Betti numbers for this family were determined in [**2**]: the i^{th} Betti number $b_i(\mathfrak{p}_t)$ given by

$$b_i(\mathfrak{p}_t) = \binom{t+1}{\lfloor \frac{i+1}{2} \rfloor} \binom{t}{\lfloor \frac{i}{2} \rfloor} \tag{1}$$

for each $0 \leqslant i \leqslant 2t+1$ and $t \in \mathbb{N}$, where $\lfloor x \rfloor$ denotes the integer part of x. In particular the sequence $\{b_i(\mathfrak{p}_t)\}_{i=0}^{\dim(\mathfrak{p}_t)}$ is unimodal for each $t \in \mathbb{N}$.

In this paper we investigate the second case in our applications of Theorem 1. Here one is interested in the family $\{\mathfrak{f}_n \colon n \in \mathbb{N}\}$ of $(n+1)$-dimensional filiform-nilpotent Lie algebras, where each $\mathfrak{f}_n$ has basis $\{x_1, \dots, x_n, z\}$ and non-zero relations $[z, x_j] = x_{j+1}$ for each $1 \leqslant j \leqslant n-1$. The Betti numbers for this family have previously been computed by Bordemann [**3**], whose description we recover as an immediate corollary to Theorem 1:

Corollary 1. *The* i^{th} *Betti number* $b_i(\mathfrak{f}_n)$ *is given by*

$$b_i(\mathfrak{f}_n) = \kappa_i(\mathfrak{f}_n) + \kappa_{i-1}(\mathfrak{f}_n)$$

for each $1 \leqslant i \leqslant n+1$ *and* $n \in \mathbb{N}$, *where*

$$\kappa_i(\mathfrak{f}_n) = \# \left\{ (\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i \;\middle|\; 1 \leqslant \alpha_1 < \dots < \alpha_i \leqslant n \;\; \textit{and} \;\; \sum \alpha_j = \left\lceil \frac{i(n+1)}{2} \right\rceil \right\}$$

for each $1 \leqslant i \leqslant n$, *and* $\kappa_0(\mathfrak{f}_n) = 1$.

At the heart of Corollary 1 is the task of counting the number of partitions of an integer d into i distinct parts, each part being no larger than n. This particular partition problem has a long history in the theory of combinatorics. It may be solved explicitly for small values of i, and in the special case of Corollary 1, Bordemann [**3**] has shown that for each $n \in \mathbb{N}$, one has

$$b_1(\mathfrak{f}_n) = 2, \; b_2(\mathfrak{f}_n) = \left\lfloor \frac{n+2}{2} \right\rfloor \;\; \text{and} \;\; b_3(\mathfrak{f}_n) = \left\lfloor \binom{\frac{n+2}{2}}{2} + \frac{1}{8} \right\rfloor.$$

https://doi.org/10.1017/S0004972700021936 Published online by Cambridge University Press

The numbers $b_i(\mathfrak{f}_n)$ get progressively more complicated as i gets bigger. One finds [1] that for each $n \in \mathbb{N}$,

$$b_4(\mathfrak{f}_n) = \left\lfloor \frac{4}{3}\binom{\frac{n+2}{2}}{3} + \frac{n}{9} + \frac{17}{36} \right\rfloor .$$

According to Richard Stanley [11] it is hopeless to expect any kind of explicit formula giving the number of such partitions for general n, i and d. In Corollary 1 we are admittedly seeking partitions for integers of a special type, namely those of the form $\lceil i(n+1)/2 \rceil$, where $0 \leqslant i \leqslant n+1$. Nevertheless Robert Proctor [8] believes that even in this special case one should not be very optimistic of obtaining any such formula, and hence of obtaining general expressions for $b_i(\mathfrak{f}_n)$ of type (1). We can however prove the following:

Theorem 2. *For each $n \in \mathbb{N}$, the sequence $\{b_i(\mathfrak{f}_n)\}_{i=0}^{\dim(\mathfrak{f}_n)}$ is unimodal.*

The proof of Theorem 1 is given in sections 1 and 2, and the proof of Theorem 2 in section 3.

1. Proof of Theorem 1

We use the long exact sequence of Dixmier [5] to reduce the theorem to a computation of the dimensions of the kernels of endomorphisms defined by extending $ad(z)|_{\mathfrak{a}}$ as a derivation on the exterior algebra $\wedge\mathfrak{a} = \bigoplus_{i\geqslant 0} \wedge^i\mathfrak{a}$ of $\mathfrak{a}$. The Jordan decomposition $\mathfrak{a} = \bigoplus_{l=1}^{t} \mathfrak{a}_l$ defined with respect to $ad(z)|_{\mathfrak{a}}$ induces a grading in each $\wedge^i\mathfrak{a}$ that facilitates computation via Lemma 2, which is the key to the proof of Theorem 1. We postpone the verification of Lemma 2 until the following section.

Lemma 1. (Dixmier [5]) *Let u_i denote the endomorphism of the $\mathfrak{g}$-module $H^i(\mathfrak{a}, \mathbb{C})$ induced by the action of z. The i^{th} Betti number $b_i(\mathfrak{g})$ is given by*

$$b_i(\mathfrak{g}) = \dim(\ker(u_i)) + \dim(\ker(u_{i-1}))$$

for each $0 \leqslant i \leqslant \dim(\mathfrak{g})$.

Clearly $H^i(\mathfrak{a}, \mathbb{C})$ and $\wedge^i\mathfrak{a}$ are isomorphic as vector spaces, indeed as algebras, since $\mathfrak{a}$ is Abelian. Denote by X_i the endomorphism of $\wedge^i\mathfrak{a}$ defined by extending $ad(z)|_{\mathfrak{a}}$ as a derivation on $\wedge\mathfrak{a}$. Then clearly $\dim(\ker(X_i)) = \dim(\ker(u_i))$.

Corollary 2. *For each $0 \leqslant i \leqslant \dim(\mathfrak{g})$, one has*

$$b_i(\mathfrak{g}) = \dim(\ker(X_i)) + \dim(\ker(X_{i-1})).$$

Choose a basis $\{x_1^1, \dots, x_{n_1}^1, \dots, x_1^t, \dots, x_{n_t}^t\}$ for $\mathfrak{a}$ with respect to which the matrix representation of $ad(z)|_{\mathfrak{a}}$ is in lower triangular Jordan canonical form. Then the bracket structure of $\mathfrak{g}$ becomes explicit via the relations $[z, x^l_j] = x^l_{j+1}$ for each $1 \leqslant j \leqslant n_l - 1$ and $1 \leqslant l \leqslant t$. Moreover one has the decomposition $\mathfrak{a} = \bigoplus_{l=1}^{t} \mathfrak{a}_l$, where each $\mathfrak{a}_l = \langle x^l_1, \dots, x^l_{n_l} \rangle$. This induces a grading of $\wedge^i \mathfrak{a}$ as follows:

$$\wedge^i \mathfrak{a} = \bigoplus_{\substack{\underline{k}=(k_1,\dots,k_t)\in \mathbb{Z}^t \\ k_1+\dots+k_t=i \\ 0\leqslant k_l \leqslant n_l}} \left(\wedge^i \mathfrak{a}\right)_{\underline{k}} \tag{2}$$

where $\left(\wedge^i \mathfrak{a}\right)_{\underline{k}} = \wedge^{k_1} \mathfrak{a}_1 \otimes \dots \otimes \wedge^{k_t} \mathfrak{a}_t$. Clearly one obtains $\dim(\ker(X_i))$ from (2) via the equation

$$\dim(\ker(X_i)) = \sum_{\substack{\underline{k}=(k_1,\dots,k_t)\in \mathbb{Z}^t \\ k_1+\dots+k_t=i \\ 0\leqslant k_l \leqslant n_l}} \dim\left(\ker\left(X_i|_{(\wedge^i \mathfrak{a})_{\underline{k}}}\right)\right). \tag{3}$$

It now remains to determine the summands in (3). To facilitate computation we introduce the following decomposition of each $\left(\wedge^i \mathfrak{a}\right)_{\underline{k}}$:

$$\left(\wedge^i \mathfrak{a}\right)_{\underline{k}} = \bigoplus_{s=\min(i,\underline{k})}^{\max(i,\underline{k})} V^i_{\underline{k}}(s) \tag{4}$$

where

$$V^i_{\underline{k}}(s) = \left\langle x^1_{\alpha_{1,1}} \wedge \dots \wedge x^1_{\alpha_{k_1,1}} \otimes \dots \otimes x^t_{\alpha_{1,t}} \wedge \dots \wedge x^t_{\alpha_{k_t,t}} \;\middle|\; \begin{matrix} 1 \leqslant \alpha_{1,l} < \dots < \alpha_{k_l,l} \leqslant n_l \\ \sum \alpha_{k,l} = s \end{matrix} \right\rangle$$

and $\min(i,\underline{k}) = \frac{1}{2}\sum_{l=1}^{t} k_l(k_l+1)$ and $\max(i,\underline{k}) = \left(\sum_{l=1}^{t} k_l(n_l+1)\right) - \min(i,\underline{k})$. Calculations verify that $X_i\left(\left(\wedge^i \mathfrak{a}\right)_{\underline{k}}\right) \subseteq \left(\wedge^i \mathfrak{a}\right)_{\underline{k}}$ and $X_i\left(V^i_{\underline{k}}(s)\right) \subseteq V^i_{\underline{k}}(s+1)$, so in each $\left(\wedge^i \mathfrak{a}\right)_{\underline{k}}$ there is the sequence

$$\begin{aligned} &V^i_{\underline{k}}(\min(i,\underline{k})) \xrightarrow{X_i} V^i_{\underline{k}}(\min(i,\underline{k})+1) \xrightarrow{X_i} \dots \\ &\qquad \dots \xrightarrow{X_i} V^i_{\underline{k}}(s) \xrightarrow{X_i} V^i_{\underline{k}}(s+1) \xrightarrow{X_i} \dots \xrightarrow{X_i} V^i_{\underline{k}}(\max(i,\underline{k})). \end{aligned}$$

The key to establishing Theorem 1 is the following:

LEMMA 2.

$$X_i|_{V^i_{\underline{k}}(s)} \text{ is } \begin{cases} \textit{injective}, & \textit{for } s < \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1) \\ \textit{surjective}, & \textit{for } s \geqslant \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1). \end{cases}$$

We postpone the proof of Lemma 2 until the next section. Meanwhile we finalise the proof of Theorem 1. First note that for each $1 \leqslant i \leqslant \dim(\mathfrak{g})$, one has

$$\dim(\ker(X_i)) = \sum_{\substack{(k_1,\dots,k_t)\in \mathbf{Z}^t \\ k_1+\dots+k_t=i \\ 0\leqslant k_l \leqslant n_l}} \dim\left(V^i_{\underline{k}}\left(\left\lceil \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1)\right\rceil\right)\right). \tag{5}$$

Indeed, by Lemma 2 it is clear that $\dim\left(\ker\left(X_i|_{V^i_{\underline{k}}(s)}\right)\right)$ is equal to

$$\begin{cases} 0, & \text{if } s < \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1) \\ \dim\left(V^i_{\underline{k}}(s)\right) - \dim\left(V^i_{\underline{k}}(s+1)\right), & \text{if } s \geqslant \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1). \end{cases}$$

So in using (4) one has

$$\begin{aligned} \dim\left(\ker\left(X_i|_{(\wedge^i\mathfrak{a})_{\underline{k}}}\right)\right) &= \sum_{s=\min(i,\underline{k})}^{\max(i,\underline{k})} \dim\left(\ker\left(X_i|_{V^i_{\underline{k}}(s)}\right)\right) \\ &= \sum_{s=\left\lceil \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1)\right\rceil}^{\max(i,\underline{k})} \left(\dim\left(V^i_{\underline{k}}(s)\right) - \dim\left(V^i_{\underline{k}}(s+1)\right)\right) \\ &= \dim\left(V^i_{\underline{k}}\left(\left\lceil \frac{1}{2}\sum_{l=1}^{t} k_l(n_l+1)\right\rceil\right)\right). \end{aligned}$$

This in conjunction with (3) verifies (5). Theorem 1 follows at once from (5) and the fact that $\dim\left(V^i_{\underline{k}}(s)\right)$ is equal to the cardinality of the set

$$\left\{ \left((\alpha_{1,1},\dots,\alpha_{k_1,1}),\dots,(\alpha_{1,t},\dots,\alpha_{k_t,t}) \right) \in \mathbb{Z}^i \;\middle|\; \begin{array}{c} 1 \leqslant \alpha_{1,l} < \dots < \alpha_{k_l,l} \leqslant n_l \\ \sum \alpha_{k,l} = s \end{array} \right\}.$$

2. Proof of Lemma 2

Results similiar to Lemma 2 are well known in combinatorics. We mention in particular Richard Stanley's school at MIT and the excellent survey papers [**13**] and [**14**]. A standard method to prove a result of this type is to show that X_i appears as an "x-part" of a suitable $\mathfrak{sl}(2,\mathbb{C})$ action. This method has been worked out by Proctor in [**9**] and [**10**] and rests on a simplification of the techniques used by Stanley in [**12**]. This "$\mathfrak{sl}(2,\mathbb{C})$-trick" can also be applied in our situation.

Recall that $\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra over $\mathbb{C}$ with basis $\{x,y,h\}$ and relations $[x,y]=h$, $[h,x]=2x$ and $[h,y]=-2y$. Consider the endomorphisms of $\mathfrak{a}$ defined by

$$Y_1\left(x_j^l\right)=(j-1)(n_l+1-j)x_{j-1}^l,\ H_1\left(x_j^l\right)=(2j-(n_l+1))x_j^l$$

for each $1\leqslant j\leqslant n_l$ and $1\leqslant l\leqslant t$. Denote by Y_i and H_i respectively the endomorphisms of $\wedge^i\mathfrak{a}$ defined by extending Y_1 and H_1 as derivations on $\wedge\mathfrak{a}$. A short calculation verifies that $Y_i\left(\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}\right)\subseteq\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}$ and $H_i\left(\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}\right)\subseteq\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}$.

Lemma 3. *The linear map $\rho_{\underline{k}}^i\colon \mathfrak{sl}(2,\mathbb{C})\to End\left(\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}\right)$ defined by*

$$x\mapsto X_i,\ y\mapsto Y_i,\ h\mapsto H_i$$

is a Lie algebra homomorphism.

Proof: In the case $i=1$ it is a simple matter of verifying the bracket relations

$$[X_1,Y_1]=H_1,\ [H_1,X_1]=2X_1\ \text{ and }\ [H_1,Y_1]=-2Y_1.$$

This induces a representation for $\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}$ in the standard manner, see for instance page 159 of [**6**]. ∎

We can now apply standard results from the representation theory of $\mathfrak{sl}(2,\mathbb{C})$. We follow the presentation given in Humphreys [7]. To begin with, note that

$$H_i(\alpha)=\left(2s-\sum_{l=1}^{t}k_l(n_l+1)\right)\alpha$$

for any $\alpha\in V_{\underline{k}}^i(s)$. Therefore each $V_{\underline{k}}^i(s)\subseteq\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}$ is the weight space corresponding to the weight $2s-\sum\limits_{l=1}^{t}k_l(n_l+1)$ with respect to the action of $h\in\mathfrak{sl}(2,\mathbb{C})$. If we let

$$\left(\wedge^i\mathfrak{a}\right)_{\underline{k}}=\bigoplus_{\lambda\in\Phi}W_{\underline{k}}^i(\lambda)$$

https://doi.org/10.1017/S0004972700021936 Published online by Cambridge University Press

denote the weight decomposition with respect to $\rho^i_{\underline{k}}$, then

$$V^i_{\underline{k}}(s) = W^i_{\underline{k}}\left(2s - \sum_{l=1}^{t} k_l(n_l+1)\right)$$

and

$$W^i_{\underline{k}}(\lambda) = V^i_{\underline{k}}\left(\frac{1}{2}\left(\lambda + \sum_{l=1}^{t} k_l(n_l+1)\right)\right)$$

and for the weights $\lambda \in \Phi$, one has

$$-\sum_{l=1}^{t} k_l(n_l - k_l) \leqslant \lambda \leqslant \sum_{l=1}^{t} k_l(n_l - k_l).$$

As $\mathfrak{sl}(2,\mathbb{C})$ is semisimple it follows that $(\wedge^i \mathfrak{a})_{\underline{k}}$ is a completely reducible $\mathfrak{sl}(2,\mathbb{C})$-module, meaning there is a decomposition

$$(\wedge^i \mathfrak{a})_{\underline{k}} = \bigoplus_{m=1}^{N} U^i_{\underline{k}}(m)$$

with each $U^i_{\underline{k}}(m)$ irreducible, and $\dim\left(U^i_{\underline{k}}(m)\right) = d_m + 1$. The main theorem of the representation theory of $\mathfrak{sl}(2,\mathbb{C})$ gives us the following fact: For every $1 \leqslant m \leqslant N$, there exists $v_m \in W^i_{\underline{k}}(-d_m)$ such that $\left\{v_m, X_i(v_m), \dots, X_i^{d_m}(v_m)\right\}$ is a basis for $U^i_{\underline{k}}(m)$. It follows that $X_i^{d_m}(v_m) \in W^i_{\underline{k}}(d_m)$, and if $-d_m = 2s - \sum k_l(n_l+1)$, then $v_m \in V^i_{\underline{k}}(s)$ and $X_i^{d_m}(v_m) \in V^i_{\underline{k}}\left(\sum_{l=1}^{t} k_l(n_l+1) - s\right)$. We thus obtain:

(a) $\mathfrak{B}_s = \{X_i^l(v_m) \mid 1 \leqslant m \leqslant N,\ 0 \leqslant l \leqslant d_m\} \cap V^i_{\underline{k}}(s)$ is a basis for $V^i_{\underline{k}}(s)$.

(b) If $s < 1/2 \sum_{l=1}^{t} k_l(n_l+1)$, then $X_i(\mathfrak{B}_s) \subseteq \mathfrak{B}_{s+1}$.

(c) If $s \geqslant 1/2 \sum_{l=1}^{t} k_l(n_l+1)$, then $FX_i(\mathfrak{B}_s) \supseteq \mathfrak{B}_{s+1}$.

This finalises the proof of Lemma 2.

3. Proof of Theorem 2

Consider the general partition problem arising in connection with Corollary 1. Given $n, i, d \in \mathbb{N}$, let $K(n,i,d)$ denote the number of partitions of d into i distinct parts, each part being no larger than n. Clearly $K(n,i,d)$ is equal to the cardinality of the set

$$P(n,i,d) = \left\{(\alpha_1, \dots, \alpha_i) \in \mathbb{Z}^i \mid 1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant n \text{ and } \sum \alpha_j = d\right\}.$$

To prove Theorem 2 we use two symmetric unimodal sequences involving the $K(n,i,d)$'s to deduce that the sequence $\{\kappa_i(\mathfrak{f}_n)\}_{i=0}^{n}$ is symmetric unimodal. The theorem is then an immediate consequence of Corollary 1.

Recall that for each $n \in \mathbb{N}$, $\mathfrak{f}_n$ is the Lie algebra with basis $\{x_1, \dots, x_n, z\}$ and non-zero relations $[z, x_j] = x_{j+1}$ for each $1 \leqslant j \leqslant n-1$. Let $\mathfrak{a} = \langle x_1, \dots, x_n \rangle$ and note that $ad(z)|_{\mathfrak{a}}$ has one Jordan block. In this case the decomposition in (2) may be bypassed and (4) implemented directly to obtain

$$\wedge^i \mathfrak{a} = \bigoplus_{s=\min(i)}^{\max(n,i)} V_n^i(s)$$

where $V_n^i(s) = \langle x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_i} \in \wedge^i \mathfrak{a} \mid 1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant n \text{ and } \sum \alpha_j = s \rangle$ and $\min(i) = \binom{i+1}{2}$ and $\max(n,i) = i(n+1) - \min(i)$. In particular Lemma 2 translates to the following:

$$(6) \qquad X_i\, |_{V_n^i(s)} \text{ is } \begin{cases} \text{injective}, & \text{for } s < i(n+1)/2 \\ \text{surjective}, & \text{for } s \geqslant i(n+1)/2. \end{cases}$$

Clearly $\dim(\ker(X_i)) = \dim\left(V_n^i(\lceil i(n+1)/2 \rceil)\right) = \kappa_i(\mathfrak{f}_n)$. It is convenient to extend $K(n,i,d)$ to $i = 0$, by setting

$$K(n,0,d) = \begin{cases} 1, & \text{if } d = 0 \\ 0, & \text{if } d > 0 \end{cases}$$

for all $n \in \mathbb{N}$.

Lemma 4. *For each $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$, the sequence $\{K(n,i,d)\}_{d=\min(i)}^{\max(n,i)}$ is symmetric unimodal.*

Proof: The symmetry comes via the bijection

$$P(n,i,d) \to P(n,i,i(n+1)-d)$$
$$(\alpha_1, \dots, \alpha_i) \mapsto (n+1-\alpha_i, \dots, n+1-\alpha_1).$$

The unimodality follows from (6) since $K(n,i,d) = \dim\left(V_n^i(d)\right)$. ∎

Lemma 5. *For each $n \in \mathbb{N}$ and $d \geqslant 0$, the sequence $\{K(n,i,d+\min(i))\}_{i=0}^{n}$ is symmetric unimodal.*

Proof: The symmetry follows from Lemma 4 and the obvious identity

$$(7) \qquad K(n,i,d) = K\left(n, n-i, \binom{n+1}{2} - d\right).$$

https://doi.org/10.1017/S0004972700021936 Published online by Cambridge University Press

Indeed, for all $0 \leqslant i \leqslant n$, we see that

$$K(n,i,d+\min(i)) = K\left(n, n-i, \binom{n+1}{2} - (d+\min(i))\right), \quad \text{by (7)}$$
$$= K(n, n-i, \min(n-i)+p)$$

where $p = \binom{n+1}{2} - d - \min(i) - \min(n-i)$. Then by the symmetry in Lemma 4 one has

$$K(n,i,d+\min(i)) = K(n,n-i,\max(n,n-i)-p)$$
$$= K(n,n-i,d+\min(n-i)).$$

To verify unimodality it is enough to show that

$$(8) \qquad K(n,i,d) \leqslant K(n,i+1,d+(i+1))$$

for all $1 \leqslant i \leqslant \lfloor (n-2)/2 \rfloor$ and $\min(i) \leqslant d \leqslant \lceil i(n+1)/2 \rceil$. Indeed, using Lemma 4 one can extend (8) to hold for all values of d such that $\min(i) \leqslant d \leqslant \max(n,i)$. For this, one is required to check the cases $\lceil i(n+1)/2 \rceil < d \leqslant \lceil (i+1)(n+1)/2 \rceil - (i+1)$ and $\lceil (i+1)(n+1)/2 \rceil - (i+1) < d \leqslant \max(n,i)$. Treating the first case, one has

$$K(n,i,d) = K(n,i,i(n+1)-d), \quad \text{by symmetry in Lemma 4}$$
$$\leqslant K(n,i+1,(i(n+1)-d)+(i+1)), \quad \text{by (8)}$$
$$\leqslant K(n,i+1,d+(i+1)), \quad \text{by unimodality in Lemma 4.}$$

The remaining case follows in analogous fashion. Note that for convenience we exclude the case $i = 0$ in (8), where the result is obvious. It now remains to verify (8). We use induction on n, starting with the first non empty case n=4: unimodality is easily verified in all cases $n \leqslant 4$. Supposing that (8) is true for $n = k$, we wish to prove that

$$(9) \qquad K(k+1,i,d) \leqslant K(k+1,i+1,d+(i+1))$$

for all $1 \leqslant i \leqslant \lfloor (k-1)/2 \rfloor$ and $\min(i) \leqslant d \leqslant \lceil i(k+2)/2 \rceil$. Let $A(n,i,d)$ denote the set of all elements $(\alpha_1,\ldots,\alpha_i) \in P(n,i,d)$ satisfying $\alpha_i \neq n$, and $B(n,i,d)$ the set of all elements $(\beta_1,\ldots,\beta_i) \in P(n,i,d)$ satisfying $\beta_1 = 1$. Clearly one has the bijection

$$A(n,i,d) \to B(n,i+1,d+(i+1))$$
$$(\alpha_1,\ldots,\alpha_i) \mapsto (1,\alpha_1+1,\ldots,\alpha_i+1).$$

Now observe that there are $K(k,i-1,d-(k+1))$ elements contributing to the left hand side of (9) which do not belong to $A(k+1,i,d)$, and $K(k,i+1,d)$ elements contributing to the right hand side which do not belong to $B\big(k+1, i+1, d+(i+1)\big)$. Thus to verify (9) it is enough to show that

$$K\big(k, i-1, d-(k+1)\big) \leqslant K\big(k, i+1, d\big)$$

for all $i \leqslant \lfloor (k-1)/2 \rfloor$ and $\min(i) \leqslant d \leqslant \lceil i(k+2)/2 \rceil$. It is true that $i-1 \leqslant \lfloor (k-2)/2 \rfloor$ and $d-(k+1) \leqslant \lceil (i-1)(k+1)/2 \rceil$, meaning one can apply the inductive assumption to give

$$K\big(k, i-1, d-(k+1)\big) \leqslant K\big(k, i, d-(k+1)+i\big). \tag{10}$$

Now we wish to apply the inductive assumption a second time to the right hand side of (10). One has $d-(k+1)+i \leqslant \lceil i(k+1)/2 \rceil$, but the hypothesis $i \leqslant \lfloor (k-2)/2 \rfloor$ is violated when $i = \lfloor (k-1)/2 \rfloor$ in the case k is odd. Supposing that $i < \lfloor (k-1)/2 \rfloor$ or k even, we can use the inductive assumption to give

$$K\big(k, i, d-(k+1)+i\big) \leqslant K\big(k, i+1, d-(k+1)+2i+1\big). \tag{11}$$

Otherwise in the case $i = \lfloor (k-1)/2 \rfloor$ for k odd, one establishes (11) by noting that

$$K\big(k, i, d-(k+1)+i\big) = K\big(k, i+1, d-(k+1)+2i+1\big)$$

by the symmetry in the sequence $\big\{K\big(n, i, d+\min(i)\big)\big\}_{i=0}^{n}$ which was verified at the beginning of the proof. Finally since $d-k+2i \leqslant d \leqslant \lceil (i+1)(k+1)/2 \rceil$, Lemma 4 implies

$$K\big(k, i+1, d-(k+1)+2i+1\big) \leqslant K\big(k, i+1, d\big)$$

which completes the proof. ∎

Lemmas 4 and 5 combine to give the following:

Corollary 3. *For each $n \in \mathbb{N}$, the sequence $\big\{\kappa_i(\mathfrak{f}_n)\big\}_{i=0}^{n}$ is symmetric unimodal.*

Proof: The symmetry follows directly from identity (7), and uses the symmetry of Lemma 4 in case $i(n+1)/2$ is a non-integer. The unimodality is an immediate consequence of Lemmas 4 and 5. ∎

Theorem 2 now follows at once from Corollary 1.

Remarks. For all nilpotent Lie algebras L of dimension $\leqslant 7$, the sequence $\big\{b_i(L)\big\}_{i=0}^{\dim(L)}$ is unimodal, see for instance [4]. However unimodality is not a property shared by nilpotent Lie algebras in general, see for instance [2]. Nevertheless one may ask whether it is a property shared by all nilpotent Lie algebras containing an Abelian ideal of codimension one. Our computer experiments have verified that this is true for all such algebras of dimension $\leqslant 100$.

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https://doi.org/10.1017/S0004972700021936 Published online by Cambridge University Press