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ON CERTAIN CLASSES OF BOUNDED LINEAR OPERATORS

BY
C-S LIN

1. Let $T-c$ be a Fredholm operator, where T is a bounded linear operator on a complex Banach space and c is a scalar, the set of all such scalars is called the Φ-set of T [2] and was studied by many authors. In this connection, the purpose of the present paper is to investigate some classes $\Phi(V)$ of all such operators for any subset V of the complex plane.

2. Let X be a Banach space over the field C of complex numbers with dim $X=\infty$, unless otherwise stated, $B(X)$ the Banach algebra of all bounded linear operators and $K(X)$ the closed two-sided ideal of all compact operators on X. As usual, $T\in B(X)$ is said to be a Fredholm operator if both the dimension of the null space of T and the codimension of the range of T are finite, and is said to be a Riesz operator if $T-c$ is a Fredholm operator for every nonzero scalar c [1]. We shall write $\Phi(V)=\{T\in B(X): T-c \text{ is a Fredholm operator}, \forall c\in V\}$, where V is a proper subset of C. Thus the set of all Fredholm operators is $\Phi(\{0\})$, and $\Phi(C\backslash\{0\})$ the set of all Riesz operators. Clearly every nonzero scalar is a Fredholm operator, and if $c\in C$, $c\notin V$ iff $c\in\Phi(V)$. We shall write $\Phi(\phi)=B(X)$, where ϕ is the empty set and this expression is justifiable by

THEOREM 1. *If V and W are proper subsets of C, $V\subseteq W$ iff $\Phi(W)\subseteq\Phi(V)$.*

Proof. Let $V\subseteq W$ and $T\in\Phi(W)$, then $T-c\in\Phi(\{0\})$ for every $c\in W$, and hence for every $c\in V$, $T\in\Phi(V)$. Conversely, if $V\nsubseteq W$, then there is a $c\in V$ with $c\notin W$. Thus $c\notin\Phi(V)$ and $c\in\Phi(W)$, $\Phi(W)\nsubseteq\Phi(V)$.

Let $T\in B(X)$, we shall denote by π the canonical homomorphism of $B(X)$ onto the (quotient) Banach algebra $B(X)/K(X)$, $\sigma(T)$ and $\rho(T)$ (resp. $\sigma(\pi(T))$ and $\rho(\pi(T))$) the spectrum and the resolvent set of T (resp. $\pi(T)$). A characterization of the Fredholm operators due to F. V. Atkinson says that $T\in\Phi(\{0\})$ iff $\pi(T)$ is invertible in $B(X)/K(X)$. In this case, let $\pi(\bar{T})$ be its inverse.

LEMMA 1. *Let W be a proper subset of C, $S\in B(X)$, $T\in\Phi(\{0\})$ and $\bar{T}$ as stated above. Then $S-cT\in\Phi(\{0\})$ for every $c\in W$, iff $S\bar{T}\in\Phi(W)$.*

Proof. Let $c\in W$. $S-cT\in\Phi(\{0\})$, iff $\pi(S-cT)$ is invertible, iff $\pi((S-cT)\bar{T}) =\pi(S-cT)\pi(\bar{T})$ is invertible, iff $\pi(S\bar{T}-c)$ is invertible, iff $S\bar{T}\in\Phi(\{c\})$.

REMARK 1. Notation as in Lemma 1, we see that $S\bar{T}\in\Phi(W)$ iff $\bar{T}S\in\Phi(W)$. Also $S-c\bar{T}\in\Phi(\{0\})$ for every $c\in W$, iff $ST\in\Phi(W)$. In order to see what the set

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$\Phi(C)$ is, we shall give a simple proof of Theorem 3.2 [2] and of its converse as well.

LEMMA 2. (*Theorem* 3.2 [2]): $\Phi(C)\neq\phi$ *iff* dim $X<\infty$.

Proof. $T\in\Phi(C)$, iff $\pi(T-c)=\pi(T)-c$ is invertible for every $c\in C$, iff $\sigma(\pi(T))=\phi$, iff $K(X)=B(X)$, iff the identity operator on X is compact, iff dim $X<\infty$.

REMARK 2. S and $T\in\Phi(W)$ iff $W\subseteq\rho(\pi(S))\cap\rho(\pi(T))$. The so-called Gelfand–Mazur theorem says that if in a complex Banach algebra with unit element A every nonzero element is invertible, then A is one dimensional. In the case of $B(X)$, we have the following more general statement.

THEOREM 2. *$B(X)=C$, iff $B(X)\backslash\{c\}\subseteq\Phi(\{c\})$ for any $c\in C$.*

Proof. The "only if," part is clear. To show the "if" part, let $T\in B(X)$, then there exists $b\in C$ such that $T\notin\Phi(\{b\})$ by Lemma 2. Thus $T-b+c\notin\Phi(\{c\})$, $T-b+c=c$ by assumption, $T=b$ and hence $B(X)=C$.

LEMMA 3. *If V and W are any subsets of C, then*

(1) $\Phi(V)\cap\Phi(W)=\Phi(V\cup W)$.

(2) $\Phi(V)\cup\Phi(W)\subseteq\Phi(V\cap W)$. Equality holds if either $V\subseteq W$ or $W\subseteq V$.

The proof follows easily and may be omitted. The opposite inclusion relation in (2) is not valid in general. In order to show this it suffices to take a linear bounded operator $T\notin\Phi(\{0\})$ with finite dimensional null space and such that its range be closed and of infinite codimension. By Theorem 7.1 [2] then there exists a number $b>0$ such that for every $S\in B(X)$ with $\|S\|<b$, $T+S\notin\Phi(\{0\})$. Thus, for $c_0\neq0$ with $|c_0|<b$, one obtains $T\notin\Phi(\{c_0\})$. Accordingly

$$\Phi(\{0\})\cup\Phi(\{c_0\})\subsetneqq\Phi(\{0\}\cap\{c_0\})=\Phi(\phi)=B(X).$$

REMARK 3. Since $\Phi(C\backslash\{0\})\cap\Phi(\{0\})=\Phi(C)$, a Riesz (resp. a Fredholm) operator is a Fredholm (resp. a Riesz) operator iff X is of finite dimension.

COROLLARY 1. *If X is of infinite dimension and $L=\{\Phi(V):V\subseteq C\}$, then the system $\{L,\cap,\subseteq\}$ is a complete and complemented lower semilattice with respect to the set intersection and inclusion relation, and $\Phi(C)=\phi$ is the smallest element in L. Moreover, X is of finite dimension iff the system $\{L,\cap,\cup,\subseteq\}$ is the lattice with only two elements, $B(X)$ and ϕ.*

The proof follows easily and may be omitted. The lower semilattice is atomic, since each element $\Phi(C\backslash\{c\})$ covers $\Phi(C)$.

THEOREM 3 (1). *If W is a nonempty subset of C and $T\in\Phi(C\backslash W)$, then there is a nonempty subset $V\subseteq W$ such that $V\subseteq\sigma(\pi(T))\subseteq\sigma(T)$.*

(2) *If W is a subset of C with $W\supseteq\sigma(T)$, then $T\in\Phi(C\backslash W)$.*

Proof. (1) $T\notin\Phi(W)$, since otherwise dim $X<\infty$. Hence there is a nonempty subset $V \subseteq W$ such that $\pi(T-c)=\pi(T)-c$ is not invertible for every $c \in V$, $V \subseteq \sigma(\pi(T))$. $\rho(T) \subseteq \rho(\pi(T))$, since π carries an invertible element into an invertible element. (2) By the last argument, $T \in \Phi(\rho(T))$ and $T \notin \Phi(\sigma(T))$ for any $T \in B(X)$.

REMARK 4. If $c \in \rho(T)$, then $T-c$ and $(T-c)^{-1} \in \Phi(\{0\})$. Thus if either $T-b$ or $(T-d)^{-1}$ is a Riesz operator for some b or d in $\rho(T)$, then $\dim X < \infty$.

REMARK 5. Some direct consequences of Theorem 3 are: the spectrum of a Riesz operator contains the zero, $\sigma(T) \neq \phi$ for any $T \in B(X)$, and every quasi-nilpotent operator is a Riesz operator.

REMARK 6. By Lemma 3 and Theorem 3, if $\Phi(W) \subseteq \Phi(V)$, $T \in \Phi(V)$ and $C \setminus (W \setminus V) \supseteq \sigma(T)$, then $T \in \Phi(W)$.

3. Let $T \in B(X)$ and $r(T)$ be its lower bound [3]. It is known that the range of T is closed iff $r(T) > 0$. $T \in \Phi(\{0\})$ implies $r(T) > 0$. Also if $T \in \Phi(\{0\})$, $S \in B(X)$ and $\|S\| < r(T)$, then $T+S \in \Phi(\{0\})$.

REMARK 7. Clearly $T \in \Phi(\{c\})$ if $\|T\| < |c| = r(c)$. This condition may be weakened by that $\|\pi(T)\| < |c|$, because in this case T can be written as $T=S+A$, where $S \in B(X)$, $\|S\| < |c|$ and $A \in K(X)$, and since $\pi(T-c)=\pi(S+A-c)=\pi(S-c)$ is invertible, $T \in \Phi(\{c\})$.

LEMMA 4. *If W is a finite subset of C, then $\Phi(W)$ is an open subset of $B(X)$.*

Proof. Let $T \in \Phi(W)$ and $r(T-b)=\min\{r(T-c): c \in W\} \neq 0$. if $S \in B(X)$ is such that $\|\pi(S-T)\| < r(T-b)$, then $S-b=(S-T)+(T-b) \in \Phi(\{0\})$ and hence $S \in \Phi(W)$.

LEMMA 5. *If b and c are nonzero scalars and d is any scalar, then*

$$b\Phi(\{c\}) = c\Phi(\{b\}) \quad \text{and} \quad b\Phi(\{d\}) = \Phi(\{bd\}).$$

In particular, $d\Phi(C \setminus \{0\}) = \Phi(C \setminus \{0\})$.

Proof. $T \in b\Phi(\{c\})$, iff $T/b - c \in \Phi(\{0\})$, iff $T/c - b \in \Phi(\{0\})$, iff $T \in c\Phi(\{b\})$, and hence $b\Phi(\{c\}) = c\Phi(\{b\})$. The remainder of the proof follows similarly.

LEMMA 6. *Let $W \subsetneq C$, $T \in \Phi(C \setminus \{0\})$, $S \in \Phi(W)$ and $TS-ST \in K(X)$, then $T+S \in \Phi(W)$. Moreover, TS and $ST \in \Phi(V)$ for any subset $V \subseteq C \setminus \{0\}$.*

Proof. Let $c \in W$, $T(S-c)-(S-c)T = TS-ST \in K(X)$, then $T+S-c \in \Phi(\{0\})$ [4, Theorem 9]. But $c \in W$ was arbitrary, $T+S \in \Phi(W)$. TS and $ST \in \Phi(C \setminus \{0\})$ [4, Lemma 5].

Let $W \subsetneq C$ and $Y(W) = \{T \in \Phi(C \setminus \{0\}): TS-ST \in K(X), \forall S \in \Phi(W)\}$, then, $W_0 \subseteq W_1$ implies $Y(W_0) \subseteq Y(W_1)$, and $Y(\{c\}) = Y(\{0\})$ due to a simple fact that $\Phi(\{0\}) = \{T-c: \forall T \in \Phi(\{c\})\}$ and $\Phi(\{c\}) = \{T+c: \forall T \in \Phi(\{0\})\}$.

LEMMA 7. *If $W \subsetneq C$, then $Y(W)$ is a linear manifold of $B(X)$ such that $K(X) \subseteq Y(W)$. Moreover, $Y(W)$ is closed under multiplication.*

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Proof. Clearly $K(X) \subseteq Y(W)$. To show the closedness under addition, let T and $T' \in Y(W)$ and $S \in \Phi(W)$, then $T+S \in \Phi(W)$. $T'(T+S)-(T+S)T' \in K(X)$, i.e. $(T'T-TT')+(T'S-ST') \in K(X)$ and hence $T'T-TT' \in K(X)$. Thus

$$T+T' \in \Phi(C\backslash\{0\}), \qquad (T+T')S-S(T+T') = (TS-ST)+(T'S-ST') \in K(X)$$

for every $S \in \Phi(W)$, $T+T' \in Y(W)$. Now since $T'T-TT' \in K(X)$, TT' and $T'T \in \Phi(C\backslash\{0\})$. $TT'S-STT' = T(T'S-ST')-(ST-TS)T' \in K(X)$ for every $S \in \Phi(W)$. Hence $TT' \in Y(W)$, and $T'T \in Y(W)$ follows similarly.

It was proved in [1] that if $\{T_n\}$ is a sequence in $\Phi(C\backslash\{0\})$ and $T_n \to T$ in $B(X)$, where $T_nT=TT_n$ for all sufficiently large n, then $T \in \Phi(C\backslash\{0\})$. The next theorem extends this result.

REMARK 8. $T \in \Phi(C\backslash\{d\})$ iff $T-d \in \Phi(C\backslash\{0\})$.

THEOREM 4. *Let $\{T_n\}$ be a sequence in $\Phi(C\backslash\{d\})$ and $T_n \to T$ convergence in norm with $T \in B(X)$. If $TT_n-T_nT \in K(X)$ for all sufficiently large n, then $T \in \Phi(C\backslash\{d\})$.*

Proof. For a nonzero scalar c there is a sufficiently large n such that $\|T-T_n\| < r(c)$. Hence $T-T_n \in \Phi(\{c\})$. But $T_n-d \in \Phi(C\backslash\{0\})$ for every n, and

$$(T-T_n)(T_n-d)-(T_n-d)(T-T_n) = TT_n-T_nT \in K(X),$$
$$T-d = (T-T_n)+(T_n-d) \in \Phi(\{c\})$$

by Lemma 6. But $c \neq 0$ was arbitrary, $T-d \in \Phi(C\backslash\{0\})$ and hence $T \in \Phi(C\backslash\{d\})$.

We may apply the same method to prove

COROLLARY 2. *Let $\{T_n\}$ be a sequence in $B(X)$ and $T \in B(X)$ with $T \in \Phi(C\backslash\{d\})$. If $T_n \to T$ convergence in norm and $TT_n-T_nT \in K(X)$ for all sufficiently large n, then $T_n \in \Phi(C\backslash\{d\})$ for all such n.*

THEOREM 5. *$Y(\phi)$ and $Y(\{c\})$ are Banach algebras.*

Proof. In virtue of Theorem 4 and the fact that $T_nS-ST_n \to TS-ST$ for every $S \in \Phi(\phi)$ provided $T_n \to T$, $Y(\phi)$ is closed. By Lemma 7, $Y(\phi)$ is a Banach algebra with the same norm as in $B(X)$. To show the second part, let $T_n \to T$ in $B(X)$, then $T-b=T' \in \Phi(\{0\})$ for some $b \neq 0$ by Remark 7.

$$\begin{aligned} T_nT-TT_n &= T_n(T'+b)-(T'+b)T_n = T_nT'-T'T_n \\ &= T_n(T'+c)-(T'+c)T_n \in K(X). \end{aligned}$$

Hence $T \in \Phi(C\backslash\{0\})$ by Theorem 4. The remainder of the proof follows as above.

REMARK 9. T is a Fredholm operator, iff the adjoint T^* of T is a Fredholm operator [3]. Hence if $V \subseteq C$, and since $(T-c)^* = T^*-c$ is the Banach space adjoint of $T-c$, we have $T \in \Phi(V)$ iff $T^* \in \Phi(V)$. Thus all above statements and proofs are true if we are dealing with the adjoint space and adjoint operators.

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Queen's University,
Kingston, Ontario