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Approximate Fixed Point Sequences of Nonlinear Semigroups in Metric Spaces

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Abstract. In this paper, we investigate the common approximate fixed point sequences of nonexpansive semigroups of nonlinear mappings $\{T_t\}_{t\geq 0}$, *i.e.,* a family such that $T_0(x) = x$, $T_{s+t} = T_s(T_t(x))$, where the domain is a metric space (M, d). In particular, we prove that under suitable conditions the common approximate fixed point sequences set is the same as the common approximate fixed point sequences set of two mappings from the family. Then we use the Ishikawa iteration to construct a common approximate fixed point sequence of nonexpansive semigroups of nonlinear mappings.

1 Introduction

The purpose of this paper is to prove the existence of approximate fixed points for semigroups of nonlinear mappings acting in metric spaces. Note that from a numerical point of view, approximate fixed points are very useful, since exact fixed points may be hard to find. We will also give an algorithm of how to build such approximate fixed points in the case of hyperbolic metric spaces. Let us recall that a family $\{T_t\}_{t\geq 0}$ of mappings forms a semigroup if $T_0(x) = x$ and $T_{s+t} = T_s \circ T_t$. Such a situation is quite typical in mathematics and applications. For instance, in the theory of dynamical systems, the vector function space would define the state space, and the mapping $(t, x) \to T_t(x)$ would represent the evolution function of a dynamical system. The question about the existence of common fixed points and about the structure of the set of common fixed points can be interpreted as asking whether there exist points that are fixed during the state space transformation T_t at any given point of time t, and if yes, what does the structure of a set of such points look like. In the setting of this paper, the state space is a nonlinear metric space.

The existence of common fixed points for families of contractions and nonexpansive mappings in Banach spaces has been the subject of intense research since the early 1960s, as investigated by Belluce and Kirk [1, 2], Browder [3], Bruck [4], DeMarr [8], and Lim [19]. It is worthwhile mentioning the recent studies on the special case, when the parameter set for the semigroup is equal to $\{0, 1, 2, 3, \dots\}$, and $T_n = T^n$, the n-th iterate of an asymptotic pointwise nonexpansive mapping.

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Kirk and Xu [17] proved the existence of fixed points for asymptotic pointwise contractions and asymptotic pointwise nonexpansive mappings in Banach spaces, while Hussain and Khamsi [12] extended this result to metric spaces.

2 Main Results

Recall the definition of a nonexpansive mapping defined in a metric space.

Definition 2.1 Let (M,d) be a metric space and $C \subset M$ be a nonempty subset. A mapping $T\colon C \to M$ is said to be *nonexpansive* if

$$d(T(x), T(y)) \leq d(x, y)$$

for any $x, y \in C$. A point $x \in C$ is called a *fixed point* of T if $T(x) = x$. The set of fixed points of T will be denoted by $\operatorname{Fix}(T)$. A sequence $\{x_n\}$ in C is called an *approximate fixed point sequence* of T if $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. The set of approximate point sequences of T will be denoted by $AFPS(T)$.

Recall that in Banach spaces a nonexpansive mapping defined on a nonempty closed bounded convex subset has an approximate fixed point sequence and may not have a fixed point. This definition is now extended to a one parameter family of mappings.

Definition 2.2 Let (M,d) be a metric space and $C \subset M$ be a nonempty subset. A one-parameter family $\mathcal{F} = \{T_t\ ;\ t \geq 0\}$ of mappings from C into itself is said to be a *nonexpansive semigroup* on C if $\mathcal{F}$ satisfies the following conditions:

(i) $T_0(x) = x$ for $x \in C$;
(ii) $T_{t+s}(x) = T_t(T_s(x))$ for $x \in C$ and $t, s \in [0,\infty)$;
(iii) for each $t \geq 0$, T_t is a nonexpansive mapping.

Define the set of all common fixed points of $\mathcal{F}$ as $\operatorname{Fix}(\mathcal{F}) = \bigcap_{t\geq 0} \operatorname{Fix}(T_t)$. Similarly, define the set of approximate point sequences of $\mathcal{F}$, denoted by $AFPS(\mathcal{F})$, as

$$AFPS(\mathcal{F}) = \bigcap_{t\geq 0} AFPS(T_t).$$

The concept of continuity for semigroups of mappings is important. Next we give the definitions that will be needed throughout.

Definition 2.3 Let (M,d) be a metric space and $C \subset M$ be nonempty. A one-parameter family $\mathcal{F} = \{T_t\ ;\ t \geq 0\}$ of mappings from C into M is said to be:

(i) *continuous* on C if for any $x \in C$, the mapping $t \to T_t(x)$ is continuous, *i.e.*, for any $t_0 \geq 0$, we have $\lim_{t\to t_0} d\big(T_t(x), T_{t_0}(x)\big) = 0$, for any $x \in C$;
(ii) *strongly continuous* on C if for any bounded nonempty subset $K \subset C$, we have

$$\lim_{t\to t_0} \sup_{x\in K} \big(d(T_t(x), T_{t_0}(x))\big) = 0.$$

Recall the following lemma, which can be found in any introductory course on real analysis.

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Lemma 2.4 ([23]) *Let G be a nonempty additive subgroup of $\mathbb{R}$. Then G is either dense in $\mathbb{R}$ or there exists $a > 0$ such that $G = a \cdot \mathbb{Z} = \{an ; n \in \mathbb{Z}\}$. Therefore, if α and β are two real numbers such that $\frac{\alpha}{\beta}$ is irrational, then the set*

$$G(\alpha, \beta) = \{\alpha n + \beta m ; n, m \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. In particular, the set $G_+(\alpha, \beta) = G(\alpha, \beta) \cap [0, +\infty)$ is dense in $[0, +\infty)$.

The following technical lemmas will be useful to prove the main result of this section.

Lemma 2.5 *Let (M, d) be a metric space. Let C be a nonempty subset of M. Let $T \colon C \to C$ be a nonexpansive mapping. Then we have $AFPS(T) \subset AFPS(T^m)$ for any $m \geq 2$.*

Proof Without loss of generality we may assume that $AFPS(T)$ is not empty. Let $\{x_n\} \in AFPS(T)$. Then we have $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. Fix $m \geq 2$. Then we have

$$d(T^m(x_n), x_n) \leq \sum_{k=1}^{m} d(T^k(x_n), T^{k-1}(x_n)) \leq md(T(x_n), x_n)$$

for any $n \geq 1$. Since m is fixed and $\{x_n\} \in AFPS(T)$, we get

$$\lim_{n\to\infty} d(T^m(x_n), x_n) = 0,$$

i.e., $\{x_n\} \in AFPS(T^m)$. ■

Lemma 2.6 *Let (M, d) be a metric space and $C \subset M$ be nonempty. Let $\mathcal{F} = \{T_t ; t \geq 0\}$ be a one-parameter nonexpansive semigroup of mappings from C into C. Let α and β be two positive real numbers. Then we have*

$$AFPS(T_\alpha) \cap AFPS(T_\beta) \subset \bigcap_{t \in G_+(\alpha,\beta)} AFPS(T_t).$$

Proof Without loss of generality we may assume that $AFPS(T_\alpha) \cap AFPS(T_\beta)$ is not empty. Let $\{x_n\} \in AFPS(T_\alpha) \cap AFPS(T_\beta)$. Recall that

$$G_+(\alpha, \beta) = \{m\alpha + k\beta \geq 0 ; m, k \in \mathbb{Z}\}.$$

Let $t \in G_+(\alpha, \beta)$. Then we have two cases. First assume that $t = m\alpha + k\beta$, where $m, k \geq 0$. Then

$$d(T_t(x_n), x_n) = d\big(T_{m\alpha+k\beta}(x_n), x_n\big) = d\big(T_\alpha^m(T_\beta^k(x_n)), x_n\big),$$

which implies that

$$\begin{aligned} d\big(T_t(x_n), x_n\big) &\leq d\big(T_\alpha^m(T_\beta^k(x_n)), T_\alpha^m(x_n)\big) + d\big(T_\alpha^m(x_n)), x_n\big) \\ &\leq d\big(T_\beta^k(x_n), x_n\big) + d\big(T_\alpha^m(x_n)), x_n\big), \end{aligned}$$

for any $n \geq 1$. Using Lemma 2.5 , we get $\lim_{n\to\infty} d(T_t(x_n), x_n) = 0$, *i.e.*, $\{x_n\} \in AFPS(T_t)$. Next assume that $t = m\alpha + k\beta$, where either m or k is negative. Without loss of generality assume that $t = m\alpha - k\beta$, where $m, k \geq 0$. We have

$$d(T_t(x_n), x_n) = d(T_{m\alpha-k\beta}(x_n), x_n) \leq d(T_{m\alpha-k\beta}(x_n), T_{m\alpha}(x_n)) + d(T_{m\alpha}(x_n), x_n).$$

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Since $T_{m\alpha} = T_{m\alpha-k\beta} \circ T_{k\beta}$, and $T_{m\alpha-k\beta}$ is nonexpansive, we get

$$d(T_t(x_n), x_n) \le d(x_n, T_{k\beta}(x_n)) + d(T_{m\alpha}(x_n), x_n) = d(x_n, T^k_\beta(x_n)) + d(T^m_\alpha(x_n), x_n)$$

for any $n \ge 1$. Again using Lemma 2.5, we get $\lim_{n\to\infty} d(T_t(x_n), x_n) = 0$, *i.e.*, $\{x_n\} \in AFPS(T_t)$. Hence

$$\{x_n\} \in \bigcap_{t\in G_+(\alpha,\beta)} AFPS(T_t). \qquad \blacksquare$$

Now we are ready to give the main result of this section.

Theorem 2.7 *Let (M, d) be a metric space and $C \subset M$ be nonempty and bounded. Let $\mathcal{F} = \{T_t\,;\, t \ge 0\}$ be a one-parameter nonexpansive semigroup of mappings from C into C. Assume that $\mathcal{F}$ is strongly continuous. Let α and β be two positive real numbers such that $\frac{\alpha}{\beta}$ is irrational. Then we have*

$$AFPS(T_\alpha) \cap AFPS(T_\beta) = AFPS(\mathcal{F}).$$

Proof Since $AFPS(\mathcal{F}) \subset AFPS(T_\alpha) \cap AFPS(T_\beta)$, it is enough to prove $AFPS(T_\alpha) \cap AFPS(T_\beta) \subset AFPS(\mathcal{F})$. Without loss of generality, assume that $AFPS(T_\alpha) \cap AFPS(T_\beta)$ is not empty. Let $\{x_n\} \in AFPS(T_\alpha) \cap AFPS(T_\beta)$. Lemma 2.6 implies that

$$\{x_n\} \in \bigcap_{t\in G_+(\alpha,\beta)} AFPS(T_t).$$

From Lemma 2.4, we know that $G_+(\alpha,\beta) = G(\alpha,\beta) \cap [0,+\infty)$ is dense in $[0,+\infty)$. Let $t \in [0,+\infty)$. Then there exists $t_m \in G_+(\alpha,\beta)$, $m \ge 1$, such that $\lim_{m\to\infty} t_m = t$. We have

$$\begin{aligned} d\big(x_n, T_t(x_n)\big) &\le d\big(x_n, T_m(x_n)\big) + d\big(T_m(x_n), T_t(x_n)\big) \\ &\le d\big(x_n, T_m(x_n)\big) + \sup_{x\in C} d\big(T_m(x), T_t(x)\big). \end{aligned}$$

Let $\varepsilon > 0$. Since $\mathcal{F}$ is strongly continuous, there exists $m_0 \ge 1$ such that for any $m \ge m_0$, we have

$$\sup_{x\in C} d\big(T_m(x), T_t(x)\big) < \varepsilon.$$

Since $\{x_n\} \in AFPS(T_{m_0})$ from Lemma 2.6, there exists $n_0 \ge 1$ such that $d(x_n, T_{m_0}(x_n)) < \varepsilon$, for any $n \ge n_0$. Hence

$$d\big(x_n, T_t(x_n)\big) \le d\big(x_n, T_{m_0}(x_n)\big) + \sup_{x\in C} d\big(T_{m_0}(x), T_t(x)\big) < 2\varepsilon$$

for any $n \ge n_0$. Since ε was arbitrarily positive, we conclude that

$$\lim_{n\to\infty} d(T_t(x_n), x_n) = 0, \quad \textit{i.e.}, \{x_n\} \in AFPS(T_t). \qquad \blacksquare$$

As a corollary we get the following.

Corollary 2.8 *Let (M, d) be a metric space and $C \subset M$ be nonempty and bounded. Let $\mathcal{F} = \{T_t\,;\, t \ge 0\}$ be a one-parameter nonexpansive semigroup of mappings from C into C. Assume that $\mathcal{F}$ is strongly continuous. Then we have*

$$AFPS(T_1) \cap AFPS(T_\pi) = AFPS(T_1) \cap AFPS(T_{\sqrt{2}}) = AFPS(\mathcal{F}).$$

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In the next section, we give an algorithm of how to construct an approximate fixed point sequence of two maps in a metric space.

3 Common Approximate Fixed Point Sequence of Two Nonexpansive Mappings

In this section we will discuss a construction of a common approximate fixed point sequence of two nonexpansive mappings defined on a hyperbolic metric space as defined by Reich and Shafrir [21] (see also [10]). This class of metric spaces includes all normed linear spaces that are hyperbolic spaces. As nonlinear examples, one can consider the Hadamard manifolds [5], the Hilbert open unit ball equipped with the hyperbolic metric [10], and the CAT(0) spaces [14–16, 18].

Let (M, d) be a metric space. Suppose that there exists a family $\mathcal{S}$ of metric segments such that any two points x, y in M are endpoints of a unique metric segment $[x, y] \in \mathcal{S}$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x, y]$ that satisfies

$$d(x, z) = \beta d(x, y), \quad \text{and} \quad d(z, y) = (1-\beta) d(x, y).$$

Such metric spaces are usually called *convex metric spaces* [20]. Moreover, if we have

$$d\big(\beta p \oplus (1-\beta)x, \beta p \oplus (1-\beta)y\big) \le (1-\beta) d(x, y)$$

for all p, x, y in M, and $\beta \in [0, 1]$, then M is said to be a *hyperbolic metric space* (see [21]).

Definition 3.1 Let (M, d) be a hyperbolic metric space. We say that M is *uniformly convex* (UC) if for any $a \in M$, for every $r > 0$, and for each $\epsilon > 0$

$$\delta(r, \varepsilon) = \inf\Big\{ 1 - \frac{1}{r} d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big) \ ; d(x, a) \le r, d(y, a) \le r, d(x, y) \ge r\varepsilon \Big\} > 0.$$

The definition of uniform convexity finds its origin in Banach spaces [6]. To the best of our knowledge, the first attempt to generalize this concept to metric spaces was made in [11]. The reader may also consult [10, 13, 21].

From now on we assume that M is a hyperbolic metric space, and if (M, d) is uniformly convex, then for every $s \ge 0$, $\epsilon > 0$, there exists $\eta(s, \epsilon) > 0$ depending on s and ϵ such that

$$\delta(r, \varepsilon) > \eta(s, \epsilon) > 0 \text{ for any } r > s.$$

The following technical lemmas will be useful throughout.

Lemma 3.2 ([9, 22]) *Let X be a uniformly convex hyperbolic space. Then for arbitrary positive numbers $\varepsilon > 0$ and $r > 0$, and $\alpha \in [0, 1]$, we have*

$$d\big(a, \alpha x \oplus (1-\alpha) y\big) \le r\big(1 - \delta(r, 2\min\{\alpha, 1-\alpha\}\varepsilon)\big),$$

for all $a, x, y \in X$, such that $d(z, x) \le r$, $d(z, y) \le r$, and $d(x, y) \ge r\varepsilon$.

Using the above lemma we obtain the following result.

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Lemma 3.3 ([9, 13]) *Let (M,d) be a uniformly convex hyperbolic metric space. Assume that there exists $R \in [0,+\infty)$ such that*

$$\begin{cases} \limsup_{n\to\infty} d(x_n,a) \le R, \\ \limsup_{n\to\infty} d(y_n,a) \le R, \\ \lim_{n\to\infty} d(a,\sigma_n x_n \oplus (1-\sigma_n)y_n) = R, \end{cases}$$

where $\sigma_n \in [\alpha,\beta]$, with $0<\alpha\le\beta<1$. Then $\lim_{n\to\infty} d(x_n,y_n)=0$.

Let (M,d) be a hyperbolic metric space and $C \subset M$ be a nonempty convex subset. Let $T,S\colon C\to C$ be two mappings. Fix $x_1 \in C$. Das and Debata [7] studied the strong convergence of Ishikawa iterates $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n S\big(\beta_n T(x_n) \oplus (1-\beta_n)x_n\big) \oplus (1-\alpha_n)x_n \tag{3.1}$$

where $\alpha_n, \beta_n \in [0,1]$. Under suitable assumptions, we will show that $\{x_n\}$ is an approximate fixed point sequence of both T and S. Assume that T and S are nonexpansive and have a common fixed point $p \in C$. Then we have

$$\begin{aligned} d(x_{n+1},p) &= d\big(\alpha_n S(y_n) \oplus (1-\alpha_n)x_n, p\big) \\ &\le \alpha_n d\big(S(y_n),p\big) + (1-\alpha_n)d(x_n,p) \\ &\le \alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p) \\ &= \alpha_n d\big(\beta_n T(x_n) \oplus (1-\beta_n)x_n, p\big) + (1-\alpha_n)d(x_n,p) \\ &\le \alpha_n\big[\beta_n d(T(x_n),p) + (1-\beta n)d(x_n,p)\big] + (1-\alpha_n)d(x_n,p) \\ &\le d(x_n,p), \end{aligned}$$

where $y_n = \beta_n T(x_n) \oplus (1-\beta_n)x_n$. This proves that $\{d(x_n,p)\}$ is decreasing, which implies that $\lim_{n\to\infty} d(x_n,p)$ exists. Using the above inequalities, we get

$$\begin{aligned} \lim_{n\to\infty} d(x_n,p) &= \lim_{n\to\infty} d\big(\alpha_n Sy_n \oplus (1-\alpha_n)x_n, p\big) \\ &= \lim_{n\to\infty} \big[\alpha_n d(Sy_n,p) + (1-\alpha_n)d(x_n,p)\big] \\ &= \lim_{n\to\infty} \big[\alpha_n d(y_n,p) + (1-\alpha_n)d(x_n,p)\big] \\ &= \lim_{n\to\infty} \Big[\alpha_n d\big(\beta_n T(x_n) \oplus (1-\beta_n)x_n, p\big) + (1-\alpha_n)d(x_n,p)\Big] \\ &= \lim_{n\to\infty} \Big[\alpha_n\big(\beta_n d(T(x_n),p) + (1-\beta_n)d(x_n,p)\big) + (1-\alpha_n)d(x_n,p)\Big]. \end{aligned}$$

The following result is the main theorem of this section.

Theorem 3.4 *Let C be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space (M,d). Let $S,T\colon C\to C$ be nonexpansive mappings such that $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\neq\varnothing$. Fix $x_1\in C$ and generate $\{x_n\}$ by (3.1). Assume that $\alpha_n,\beta_n\in[\alpha,\beta]$, with $0<\alpha\le\beta<1$, then*

$$\lim_{n\to\infty} d\big(x_n,S(x_n)\big)=0, \quad \textit{and} \quad \lim_{n\to\infty} d(x_n,T(x_n))=0.$$

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Proof Let $p \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$. Then the sequence $\{d(x_n, p)\}$ is decreasing. Set $c = \lim_{n\to\infty} d(x_n, p)$. If $c = 0$, then all the conclusions are trivial. Therefore we will assume that $c > 0$. Note that we have

$$d(x_{n+1}, p) \le \alpha_n d\big(S(y_n), p\big) + (1-\alpha_n) d(x_n, p) \tag{3.2}$$

and

$$d(S(y_n), p) \le d(y_n, p) \le \beta_n d(T(x_n, p)) + (1-\beta_n) d(x_n, p) \le d(x_n, p), \tag{3.3}$$

for any $n \ge 1$. From inequalities (3.2) and (3.3), we get

$$\begin{aligned} d(x_{n+1}, p) = d\big(\alpha_n S(y_n) \oplus (1-\alpha_n)x_n, p\big) &\le \alpha_n d\big(S(y_n), p\big) + (1-\alpha_n) d(x_n, p) \\ &\le d(x_n, p), \end{aligned}$$

which implies $\lim_{n\to\infty} d(S(y_n), p) = c$. Indeed, let $\mathcal{U}$ be a nontrivial ultrafilter over $\mathbb{N}$. Then we have $\lim_{\mathcal{U}} \alpha_n = \alpha_\infty \in [\alpha, \beta]$ and $\lim_{\mathcal{U}} d(x_n, p) = \lim_{\mathcal{U}} d(x_{n+1}, p) = c$. Hence

$$c \le \alpha_\infty \lim_{\mathcal{U}} d(Sy_n, p) + (1-\alpha_\infty)c \le c.$$

Since $\alpha_\infty \ne 0$, we get $\lim_{\mathcal{U}} d(Sy_n, p) = c$. Since $\mathcal{U}$ was arbitrary, we get

$$\lim_{n\to\infty} d(S(y_n), p) = c$$

as claimed. Therefore, we have

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d\big(Sy(_n), p\big) = \lim_{n\to\infty} d\big(\alpha_n S(y_n) \oplus (1-\alpha_n)x_n, p\big) = c.$$

Using Lemma 3.3, we get $\lim_{n\to\infty} d(S(y_n), x_n) = 0$. Next from (3.2) and (3.3), we get

$$d(x_{n+1}, p) \le \alpha_n d(y_n, p) + (1-\alpha_n) d(x_n, p) \le d(x_n, p)$$

which implies $\lim_{n\to\infty} [\alpha_n d(y_n, p) + (1-\alpha_n) d(x_n, p)] = c$. Since $\liminf_{n\to\infty} \alpha_n \ge \alpha > 0$, we conclude that $\lim_{n\to\infty} d(y_n, p) = c$. Since $\beta_n \ge \alpha > 0$, we get $\lim_{n\to\infty} d(T(x_n), p) = c$ in a similar fashion. Therefore we have

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d\big(T(x_n), p\big) = \lim_{n\to\infty} d\big(\beta_n T(x_n) \oplus (1-\beta_n)x_n, p\big) = c.$$

Using Lemma 3.3, we get $\lim_{n\to\infty} d(T(x_n), x_n) = 0$. Finally, since

$$\begin{aligned} d\big(x_n, S(x_n)\big) &\le d\big(x_n, S(y_n)\big) + d\big(S(y_n), S(x_n)\big) \\ &\le d\big(x_n, S(y_n)\big) + d(y_n, x_n) \\ &= d\big(x_n, S(y_n)\big) + \beta_n d\big(T(x_n), x_n\big) \\ &\le d\big(x_n, S(y_n)\big) + d\big(T(x_n), x_n\big), \end{aligned}$$

we conclude that $\lim_{n\to\infty} d(x_n, S(x_n)) = 0$. ■

Remark 3.5 The existence of a common fixed point of T and S is crucial. If one assumes that T and S commute, *i.e.*, $S \circ T = T \circ S$, then a common fixed point exists under the assumptions of Theorem 3.4 if we assume that C is bounded. Indeed, fix $x_0 \in C$ and define

$$T_n(x) = \frac{1}{n} x_0 \oplus \Big(1 - \frac{1}{n}\Big) T(x),$$

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for $x \in C$, and $n \geq 1$. Then

$$d\big(T_n(x), T_n(y)\big) = d\Big(\frac{1}{n}x_0 \oplus \Big(1-\frac{1}{n}\Big)T(x), \frac{1}{n}x_0 \oplus \Big(1-\frac{1}{n}\Big)T(y)\Big) \leq \Big(1-\frac{1}{n}\Big)d\big(T(x),T(y)\big) \leq \Big(1-\frac{1}{n}\Big)d(x,y),$$

for any $x, y \in C$. That is, T_n is a contraction. The Banach Contraction Principle implies that T_n has a unique fixed point u_n in C. Since C is bounded and

$$d\big(u_n, T(u_n)\big) = d\Big(\frac{1}{n}x_0 \oplus \Big(1-\frac{1}{n}\Big)T(u_n), T(u_n)\Big) \leq \frac{1}{n}d\big(x_0, T(u_n)\big),$$

we get $d(u_n, T(u_n)) \to 0$. Define the function

$$\tau(x) = \limsup_{n\to\infty} d(u_n, x).$$

Since M is uniformly convex, τ has a unique minimum point $p \in C$, *i.e.,*

$$\tau(p) = \inf\{\tau(x)\ ;\ x \in C\},$$

and $\tau(p) < \tau(x)$, for any $x \neq p$. Since $\{u_n\}$ is an approximate fixed point sequence of T, we have

$$\limsup_{n\to\infty} d\big(u_n, T(p)\big) = \limsup_{n\to\infty} d\big(T(u_n), T(p)\big) \leq \limsup_{n\to\infty} d(u_n, p).$$

Hence $\tau(T(p)) \leq \tau(p)$, which implies $p = T(p)$. Since M is strictly convex, $\mathrm{Fix}(T)$ is a nonempty convex subset of M. Since T and S commute, we have $S(\mathrm{Fix}(T)) \subset \mathrm{Fix}(T)$. The above proof shows that S has a fixed point in $\mathrm{Fix}(T)$; *i.e.,* T and S have a common fixed point.

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