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Differential inequalities and the matrix Riccati equation

A.N. Stokes

The symmetric matrix Riccati equation has the property that its solutions tend to preserve their ordering with respect to positive definiteness as their arguments changes. The thesis explores this property, places it in a general context, and makes some deductions, particularly concerning disconjugacy in linear hamiltonian systems.

Chapter 1 extends the usual theory of differential inequalities for first-order systems (Coppel [1]) which applies given the component-wise ordering of vectors, to derive a necessary and sufficient condition for the preservation of order of solutions of differential inequalities, with order being more generally defined. The immediate application is to the case of symmetric matrices ordered by positive definiteness, although there are also interesting consequences for the Lorentz ordering, with a right circular positive cone of vectors.

In the second chapter it is shown that, under unrestrictive conditions, only the Riccati equation preserves the ordering of symmetric matrix solutions as the argument changes in either direction. The symmetric Riccati equation is

$$R[W] = W' + A(t) + B^*(t)W + WB(t) + WC(t)W = 0 , \tag{1}$$

$[A(t) = A^*(t), \; C(t) = C^*(t)$, all matrices $n \times n]$.

Solutions of (1) are closely related to solutions of the hamiltonian system

$$Y' = B(t)Y + C(t)Z$$

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(2) $$Z = -A(t)Y - B^*(t)Z ,$$

[Y and Z $n \times n$ matrices].

If $C(t) \geq 0$ on an interval, then principal solutions (Hartman [2]) can be defined. They are determined by extremal (that is, maximal or minimal) solutions of (1).

In Chapter 3, it is first shown that if any solution of (1) exists on an interval, then extremal solutions can be identified (although near end-points they may be infinite-valued). The solutions so identified still characterise principal solutions of (2), and so it is possible to dispense with a non-degeneracy condition on $C(t)$, called controllability by Coppel [1], and to extend consideration of principal solutions to a level more general than that of Reid [3].

In Chapter 4 a continued fraction expansion associated with solutions of a Riccati equation is given. It is shown that the convergents form good approximations near the point about which the expansion is made, but not that the fraction converges. However it is shown that the sequence of convergents is an improving sequence of bounds to a solution. Based on this, a sequence of increasingly critical necessary conditions for disconjugacy (or oscillation criteria), is given, and also a criterion requiring an assumption of positivity about the sign of only one coefficient.

Chapter 5 is concerned with asymptotic behaviour of the Riccati equation and the associated linear system. It is shown that the arguments used to prove exponential stability for certain solutions of the uniformly observable and controllable linear regulator problem can be used to show stability of some specifiable sort in many other cases. Deductions are made about the tendency of solutions of the Riccati equation to aggregate at infinity.

References

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