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ON THE NON-EXISTENCE OF CONJUGATE POINTS

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In this paper we consider the types of pairs of multiple zeros which a solution to the differential equation

$$D_n\ y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

can possess on an interval I of the real line. The results obtained generalize those in [2] and (for $n=3$) in [3].

I. Let f satisfy the condition

$$u_0 f(t, u_0, u_1, \ldots, u_{n-1}) > 0 \tag{1.1}$$

for all $t \in I$, $u_0 \neq 0$, and all $u_1, \ldots, u_{n-1}$.

DEFINITION. *The points $a<b$ in I are said to form a (μ, ν) conjugate pair (with respect to solutions of D_n on I) in case there exists a non-trivial solution y of D_n on $[a, b]$ with*

$$y(a) = y'(a) = \cdots = y^{(\mu-1)}(a) = 0 \neq y^{(\mu)}(a)$$

and

$$y(b) = y'(b) = \cdots = y^{(\nu-1)}(b) = 0 \neq y^{(\nu)}(b).$$

THEOREM 1.1. *Let f satisfy (1.1), let $n=2k+1$ where k is a positive integer, and let μ, ν be positive integers. Then there do not exist any (μ, ν) conjugate pairs in I if*

(a) *k is odd, $\mu \geq k+1$, and $\nu \geq k$,*

or

(b) *k is even, $\mu \geq k$, and $\nu \geq k+1$.*

Proof. Let y be a non-trivial solution to D_n on $[a, b]$, with $a<b$, satisfying $y(t)=y'(t)= \cdots =y^{(k-1)}(t)=0$ for $t=a$ and $t=b$. Define

$$v(t) = \sum_{j=0}^{k-1} (-1)^j y^{(2k-j)}(t) y^{(j)}(t) + (-1)^k (y^{(k)}(t))^2/2.$$

Then $v'(t)=y^{(2k+1)}(t)y(t)>0$ if $y(t)\neq 0$ by (1.1). Now $v(t)=(-1)^k(y^{(k)}(t))^2/2$ for $t=a$ and $t=b$. If k is odd and $y^{(k)}(a)=0$, then $v(a)=0$ and $v(b)\leq 0$ which implies $y(t)\equiv 0$ in $[a, b]$. Likewise, if k is even and $y^{(k)}(b)=0$, then $v(b)=0$ and $v(a)\geq 0$ so that again we conclude $y(t)\equiv 0$ on $[a, b]$. This proves (a) and (b).

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Note that in the above proof to get $v(a)=0$ (or $v(b)=0$), it would suffice to have $y^{(2k-j)}(t)y^{(j)}(t)=0$ for $j=0, 1, 2, \ldots, k$ and for $t=a$ (or $t=b$). Hence we have the following corollary to the proof of Theorem 1.1.

COROLLARY 1.2. *Let f satisfy (1.1) and let $n=2k+1$. Then there do not exist points $a<b$ in I and a non-trivial solution y of D_n on $[a, b]$ satisfying*

$$y^{(2k-j)}(a)y^{(j)}(a) = 0 = y^{(2k-j)}(b)y^{(j)}(b) \quad \text{for } j = 0, 1, \ldots, k-1$$

and either $y^{(k)}(a)=0$ if k is odd, or $y^{(k)}(b)=0$ if k is even.

THEOREM 1.3. *Let f satisfy (1.1) and let $n=2k$, where k is an odd positive integer. Then there are no (μ, ν) conjugate pairs in I where $\mu \geq k$ and $\nu \geq k$.*

Proof. Let $v(t)=\sum_{j=0}^{k-1}(-1)^j y^{(2k-1-j)}(t)y^{(j)}(t)$, note that $v'(t)=y^{(2k)}(t)y(t)+(y^{(k)}(t))^2$, and proceed as in the proof of Theorem 1.1.

As a corollary to the proof of Theorem 1.3 we have

COROLLARY 1.4. *Let f satisfy (1.1) and let $n=2k$ where k is an odd positive integer. Then there do not exist points $a<b$ in I and a non-trivial solution y of D_n on $[a, b]$ satisfying*

$$y^{(2k-1-j)}(a)y^{(j)}(a) = 0 = y^{(2k-1-j)}(b)y^{(j)}(b) \quad \text{for } j = 0, 1, \ldots, k-1.$$

If condition (1.1) is replaced by

$$u_0 f(t, u_0, u_1, \ldots, u_{n-1}) < 0 \tag{1.2}$$

for all $t \in I$, $u_0 \neq 0$, and all $u_1, u_2, \ldots, u_{n-1}$, then results similar to those given above are valid. We here state only the results analogous to Theorems 1.1 and 1.3.

THEOREM 1.5. *Let f satisfy (1.2), let $n=2k+1$ where k is a positive integer, and let μ, ν be positive integers. Then there do not exist any (μ, ν) conjugate pairs in I if*

(a) *k is odd, $\mu \geq k$, and $\nu \geq k+1$.*

or

(b) *k is even, $\mu \geq k+1$, and $\nu \geq k$.*

THEOREM 1.6. *Let f satisfy (1.2) and let $n=2k$ where k is an even positive integer. Then there are no (μ, ν) conjugate pairs in I where $\mu \geq k$, $\nu \geq k$.*

We shall give examples in Section 2 to show that one may not allow k to be even in Theorem 1.3 or odd in Theorem 1.6.

II. In this section we will show that Theorem 1.1 can be generalized to a much larger class of conjugate pairs, provided an additional assumption is made regarding solutions of D_n. Examples are also given to show that the theorem is not true for the remaining conjugate pairs. The proof will not make use of any auxiliary function $v(t)$.

THEOREM 2.1. *Let f satisfy (1.1) and assume that no solution of D_n has more than a* *finite number of zeros on any interval* $[a, b]\subseteq I$. *Let the positive integers* μ, ν *satisfy* $\mu+\nu\geq n$ *with* ν *odd in case equality holds. Then there are no* (μ, ν) *conjugate pairs in* I.

Proof. We shall first assume that $\mu+\nu=n$ and that $\mu\leq\nu$. If the theorem is false, let y be a non-trivial solution of D_n satisfying

$$\begin{aligned} y(a) &= y'(a) = \cdots = y^{(\mu-1)}(a) = 0 \neq y^{(\mu)}(a) \\ y(b) &= y'(b) = \cdots = y^{(\nu-1)}(b) = 0 \neq y^{(\nu)}(b). \end{aligned} \tag{2.1}$$

Let $a=a_1<a_2<\cdots<a_m=b$ be the $m(\geq 2)$ zeros of y on $[a, b]$. If $\mu=1$, the Mean-Value Theorem implies that y' has at least m zeros on $[a, b]$. If $\mu>1$, then, for $1\leq j\leq\mu-1$, the Mean-Value Theorem implies that $y^{(j)}(t)$ will have at least $m+j$ zeros on $[a, b]$ at the points

$$a = a(1,j) < a(2,j) < \cdots < a(m+j,j) = b. \tag{2.2}$$

It follows also that $a(i-1,j-1)<a(i,j)<a(i,j-1)$ for $2\leq i\leq m+j-1$, $1\leq j\leq\mu-1$. Now if $\mu<\nu$, then $y^{(\mu)}$ will have at least $m+\mu-1$ zeros at the points

$$a(1, \mu) < a(2, \mu) < \cdots < a(m+\mu-1, \mu) = b. \tag{2.3}$$

Inductively, for $\mu\leq j\leq\nu-1$, $y^{(j)}$ will have at least $m+\mu-1$ zeros at the points

$$a(1,j) < a(2,j) < \cdots < a(m+\mu-1,j) = b$$

where

$$\begin{aligned} a(1,j-1) < a(1,j) < a(2,j-1) < \cdots < a(m+\mu-2,j-1) &< a(m+\mu-2,j) \\ &< a(m+\mu-1,j-1) \\ &= a(m+\mu-1,j) = b. \end{aligned} \tag{2.4}$$

Therefore, $y^{(\nu)}$ will have at least $m+\mu-2$ zeros at the points

$$a(1, \nu) < a(2, \nu) < \cdots < a(m+\mu-2, \nu) \tag{2.5}$$

where

$$a < a(1, \nu-1) < a(1, \nu) < a(2, \nu-1) < \cdots < a(m+\mu-2, \nu) < b. \tag{2.6}$$

In case $\mu=\nu$, we may use (2.2) to see that that $y^{(\nu)}$ will have at least $m+\mu-2$ zeros satisfying (2.5) and (2.6). Now for $j=\nu+1, \nu+2, \ldots, n-1$, applying the Mean-Value Theorem successively we conclude that $y^{(j)}$ will have at least $m+n-j-2$ zeros at the points

$$a(1,j) < a(2,j) < \cdots < a(m+n-j-2,j)$$

where

$$\begin{aligned} a < a(1,j-1) < a(1,j) < a(2,j-1) < \cdots &< a(m+n-j-3,j-1) \\ &< a(m+n-j-2,j) \\ &< a(m+n-j-2,j-1) < b. \end{aligned} \tag{2.7}$$

Hence, $y^{(n-1)}$ will have at least $m-1$ zeros in (a, b). If $m \geq 3$ and if two of the zeros of $y^{(n-1)}$ lie in some interval $[a_j, a_{j+1}]$, then $y^{(n)}$ will have a zero at a point α_j, $a_j < \alpha_j < a_{j+1}$ which contradicts (1.1).

Therefore, we must have

$$a = a_1 < a(1, n-1) < a_2 < \cdots < a_{m-1} < a(m-1, n-1) < b \tag{2.8}$$

Moreover, by our observations (2.4), (2.6) and (2.7) we see that

$$a_1 < a(1, \mu) < a(1, \mu+1) < \cdots < a(1, n-1) < a_2. \tag{2.9}$$

Now let $\gamma \geq 1$ be such that $(-1)^\gamma y^{(\mu)}(a) > 0$. It follows that there is an α_1, $a < \alpha_1 < a(1, \mu)$ with

$$(-1)^\gamma y^{(\mu+1)}(\alpha_1) < 0$$

and hence, $(-1)^{\gamma+1} y^{(\mu+1)}(\alpha_1) > 0$. Proceeding inductively, we conclude the existence of a point α_k, $a < \alpha_k < a(1, \mu+k-1)$, with

$$(-1)^{\gamma+k} y^{(\mu+k)}(\alpha_k) > 0.$$

Hence, for $k = n-\mu = \nu$, $(-1)^{\gamma+\nu} y^{(n)}(\alpha_\nu) > 0$. But since $(-1)^\gamma y^{(\mu)}(a) > 0$, it follows that $(-1)^\gamma y(t) > 0$ on (a, a_2). Thus

$$(-1)^{2\gamma+\nu} y^{(n)}(\alpha_\nu) y(\alpha_\nu) > 0,$$

a contradiction to (1.1).

For the case $\mu+\nu = n$ and $\mu > \nu$, a similar proof holds. One can show that $y^{(n-1)}$ has at least $m-1$ zeros in (a, b) and hence (2.8) will hold. In addition, (2.9) will hold and then the remainder of the proof is the same.

It is also clear that if $\mu+\nu > n$, then one can show that $y^{(n-1)}$ has at least m zeros in (a, b) and hence two of them must lie in some interval $[a_j, a_{j+1}]$. This implies that $y^{(n)}$ has a zero in (a_j, a_{j+1}), contradicting (1.1).

From the proof of Theorem 2.1 we have

COROLLARY 2.2. *Let f satisfy (1.2) and assume that no non-trivial solution of D_n has more than a finite number of zeros on any interval $[a, b] \subseteq I$. Let the positive integers μ, ν satisfy $\mu+\nu \geq n$ with ν even in case equality holds. Then there are no (μ, ν) conjugate pairs in I.*

Remark 2.3. Consider now a pair of integers μ, $\nu \geq 1$ where $\mu+\nu = n$ and ν is even. Defining the function $f(t, u_0, u_1, \ldots, u_{n-1})$ by

$$f = \begin{cases} n!, & u_0 \geq 0, \quad \text{all } t, u_1, \ldots, u_{n-1} \\ -n!, & u_0 < 0, \quad \text{all } t, u_1, \ldots, u_{n-1} \end{cases}$$

we see that (1.1) holds. Moreover, on the interval [0, 1] the function

$$y(t) \equiv t^\mu (t-1)^\nu$$

is a solution of D_n which has a (μ, ν) conjugate pair.

If one requires that the function f be continuous, then examples can still be given to show that Theorem 2.1 cannot, in general, be extended to include additional conjugate pairs. To see this, consider the simple linear differential equation

$$y^{(n)} = y. \tag{2.10}$$

For $n=3$ one can show that there is a non-trivial solution of (2.10) with a simple zero at τ, $-\sqrt{3}\,\pi<\tau<0$, and a double zero at the origin. Also, for $n=4$, there is a non-trivial solution of (2.10) having a double zero at the origin and another double zero at τ, where $3\pi/2<\tau<2\pi$.

In conjunction with this, it is interesting to compare our results with those obtained by Sherman ([4]) for the linear differential equation

$$y^{(n)} = p(t)y, \quad t \in I. \tag{2.11}$$

where $p(t)$ is continuous and satisfies

$$|p(t)| > 0 \text{ on } I. \tag{2.12}$$

For any $a \in I$ let $\eta_1(a)$, the first conjugate point of a, be the smallest $b>a$ such that there is a non-trivial solution of (2.11) with n zeros on $[a, b]$ (counting multiplicities). Suppose now that $y(t)$ is a non-trivial solution of (2.11) with n simple zeros on $[a, \eta_1(a)]\subseteq I$. Then by Theorem 5 of [4], there exist solutions $y_1, y_2, \ldots, y_{n-1}$ of (2.11), not necessarily distinct, such that y_k has a zero at a of order at least $n-k$ and a zero at $\eta_1(a)$ of order at least k. This contradicts Theorem 2.1 or Corollary 2.2. Thus, if $\eta_1(a)<+\infty$, any solution of (2.11) with n zeros on $[a, \eta_1(a)]$ has at least one multiple zero. However, in [5] is it shown that for any $\epsilon>0$ there is a solution of (2.11) with n simple zeros on $[a, \eta_1(a)+\epsilon)$.

Remark 2.4. Results analogous to Theorem 2.1 and Corollary 2.2 can be obtained if one assumes instead of (1.1) that the following condition holds for some j, $1\le j\le n-1$:

$$u_j f(t, u_0, u_1, \ldots, u_{n-1}) > 0 \quad \text{if } u_j \neq 0. \tag{2.13)j}$$

As an example of what is true here, we state

THEOREM 2.5. *Let f satisfy (2.13)j and also assume that no solution y of D_n is such that $y^{(j)}$ has an infinite number of zeros on some interval $[a, b]\subseteq I$ and $y^{(j)}\not\equiv 0$ on $[a, b]$. Let the positive integers μ, ν satisfy $\mu+\nu\ge n-j$ with ν odd in case equality holds. Then all solutions y of D_n which are such that $y^{(j)}$ has a (μ, ν) conjugate pair belong to the class of polynomials in t of degree $\le j-1$.*

Remark 2.5. We note also that Theorem 1.1, Corollary 1.2, Theorem 1.3, Corollary 1.4 and Theorem 2.1 are true, as stated, for solutions of the differential inequality

$$y^{(n)} \ge f(t, y, y', \ldots, y^{(n-1)}). \tag{2.13}$$

Likewise, Theorems 1.5 and 1.6 and Corollary 2.2 are valid for solutions of the differential inequality

$$y^{(n)} \leq f(t, y, y', \ldots, y^{(n-1)}). \tag{2.14}$$

III. In this section we shall show that several results obtained in [3] for the case $n=3$ can be generalized to arbitrary $n \geq 2$. We assume f satisfies the following conditions:

(3.1) f is continuous on $I \times R^n$ where $n \geq 2$ with $f(t, 0, u_1, \ldots, u_{n-3}, 0, 0) \equiv 0$;

$$\begin{aligned} f(t, u_0, u_1, \ldots, u_{n-1}) &\geq f(t, 0, u_1, \ldots, u_{n-1}) \quad \text{if } u_0 > 0 \quad \text{and} \\ f(t, u_0, u_1, \ldots, u_{n-1}) &\leq f(t, 0, u_1, \ldots, u_{n-1}) \quad \text{if } u_0 < 0, \end{aligned} \tag{3.2}$$

the inequality holding for all $t \in I$ and all $u_1, \ldots, u_{n-1}$.

(3.3) $f(t, 0, u_1, \ldots, u_{n-1})$ is non-decreasing in u_{n-2} for fixed $t, 0, u_2, \ldots, u_{n-1}$ and satisfies a Lipschitz condition with respect to u_{n-1} on compact subsets of $I \times R^n$.

THEOREM 3.1. *Assume conditions (3.1), (3.2) and (3.3) hold, let y be a non-trivial solution of D_n which has a zero of order $n-1$ at the point $a \in I$, and assume a is not an accumulation point of zeros of y. Then y has no zeros to the right of a in I.*

Proof. We shall be quite brief in this proof since it is a straightforward generalization of the proof of Theorem 2 in [3]. In addition, we shall assume $n \geq 3$ since the proof for $n=2$ will be obvious. Let y satisfy

$$y(a) = y'(a) = \cdots = y^{(n-2)}(a) = y(b) = 0 \quad \text{with } a < b.$$

By repeated application of Rolle's theorem there is a point c in (a, b) with $y^{(n-2)}(c)=0$. Define

$$G(t, u, u') \equiv f(t, 0, y'(t), \ldots, y^{(n-3)}(t), u, u').$$

Assume, to be specific, that $y>0$ on (a, c). Then by (3.2) we have

$$\begin{aligned} (y^{(n-2)}(t))'' &= f(t, y(t), y'(t), \ldots, y^{(n-3)}(t), y^{(n-2)}(t), y^{(n-1)}(t)) \\ &\geq G(t, y^{(n-2)}(t), (y^{(n-2)}(t))'), \end{aligned}$$

so that $y^{(n-2)}(t)$ is a subfunction with respect to solutions of $u''=G(t, u, u')$ on (a, c) (see [1] p. 1056). Since $u \equiv 0$ is a solution of $u''=G(t, u, u')$, $y^{(n-2)}(a)=y^{(n-2)}(c)=0$ implies $y^{(n-2)}(t) \leq 0$ on (a, c). Since a is a zero of order $n-1$ of y, it follows that $y(t) \leq 0$ on (a, c), contrary to our assumption. A similar proof works in case $y(t)<0$ on (a, b) by showing that $y^{(n-2)}(t)$ is a superfunction with respect to solutions of $u''=G(t, u, u')$.

We note that $y(b)=0$ was used only to get the point $c>a$ where $y^{(n-2)}$ vanished.

Therefore, as a corollary to the proof of Theorem 3.1 we have

COROLLARY 3.2. *Under the assumptions in Theorem 3.1,* $y^{(n-2)}(t)>0$ *for* $t>a$ *if* $y(t)>0$ *for* $t>a$ *and* $y^{(n-2)}(t)<0$ *for* $t>a$ *if* $y(t)<0$ *for* $t>a$.

For the case $n=4$ we have the following corollary of the proof of Theorem 3.1:

COROLLARY 3.3. *Let a and b be successive zeros of a non-trivial solution y of* D_4 *and assume f satisfies* (*3.1*), (*3.2*) *and* (*3.3*) *for* $n=4$. *Then y does not have two strict extrema in* (a, b).

Proof. If the corollary is false, let $y(a)=y(b)=0$ and suppose $y>0$ on (a, b). If the extrema occur at $t=c$ and $t=d$ with $c<d$ then we have $y''(c)\leq 0$ and $y''(d)\leq 0$. Hence, by the proof of Theorem 3.1, $y''(t)\leq 0$ on (c, d) so that $y'(t)\equiv 0$ on $[c, d]$, contrary to assumption. If $y<0$ on (a, b), a similar argument works.

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