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SUBORDINATION IN THE SENSE OF BOCHNER AND A RELATED FUNCTIONAL CALCULUS

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Abstract

We prove a new representation of the generator of a subordinate semigroup as limit of bounded operators. Our construction yields, in particular, a characterization of the domain of the generator. The generator of a subordinate semigroup can be viewed as a function of the generator of the original semigroup. For a large class these functions we show that operations at the level of functions has its counterpart at the level of operators.

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0. Introduction

Let $\{T_t\}_{t\geq 0}$ be a contraction semigroup of type (C_0) (that is, strongly continuous semigroup) on a Banach space $(X, \|\cdot\|)$ with infinitesimal generator $(A, D(A))$. Subordination (in the sense of Bochner) is a method of getting new (C_0)-semigroups from the original one $\{T_t\}_{t\geq 0}$ by integrating $\{T_t\}_{t\geq 0}$ (as function of t) with respect to a *subordinator*, that is a vaguely continuous semigroup $\{\mu_t\}_{t\geq 0}$ of sub-probability measures on $[0,\infty)$. Thus, the *subordinate semigroup* $\{T_t^f\}_{t\geq 0}$ is given by the Bochner integral

$$T_t^f u = \int_{[0,\infty)} T_s u \mu_t(ds), \quad t \geq 0, u \in X. \tag{0.1}$$

The superscript f in T_t^f refers to the *Bernstein function* $f : (0,\infty) \to [0,\infty)$ which is the logarithm of the (one-sided) Laplace transform of μ_1,

$$(0.2)\qquad \widehat{\mu_t}(x) = \int_{[0,\infty)} e^{-sx}\,\mu_t(ds) = e^{-tf(x)}, \quad t, x \geq 0,$$

and is given by the *Lévy-Khinchine*-type *formula*

$$(0.3)\qquad f(x) = bx + \int_{(0,\infty)} (1 - e^{-tx})\,\mu(dt), \quad x > 0,$$

where $b \geq 0$ and $\int_{(0,\infty)} t/(1+t)\,\mu(dt) < \infty$. The generator of $\{T_t^f\}_{t\geq 0}$ will consequently be denoted by $(A^f, D(A^f))$. In [15] Phillips obtained the following representation formula for A^f:

$$(0.4)\qquad A^f u = bAu + \int_{(0,\infty)} (T_t u - u)\,\mu(dt), \quad u \in D(A).$$

If we write e^{tA} for T_t, this result shows—at least at a formal level—that $A^f = -f(-A)$. In fact, it gave rise to a functional calculus which is sometimes referred to as *Bochner-Phillips calculus*, see [2]. It is well-known for contractive semigroups $\{T_t\}_{t\geq 0}$ and one-sided α-stable subordinators $f(x) = x^\alpha$, $\alpha \in (0, 1)$, that A^f is indeed the fractional power $-(-A)^\alpha$ (in the sense of Balakrishnan), see [23, 13]. In [11, 2, 19] the *complete Bernstein functions*, a sufficiently rich subclass of the Bernstein functions (containing, for example, the above fractional powers), was considered and the relation $A^f = -f(-A)$ established whenever f is defined on the spectrum of $-A$, see Proposition 1.3. Here, $-f(-A)$ is characterized via its resolvent in terms of the Dunford-Taylor integral, see Section 4 and [2, 19].

In [11, 2, 19] the representation formula

$$(0.5)\qquad A^f u = bAu + \int_{(0,\infty)} A(A-\lambda)^{-1}\,\rho(d\lambda), \quad u \in D(A),$$

for A^f was used, where $f(x) = bx + \int_{(0,\infty)} x(\lambda + x)^{-1}\,\rho(d\lambda)$ is a complete Bernstein function with a representation which is particular to this class of functions. Note that (0.5) generalizes Balakrishnan's formula for fractional powers, [23, Chapter IX.11]. Both formulae (0.4) and (0.5) are only defined for $u \in D(A)$ and fail, in general, to describe A^f as a whole. There is little information on $D(A^f)$, only that $D(A^n) \subset D(A^f)$, $n \in N$, is an operator core for A^f and that $D(A^f) = D(A)$ if and only if $\lim_{x\to\infty} x^{-1} f(x) \neq 0$, see [19]. In [18, 19] we used for self-adjoint Hilbert space operators A, B and complete Bernstein functions f a Heinz-Kato type theorem so as to compare $D(A^f)$ and $D(B^f)$: if the graph norms of A and B are comparable, so are those of A^f and B^f and, in particular, $D(A^f) = D(B^f)$. This technique can be fruitful if one works in concrete situations, see [12] for an application to pseudo-differential operators.

In this paper we will give an alternative description of $(A^f, D(A^f))$ by approximating A^f through a sequence of bounded infinitesimal generators. The idea itself does not seem to be new and was employed by Westphal [22] in order to get a similar approximation of Balakrishnan's fractional powers $-(-A)^\alpha$ of an infinitesimal generator A. Westphal showed that there is a sequence $g_n(x) \to x^\alpha$ as $n \to \infty$ such that $Q_n(x) \equiv g_n(x)x^{-\alpha}$ is a Laplace transform of a measure $q_{\alpha,n}(x)1_{(0,\infty)}\,dx$. Putting formally $x = -A$, we can interpret $Q_n(-A)$ as bounded operator and approximate $(-A)^\alpha$. Since the densities $q_{\alpha,n}(x)$ can be explicitly calculated, the reasoning in [22] was straightforward and could easily be made rigorous.

We will go along similar lines, although we cannot use explicit formulae when approximating the complete Bernstein function f. Here we will construct an approximating sequence $\{f_n\}_{n\in N}$ for f consisting of bounded complete Bernstein functions. Now the operators A^{f_n} are well-defined, for example, by Phillips' formula (0.4), and the following definition makes sense:

$$(0.6)\qquad \begin{cases} \tilde{A}^f & = \text{weak-}\lim_{n\to\infty} A^{f_n} \\ D(\tilde{A}^f) & = \{u \in X : \text{the above weak limit exists}\}. \end{cases}$$

We show that $(\tilde{A}^f, D(\tilde{A}^f))$ is a closed dissipative extension of $(A^f, D(A^f))$, hence $\tilde{A}^f = A^f$.

In particular, the above characterization of $D(A^f)$ yields the following asymptotic result

$$(0.7)\qquad \|T_\epsilon u - u\| = O\Big(\frac{1}{f(\epsilon^{-1})}\Big) \quad \text{as} \quad \epsilon \to 0 \quad \text{for} \quad u \in D(A^f),$$

which allows for a comparison of the domains of A^f and A^g, where g is some other Bernstein function.

In our final section we make some first steps towards a functional calculus for generators of subordinate semigroups. We show that operations at the level of complete Bernstein functions—scalar multiplication, addition, composition, multiplication, and convergence—have their counterparts at the level of operators. In particular, we can show for complete Bernstein functions f, g,

$$(0.8)\quad \text{if } f \cdot g \text{ is a complete Bernstein function, then } A^{fg} = -A^f \circ A^g = -A^g \circ A^f$$

where equality is understood in the sense of operators. Putting $-f(-A) = A^f$ we can extend (0.8) to the case where $f \cdot g$ is no longer a Bernstein function:

$$(0.9)\qquad (fg)(-A) = f(-A) \circ g(-A) = g(-A) \circ f(-A).$$

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Hirsch derived in [11] a representation of $(A^f, D(A^f))$ for the class of complete Bernstein functions that is basically identical with (0.6) but uses a different approximation of the function f. In fact, Corollary 2.10 and Theorem 4.1 (5) can already be found there. However, the method used in [11] is quite different from ours. It is motivated by potential theoretic considerations and uses an approach via the resolvent rather than the semigroup. Since our results in sections 3 and 4 strongly depend on our approach we will, nevertheless, include our proofs in full.

1. Notations and auxiliary results

Let us recall some results from semigroup theory, see, for example, [8] and [14].

A (C_0)-*semigroup of type* $\omega_0 \in [-\infty, \infty)$ on a Banach space $(X, \|\cdot\|)$ is a one-parameter family of operators $\{T_t\}_{t\geq 0}$ on X satisfying $T_{t+s}u = T_tT_su$ for all $t, s \geq 0$, and $\lim_{t\to 0}\|T_tu - u\| = 0$ for all $u \in X$, and

$$\|T_t\| \leq M_\omega e^{\omega t}, \quad t \geq 0, \tag{1.1}$$

for all $\omega > \omega_0$ with a suitable constant $M_\omega > 0$. The *infinitesimal generator* $(A, D(A))$ of $\{T_t\}_{t\geq 0}$ is the operator

$$Au = \lim_{t\to 0}\frac{T_tu - u}{t} \quad \text{on} \quad D(A), \tag{1.2}$$

where

$$D(A) = \big\{u \in X : \text{the limit (1.2) exists strongly}\big\}. \tag{1.3}$$

The resolvent set $\rho(A)$ of A contains the complex half-plane $\{z \in \boldsymbol{C} : \operatorname{Re} z > \omega_0\}$, and the following estimate holds for $z \in \rho(A)$, $\operatorname{Re} z > \omega$, $\omega > \omega_0$:

$$\|(z - A)^{-1}u\| \leq \frac{M_\omega}{\operatorname{Re} z - \omega}\|u\|, \quad \text{for all } u \in X. \tag{1.4}$$

If A is the generator of a *contraction semigroup*, that is, a semigroup where (1.1) becomes $\|T_t\| \leq 1$ for all $t \geq 0$, it is a *dissipative* operator:

$$\|Au - zu\| \geq \operatorname{Re} z\|u\|, \quad z \in \boldsymbol{C},\ \operatorname{Re} z > 0, \tag{1.5}$$

holds for all $u \in D(A)$. In the general situation, we can always turn a semigroup $\{T_t\}_{t\geq 0}$ into a contraction semigroup with generator $\tilde{A} = A - \omega$ on $(X, |||\cdot|||))$ where $\tilde{T}_t = e^{-\omega t}T_t$ is defined on X equipped with the new norm $|||u||| = M_\omega^{-1}\sup_{s\geq 0}\|\tilde{T}_su\|$. Hence, (1.4) and (1.5) are, in this sense, the same properties. Let us recall from [21, pp. 22–24] the following result for dissipative operators.

THEOREM 1.1. *For any [closed] operator B on a Banach space $(X, \|\cdot\|)$ the following assertions are equivalent:*

(1) *B is dissipative, that is, $\|(B-\lambda)u\| \geq \operatorname{Re}\lambda\|u\|$ for all $u \in D(B)$ and some [all] $\lambda \in C$ such that $\operatorname{Re}\lambda > 0$.*

(2) *$\|(B-\lambda)u\| \geq \lambda\|u\|$ for all $u \in D(B)$ and some [all] $\lambda > 0$.*

(3) *For all $u \in D(B)$ there is a $\phi \in X^*$ such that $\|u\|^2 = \|\phi\|_*^2 = {}_X\langle u, \phi\rangle_{X^*}$ and $\operatorname{Re}{}_X\langle Bu, \phi\rangle_{X^*} \leq 0$.*

If B is closed, dissipative, and densely defined with $(\lambda - B)(D(B)) = X$ for some $\lambda > 0$, (3) holds for all $\phi \in X^*$ with $\|u\|^2 = \|\phi\|_*^2 = {}_X\langle u, \phi\rangle_{X^*}$.

By the above remark, the operator $B = A - \omega$, where $\omega > \omega_0$ and A is the generator of a (C_0)-semigroup of type ω_0, satisfies the conditions of the above theorem.

A *subordinator* is a vaguely continuous convolution semigroup $\{\mu_t\}_{t\geq 0}$ of sub-probability measures μ_t on $[0, \infty)$. By Bochner's theorem the *Laplace transform* of μ_t has the form

$$\widehat{\mu}_t(x) = \int_0^\infty e^{-sx}\,\mu_t(ds) = e^{-tf(x)}, \quad t, x \geq 0, \tag{1.6}$$

where f is a *Bernstein function* ($\mathscr{BF}$), that is a C^∞-function on $(0, \infty)$ with representation

$$f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-tx})\,\mu(dt) \tag{1.7}$$

with $a, b \geq 0$ and a measure μ on $(0, \infty)$ such that $\int_{(0,\infty)} t/(1+t)\,\mu(dt) < \infty$. We will also need the related class of *completely monotone functions* $\mathscr{CM}$ which consists of the Laplace transforms of measures in $[0, \infty)$. For an exhaustive account on $\mathscr{BF}$ and $\mathscr{CM}$ we refer to [3, §9].

If the semigroup $\{T_t\}_{t\geq 0}$ is not equi-bounded but of type ω_0, not every subordinator is eligible for subordination. If, however, $\int_{[0,\infty)} e^{s\omega}\,\mu_t(ds) < \infty$ for some $\omega > \omega_0$ from (1.1), the Bochner integral

$$T_t^f u = \int_{[0,\infty)} T_s u\,\mu_t(ds), \quad u \in X, t \geq 0, \tag{1.8}$$

makes sense and defines a new (C_0)-semigroup $\{T_t^f\}_{t\geq 0}$.

DEFINITION 1.2. Let $\{T_t\}_{t\geq 0}$ be a (C_0)-semigroup of type ω_0 and $\{\mu_t\}_{t\geq 0}$ be an admissible subordinator with $f \in \mathscr{BF}$. Then the semigroup $\{T_t^f\}_{t\geq 0}$ of (1.8) is called *subordinate* (in the sense of Bochner) with respect to $\{T_t\}_{t\geq 0}$.

The idea of subordination is probably due to Bochner [4], also [5, Chapter 4.4]. The following result characterizes those subordinators which can be used for subordination if the semigroup is not equi-bounded, see [18, Satz 2.13] or [19]. This, in particular, allows us later on to restrict ourselves to contractive semigroups.

PROPOSITION 1.3. *Let* $\{\mu_t\}_{t\geq 0}$ *be a subordinator with* $f \in \mathscr{BF}$ *as in* (1.7). *For all* $\beta \geq 0$ *the following assertions are equivalent:*

(1) $\int_{[0,\infty)} e^{s\beta}\,\mu_t(ds) < \infty$ *for all* $t \geq 0$*;*
(2) f *extends analytically onto* $\{z \in \boldsymbol{C} : \operatorname{Re} z > -\beta\}$ *and continuously up to the boundary;*
(3) $\int_{(1,\infty)} e^{s\beta}\,\mu(ds) < \infty$.

Obviously, the type ω_0^f of $\{T_t^f\}_{t\geq 0}$ is less or equal to $-f(-\omega_0) = \inf_{\omega>\omega_0}(-f(-\omega))$. Later on, we will only consider a subset of $\mathscr{BF}$.

DEFINITION 1.4. A function $f : (0,\infty) \to \boldsymbol{R}$ is a *complete Bernstein function* ($\mathscr{CBF}$), if

$$f(x) = x^2 \int_{(0,\infty)} e^{-sx}\phi(s)\,ds, \quad x > 0, \tag{1.9}$$

holds with some $\phi \in \mathscr{BF}$.

Complete Bernstein functions are sometimes also called *operator monotone functions*, see, for example, [9]. Our notation follows closely [16].

Examples for complete Bernstein functions are $x \mapsto x/(x+c)$, $c \geq 0$, $x \mapsto x^\alpha$, $(0 \leq \alpha \leq 1)$, or $x \mapsto \log(1+x)$, whereas $x \mapsto 1 - e^{-cx}$, $(c > 0)$, is contained in $\mathscr{BF} \setminus \mathscr{CBF}$. The following theorem gives a precise characterization of the class $\mathscr{CBF}$:

THEOREM 1.5. (see [18]) *Each of the following five properties of* $f : (0,\infty) \to \boldsymbol{R}$ *implies the other four:*

(1) $f \in \mathscr{CBF}$*;*
(2) $x \mapsto f(x)/x$ *is a Stieltjes transform, that is*

$$\frac{f(x)}{x} = \frac{a}{x} + b + \int_{(0,\infty)} \frac{1}{t+x}\,\sigma(dt), \quad x > 0,$$

with $a, b \geq 0$ *and a measure* σ *on* $(0,\infty)$.
(3) $f : (0,\infty) \to [0,\infty)$ *extends analytically onto* $\boldsymbol{C} \setminus (-\infty, 0]$ *such that* $\overline{f(\bar{z})} = f(z)$ *and* $\operatorname{Im} z\ \operatorname{Im} f(z) \geq 0$. (*In other words: f preserves upper and lower half-planes in* $\boldsymbol{C}$).

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(4) *$f \in \mathscr{BF}$ with representation*

$$f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-sx}) m(s)\, ds, \quad x > 0,$$

where $a, b \geq 0$, $m(s) = \int_{(0,\infty)} e^{-ts}\, \rho(dt)$, and $\int_{(0,\infty)} 1/(t(1+t))\, \rho(dt) < \infty$. (In fact, $\rho(dt) = t\sigma(dt)$ of (2)*.)*

(5) *$x/f(x) \in \mathscr{CBF}$.*

Some of the above implications can be found in [2] and [9], a detailed proof for (1)–(4) is given in [18]. Only (5) seems to be new, but it is straightforward that (1) and (3) imply (5), whereas (1) follows from (5) because of the identity $f(x) = x/(x/f(x))$.

ASSUMPTION 1.6. Throughout this paper, $\{T_t\}_{t \geq 0}$ will always be a (C_0) contraction semigroup on the Banach space $(X, \|\cdot\|)$, its generator will be denoted by $(A, D(A))$, and $\{\mu_t\}_{t \geq 0}$ will be a subordinator with Bernstein function f. The subordinate semigroup is denoted by $\{T_t^f\}_{t \geq 0}$ and its generator by $(A^f, D(A^f))$.

2. On the domain of A^f

In his 1952 paper on the generation of semigroups of linear operators [15] Phillips showed that $D(A) \subset D(A^f)$ is a core of A^f and gave the following representation formula for the generator of the subordinate semigroup

$$A^f u = -au + bAu + \int_{(0,\infty)} (T_s u - u)\, \mu(ds), \quad u \in D(A); \tag{2.1}$$

the spectrum of A^f satisfies $\sigma(A^f) \supset -f(-\sigma(A))$. Little seems to be known about the domain $D(A^f)$ of A^f. It was shown in [19, Theorem 5.1] that $D(A^f) = D(A)$ if and only if either A is bounded or if $\lim_{x\to\infty} x^{-1} f(x) = b > 0$.

LEMMA 2.1. *Let f be a Bernstein function with representation* (1.7) *where $a = b = 0$. Then f is bounded if and only if $\mu((0, \infty)) < \infty$. The operator A^f is bounded if A is bounded or if f is bounded.*

PROOF. If $\mu((0, \infty)) < \infty$, we have for all $x \geq 0$

$$f(x) = \int_{(0,\infty)} (1 - e^{-tx})\, \mu(dt) \leq \int_{(0,\infty)} \mu(dt) < \infty.$$

Conversely, if f is bounded, we find from Fatou's lemma

$$\int_{(0,\infty)} \mu(dt) \leq \liminf_{x\to\infty} \int_{(0,\infty)} (1 - e^{-tx})\, \mu(dt) = \sup_{x>0} f(x) < \infty.$$

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Suppose now that $\|A\| < \infty$. Then for $u \in D(A) = X$

$$\begin{aligned}\|A^f u\| &\le \int_{(0,\infty)} \|T_t u - u\|\, \mu(dt) \\ &\le \int_{(0,1)} \|T_t u - u\|\, \mu(dt) + \int_{[1,\infty)} \mu(dt)\|u\| \\ &\le \Big(\|A\| \int_{(0,1)} t\, \mu(dt) + \int_{[1,\infty)} \mu(dt)\Big)\|u\|,\end{aligned}$$

hence, $\|A^f\| < \infty$ (use Proposition 1.3 (3)). If f is bounded, we have by much the same calculation

$$\|A^f u\| \le \int_{(0,\infty)} \mu(dt)\|u\|,$$

hence, $\|A^f\| < \infty$. □

In what follows we will always assume that f is a complete Bernstein function $\mathscr{CBF}$. As a consequence of the above Lemma we may furthermore assume that f is not bounded, that is, that $\mu((0,1)) = \infty$ and has no linear part. Thus,

$$\begin{aligned} f(x) &= \int_{(0,\infty)} (1 - e^{-tx}) m(t)\, dt, \\ m(t) &= \int_{(0,\infty)} e^{-st}\, \rho(ds), \qquad \text{with} \quad \int_{(0,\infty)} \frac{\rho(ds)}{s(1+s)} < \infty. \end{aligned} \tag{2.2}$$

The following approximations of f will be important

$$\begin{aligned} f_k(x) &= \int_{(0,\infty)} (1 - e^{-tx}) m_k(t)\, dt, \quad k \in \mathbf{N} \\ m_k(t) &= \int_{(0,k)} e^{-st}\, \rho(ds) \qquad \text{with } \rho \text{ as in (2.2).} \end{aligned} \tag{2.3}$$

Note that $f_k \in \mathscr{CBF}$ and is bounded, since

$$f_k(x) \le \int_{(0,\infty)} m_k(t)\, dt = \int_{(0,k)} \int_{(0,\infty)} e^{-st}\, dt\, \rho(ds) = \int_{(0,k)} \frac{\rho(ds)}{s} < \infty,$$

and that $f(x) = \lim_{k\to\infty} f_k(x)$ for all $x > 0$. We use f_k in order to give an alternative definition of the operator A^f.

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DEFINITION 2.2. Let $\{T_t\}_{t\geq 0}$ and A be as in Assumption 1.6, $\{\mu_t\}_{t\geq 0}$ be a subordinator with $f \in \mathscr{BF}$ as in (2.2), and the sequence $\{f_k\}_{k\in N}$ given by (2.3). Then the Bochner integral

$$\tilde{A}^f u = \text{weak-}\lim_{k\to\infty} \int_{(0,\infty)} (T_t u - u) m_k(t)\, dt \tag{2.4}$$

defines an operator on X with domain

$$D(\tilde{A}^f) = \{u \in X : \text{the limit (2.4) exists weakly}\}. \tag{2.5}$$

Obviously, $D(A) \subset D(\tilde{A}^f)$ and $\tilde{A}^f$ extends $A^f|D(A)$. We shall show that, in fact, $A^f = \overline{A^f|D(A)} = \tilde{A}^f$. In order to do so, we need some preparations.

LEMMA 2.3. *Let f, f_k be as in (2.2), (2.3). Then there exist (possibly signed) measures γ_k (depending on a constant $b \geq 0$) such that*

$$(b + f)\widehat{\gamma_k} = b + f_k, \quad k \in N, \tag{2.6}$$

holds for the one-sided Laplace transform $\widehat{\gamma_k}$ of γ_k.

PROOF. Since $f \in \mathscr{BF}$, $f \not\equiv 0$, implies $1/(b + f) \in \mathscr{CM}$ for any $b \geq 0$, see [3, example 9.9], we find

$$\frac{1}{b + f(x)} = \int_{[0,\infty)} e^{-tx}\, \nu(dt) = \widehat{\nu}(x) \tag{2.7}$$

with a suitable measure ν on $[0, \infty)$. By the convolution theorem for Laplace transforms we have

$$\begin{aligned} \frac{b + f_k(x)}{b + f(x)} &= \left(b + \int_{(0,\infty)} m_k(t)\, dt\right) \frac{1}{b + f(x)} - \widehat{m_k}(x) \frac{1}{b + f(x)} \\ &= \left(b + \int_{(0,\infty)} m_k(t)\, dt\right) \widehat{\nu}(x) - \widehat{m_k}(x)\widehat{\nu}(x) \\ &= \left(\left(b + \int_{(0,\infty)} m_k(t)\, dt\right) \nu - m_k \star \nu\right)^{\widehat{}}(x), \end{aligned}$$

and so

$$\gamma_k(ds) = \left(b + \int_{(0,\infty)} m_k(t)\, dt\right) \nu(ds) - m_k \star \nu(s)\, ds. \tag{2.8}$$

□

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In fact, under the above assumptions on f and f_k, the measures γ_k are *positive* measures as the following Lemma shows.

LEMMA 2.4. *Let f, f_k, γ_k, and $b \geq 0$ be as in Lemma 2.3. Then the function $x \mapsto (b + f_k(x))/(b + f(x))$ is completely monotone and its representing measure γ_k is positive.*

PROOF. Assume that $b > 0$. Since $f_k \in \mathscr{CBF}$, we know from Theorem 1.5 (3) that f_k is analytic on $\boldsymbol{C} \setminus (-\infty, 0]$ and maps $\{\operatorname{Im} z > 0\}$ onto itself. Thus, $z \mapsto (f(z) - f_k(z))/(b + f_k(z))$ is an analytic function on $\boldsymbol{C} \setminus (-\infty, 0]$, maps $(0, \infty)$ into $(0, \infty)$, satisfies

$$\lim_{x \to 0+} \frac{f(x) - f_k(x)}{b + f_k(x)} = \frac{f(0) - f_k(0)}{b + f_k(0)} = 0,$$

and

$$\frac{f(\overline{z}) - f_k(\overline{z})}{b + f_k(\overline{z})} = \frac{\overline{f(z)} - \overline{f_k(z)}}{b + \overline{f_k(z)}} = \overline{\left(\frac{f(z) - f_k(z)}{b + f_k(z)}\right)},$$

and preserves the upper complex half-plane:

$$\begin{aligned} \operatorname{Im} \frac{f(z) - f_k(z)}{b + f_k(z)} &= \frac{1}{|b + f_k(z)|^2}\big(\operatorname{Re}(f(z) - f_k(z)) \operatorname{Im}(b + \overline{f_k(z)})\big) \\ &\quad + \big(\operatorname{Im}(f(z) - f_k(z)) \operatorname{Re}(b + \overline{f_k(z)})\big) \\ &= \frac{1}{|b + f_k(z)|^2}\big(\operatorname{Im}(f(z) - f_k(z)) \operatorname{Re}(b + f_k(z))\big) \\ &\quad - \big(\operatorname{Re}(f(z) - f_k(z)) \operatorname{Im} f_k(z))\big). \end{aligned}$$

Using the Stieltjes representation (see Theorem 1.5 (2)) for f and f_k we get

$$f_k(z) = \int_{(0,k)} \frac{z}{t + z}\, \rho(dt) \quad \text{and} \quad f(z) = \int_{(0,\infty)} \frac{z}{t + z}\, \rho(dt)$$

and since for $z = x + iy$

$$\operatorname{Im} \frac{z}{t + z} = \frac{ty}{(t + x)^2 + y^2} \quad \text{and} \quad \operatorname{Re} \frac{z}{t + z} = \frac{tx + x^2 + y^2}{(t + x)^2 + y^2}$$

is valid, we have

$$\operatorname{Im}\frac{f(z)-f_k(z)}{b+f_k(z)} = \frac{1}{|b+f_k(z)|^2}\int_{t\in(k,\infty)}\int_{s\in(0,k)}\Big[\frac{ty}{(t+x)^2+y^2}\Big(b+\frac{sx+x^2+y^2}{(s+x)^2+y^2}\Big)$$
$$-\frac{tx+x^2+y^2}{(t+x)^2+y^2}\,\frac{sy}{(s+x)^2+y^2}\Big]\,\rho(ds)\,\rho(dt)$$
$$=\frac{1}{|b+f_k(z)|^2}\int_{t\in(k,\infty)}\int_{s\in(0,k)}\Bigg[\frac{(t-s)y(x^2+y^2)}{((t+x)^2+y^2)((s+x)^2+y^2)}$$
$$+\frac{bty}{(t+x)^2+y^2}\Bigg]\,\rho(ds)\,\rho(dt)$$

which is positive whenever $y = \operatorname{Im} z \geq 0$. Hence, Theorem 1.5 shows that $(f - f_k)/(b + f_k) \in \mathscr{CBF}$, thus $(b+f)/(b+f_k) = (f-f_k)/(b+f_k) + 1 \in \mathscr{BF}$ and, by [3, example 9.9], its reciprocal value is completely monotone. Now Bernstein's theorem, see [3, Theorem 9.3] shows that the representing measure γ_k (from Lemma 2.3) of this completely monotone function is positive.

We still have to treat the case where $b = 0$. In this situation choose a sequence $\{b_n\}_{n\in N}$ such that $b_n > 0$ and $b_n \to 0$ and observe that the pointwise limit $f_k(x)/f(x) = \lim_{n\to\infty}(b_n + f_k(x))/(b_n + f(x))$ is again a completely monotone function. In particular, γ_k is positive. □

The measures γ_k are even (sub-)probability measures,

$$\int_{[0,\infty)} \gamma_k(dt) = \lim_{x\to 0}\widehat{\gamma_k}(x) = \lim_{x\to 0}\frac{b+f_k(x)}{b+f(x)} = 1$$

(if $b = 0$, a limiting argument as above still gives $\gamma_k([0,\infty)) \leq 1$), whose Laplace transforms approach the function $x \mapsto 1$ as $k \to \infty$. By Lévy's continuity theorem we have

$$\gamma_k \to \delta_0 \quad \text{as} \quad k \to \infty$$

vaguely, and by the uniform boundedness of the sequence $\{\gamma_k\}_{k\in N}$ also weakly. We can use this fact to construct approximations for every $u \in X$.

LEMMA 2.5. *Let $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ be as in Assumption* 1.6, *$f \in \mathscr{CBF}$ with representation* (2.2) *and $\{\gamma_k\}_{k\in N}$ as in Lemma* 2.3. *Then the Bochner integrals*

$$\int_{[0,\infty)} T_t u\,\gamma_k(dt), \quad u \in X,\ k \in N, \tag{2.9}$$

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define a sequence of bounded operators on X which strongly converges to the identity operator on X.

PROOF. Write $\alpha_k = 1/\int_{[0,\infty)} \gamma_k(dt)$ and observe that $\lim_{k\to\infty} \alpha_k = 1$. Since

$$\left\| \alpha_k \int_{[0,\infty)} T_t u\, \gamma_k(dt) \right\| \le \|u\|,$$

(2.9) defines indeed a bounded operator. For some small $\delta > 0$ we find

$$\begin{aligned}\left\| \alpha_k \int_{[0,\infty)} T_t u\, \gamma_k(dt) - u \right\| &= \left\| \alpha_k \int_{[0,\infty)} (T_t u - u)\, \gamma_k(dt) \right\| \\ &\le \alpha_k \int_{[0,\delta)} \|T_t u - u\|\, \gamma_k(dt) + 2\alpha_k \int_{[\delta,\infty)} \gamma_k(dt)\, \|u\| \\ &\le \sup_{t\le\delta} \|T_t u - u\| + 2\alpha_k \int_{[\delta,\infty)} \gamma_k(dt)\, \|u\|\end{aligned}$$

Letting first $k \to \infty$ and then $\delta \to 0$ we get both

$$\alpha_k \int_{(0,\infty)} T_t u\, \gamma_k(dt) \to u \quad \text{and} \quad \int_{(0,\infty)} T_t u\, \gamma_k(dt) \to u \quad \text{(strongly)}$$

as $k \to \infty$ since $\alpha_k \to 1$. □

Lemma 2.5 enables us to mimic the usual proof of the closedness of an infinitesimal generator of a (C_0)-semigroup.

PROPOSITION 2.6. *Let $\{T_t\}_{t\ge0}$, A, $\{\mu_t\}_{t\ge0}$ be as in Assumption 1.6, $f \in \mathscr{CBF}$ with representation (2.2), and $\{\gamma_k\}_{k\in N}$ as in Lemma 2.3. We then have the identity*

$$\begin{aligned}&\left(\int_{(0,\infty)} (T_t - 1) m_k(t)\, dt - b\right) \circ \int_{[0,\infty)} T_s u\, \gamma_l(ds) \\ &\quad = \left(\int_{(0,\infty)} (T_t - 1) m_l(t)\, dt - b\right) \circ \int_{[0,\infty)} T_s u\, \gamma_k(ds)\end{aligned} \tag{2.10}$$

for all $k, l \in N$, $b \ge 0$, and $u \in X$.

PROOF. Using the Stieltjes representation of f,

$$f(x) = \int_{(0,\infty)} \frac{x}{t(x+t)}\, \rho(dt),$$

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it is seen that for $v \in X$

$$\begin{aligned}\left\| \int_{(0,\infty)} (T_t - 1) v\, m_k(t)\, dt \right\| &\le \int_{(0,\infty)} \|T_t - 1\|\, m_k(t)\, dt\, \|v\| \\ &\le \int_{(0,k)} \int_{(0,\infty)} e^{-st}\, dt\, \rho(ds)\, \|v\| \\ &= \int_{(0,k)} \frac{\rho(ds)}{s}\, \|v\|,\end{aligned}$$

that is, both sides of the identity (2.10) are well-defined. Fixing $k, l \in N$, $k \neq l$, we observe

$$\begin{aligned}&\left(\int_{(0,\infty)} (T_t - 1) m_k(t)\, dt - b \right) \circ \int_{[0,\infty)} T_s u\, \gamma_l(ds) \\ &\quad = \int_{(0,\infty)} \int_{[0,\infty)} (T_t - 1) T_s u m_k(t) \gamma_l(ds)\, dt - b \int_{[0,\infty)} T_s u\, \gamma_l(ds) \\ &\quad = \int_{(0,\infty)} \int_{[0,\infty)} T_{t+s} u m_k(t) \gamma_l(ds)\, dt \\ &\qquad - \int_{(0,\infty)} \int_{[0,\infty)} T_s u m_k(t) \gamma_l(ds)\, dt - b \int_{[0,\infty)} T_s u\, \gamma_l(ds) \\ &\quad = \int_{[0,\infty)} T_r u (m_k \star \gamma_l)(r)\, dr - \left(b + \int_{(0,\infty)} m_k(t)\, dt \right) \int_{[0,\infty)} T_r u\, \gamma_l(dr) \\ &\quad = \int_{[0,\infty)} T_r u \left((m_k \star \gamma_l)(r) \lambda_{[0,\infty)} - \left(b + \int_{(0,\infty)} m_k(t)\, dt \right) \gamma_l \right) (dr)\end{aligned}$$

where $\lambda_{[0,\infty)}$ denotes the one-dimensional Lebesgue measure on the half-line. The Laplace transforms of

$$\left((m_k \star \gamma_l) \lambda_{[0,\infty)} - \left(b + \int_{(0,\infty)} m_k(t)\, dt \right) \gamma_l \right)^{\widehat{}}(x) = \frac{b + f_l(x)}{b + f(x)} (-b - f_k(x))$$

and of

$$\left((m_l \star \gamma_k) \lambda_{[0,\infty)} - \left(b + \int_{(0,\infty)} m_l(t)\, dt \right) \gamma_k \right)^{\widehat{}}(x) = \frac{b + f_k(x)}{b + f(x)} (-b - f_l(x))$$

coincide, therefore the measures coincide and the same calculation as above with k and l interchanged proves (2.10). □

REMARK 2.7. The key in the proof of Proposition 2.6 was to show that the representation measures of both sides of (2.10) are the same. This was done via their Laplace transforms. This, however, amounts to checking the identity (2.10) *just* for the semigroup $\{e^{-tx}\}_{t \ge 0}$, $x > 0$ fixed.

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We shall combine (2.10) with the approximation result of Lemma 2.5 to show the closedness of $(\tilde{A}^f, D(\tilde{A}^f))$.

THEOREM 2.8. *Let* $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ *be as in Assumption* 1.6, $f \in \mathscr{CBF}$ *of the form* (2.2) *and* $\{\gamma_k\}_{k\in N}$ *as in Lemma* 2.3. *Then we have* $\int_{[0,\infty)} T_t u\, \gamma_k(dt) \in D(\tilde{A}^f)$ *for all* $k \in N$ *and all* $u \in X$,

$$(2.11)\quad (\tilde{A}^f - b)\int_{[0,\infty)} T_t u\, \gamma_k(dt) = -bu + \int_{(0,\infty)} (T_t - 1)u\, m_k(t)\, dt, \quad u \in X,$$

and if $u \in D(\tilde{A}^f)$,

$$(2.12)\quad \begin{aligned}&\int_{[0,\infty)} T_t(\tilde{A}^f - b)u\, \gamma_k(dt)\\ &\qquad = -bu + \int_{(0,\infty)} (T_t - 1)u\, m_k(t)\, dt, \quad u \in D(\tilde{A}^f).\end{aligned}$$

Moreover, $D(\tilde{A}^f)$ *is dense in* X *and* $(\tilde{A}^f, D(\tilde{A}^f))$ *is a closed operator.*

PROOF. Throughout the proof we denote by $\langle \cdot, \cdot \rangle$ the dual pairing of X, X^* and ϕ always denotes an (arbitrary) vector of X^*.

In the proof of Proposition 2.6 we saw that $u \mapsto \int_{(0,\infty)} (T_t u - u)\, m_k(t)\, dt$ is a (strongly) bounded operator. Lemma 2.5 therefore allows us, for fixed $k \in N$, to pass to the weak limit $l \to \infty$ in the identity (2.10). This yields on the left-hand side

$$\begin{aligned}&\lim_{l\to\infty}\left\langle \left(\int_{(0,\infty)} (T_t - 1)m_k(t)\, dt - b\right) \circ \int_{[0,\infty)} T_s u\, \gamma_l(ds), \phi\right\rangle\\ &\qquad = \left\langle \left(\int_{(0,\infty)} (T_t - 1)m_k(t)\, dt - b\right) u, \phi\right\rangle\end{aligned}$$

for all $u \in X$ and $\phi \in X^*$. Since the limit on the right-hand side exists, the very definition of $\tilde{A}^f$ gives

$$\begin{aligned}&\lim_{l\to\infty}\left\langle \left(\int_{(0,\infty)} (T_t - 1)m_l(t)\, dt - b\right) \circ \int_{[0,\infty)} T_s u\, \gamma_k(ds), \phi\right\rangle\\ &\qquad = \left\langle (\tilde{A}^f - b)\int_{[0,\infty)} T_s u\, \gamma_k(ds), \phi\right\rangle.\end{aligned}$$

The definition of $D(\tilde{A}^f)$ now implies—note that the limit on the left side exists—that we have $\int_{[0,\infty)} T_s u\, \gamma_k(ds) \in D(\tilde{A}^f)$. Since there were no restrictions on $\phi \in X^*$, (2.11) follows; (2.12) is derived similarly. Just observe that

$$\begin{aligned}&\left(\int_{(0,\infty)} (T_t - 1)m_l(t)\, dt - b\right) \circ \int_{[0,\infty)} T_s u\, \gamma_k(ds)\\ &\qquad = \int_{[0,\infty)} T_s \left(\int_{(0,\infty)} (T_t - 1)m_l(t)\, dt - b\right) u\, \gamma_k(ds)\end{aligned}$$

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holds for all $u \in X$ and that for $u \in D(\tilde{A}^f)$

$$\text{weak-}\lim_{l\to\infty} \int_{(0,\infty)} (T_t - 1)u\, m_l(t)\, dt - bu = (\tilde{A}^f - b)u.$$

That $D(\tilde{A}^f) \subset X$ is (strongly) dense follows at once from Lemma 2.5. In order to check the closedness of $\tilde{A}^f$ on $D(\tilde{A}^f)$, we choose any sequence $\{u_n\}_{n\in N}$ in $D(\tilde{A}^f)$ satisfying

$$u_n \to u \in X \quad \text{strongly} \quad \text{and} \quad \tilde{A}^f u_n \to v \in X \quad \text{strongly} \quad \text{as} \quad n \to \infty.$$

We have to show that $u \in D(\tilde{A}^f)$ and $\tilde{A}^f u = v$. By (2.12) we have for any $\phi \in X^*$

$$\left\langle \int_{[0,\infty)} T_t(\tilde{A}^f - b)u_n\, \gamma_k(dt), \phi \right\rangle = \left\langle \int_{(0,\infty)} (T_t - 1)u_n\, m_k(t)\, dt - bu_n, \phi \right\rangle,$$

and as $n \to \infty$

$$\left\langle \int_{[0,\infty)} T_t(v - bu)\, \gamma_k(dt), \phi \right\rangle = \left\langle \int_{(0,\infty)} (T_t - 1)u\, m_k(t)\, dt - bu, \phi \right\rangle.$$

By Lemma 2.5 the strong limit

$$\text{strong-}\lim_{k\to\infty} \int_{[0,\infty)} T_t(v - bu)\, \gamma_k(dt) = v - bu$$

exists, and according to the definition of $D(\tilde{A}^f)$,

$$u \in D(\tilde{A}^f) \quad \text{and} \quad \tilde{A}^f u = v. \qquad \square$$

We can now identify the generator of the subordinate semigroup, A^f, and the operator $\tilde{A}^f$ given by (2.4), (2.5).

COROLLARY 2.9. *With the assumptions of Theorem* 2.8 *we have* $A^f = \tilde{A}^f$.

PROOF. Clearly, the operators (A^{f_k}, X), $k \in N$, generate (C_0) contraction semigroups. An application of Theorem 1.1 yields

$$\operatorname{Re}\left\langle A^{f_k} u, \phi \right\rangle \le 0 \tag{2.13}$$

for all $u \in X$ and *all* $\phi \in X^*$ (note: for all k we have $D(A^{f_k}) = X$ as $\|A^{f_k}\| < \infty$) satisfying $\|u\|^2 = \|\phi\|_*^2 = \langle u, \phi \rangle$. Passing to the limit $k \to \infty$ in (2.13) we get

$$\operatorname{Re}\left\langle \tilde{A}^f u, \phi \right\rangle \le 0 \tag{2.14}$$

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for all $u \in D(\tilde{A}^f)$ and all $\phi \in X^*$ such that $\|u\|^2 = \|\phi\|_*^2 = \langle u, \phi \rangle$. This is but the dissipativity of $\tilde{A}^f$.

Since $(A^f, D(A^f))$ generates a (C_0)-semigroup it is maximal dissipative (see [17, p. 237, Lemma 4.17], [8, p. 21, Proposition 4.1]). But $\tilde{A}^f$ is a closed, dissipative extension of $(A^f, D(A^f))$ and this is impossible unless $\tilde{A}^f = A^f$ as operators. □

COROLLARY 2.10. *Let* $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ *be as in Assumption* 1.6 *and* $f \in \mathscr{CBF}$ *as in* (2.2). *Then for the domain of* A^f *(as generator of the subordinate semigroup* $\{T_t^f\}_{t\geq 0}$*) we have*

$$D(A^f) = \left\{u \in X : \lim_{k\to\infty} \int_{(0,\infty)} (T_t u - u)\, m_k(t)\, dt \text{ exists strongly}\right\}$$
$$= \left\{u \in X : \lim_{k\to\infty} \int_{(0,\infty)} (T_t u - u)\, m_k(t)\, dt \text{ exists weakly}\right\}. \tag{2.15}$$

PROOF. Write (only in this proof) $\tilde{A}^{f,w} = \tilde{A}^f$ and $\tilde{A}^{f,s}$ for the *strong* version of Definition 2.2. We then have

$$A^f \subset \tilde{A}^{f,s} \subset \tilde{A}^{f,w}$$

where '$\subset$' means the extension in the sense of operators. By Corollary 2.9, however, we also have

$$\tilde{A}^{f,w} \subset A^f,$$

which completes the proof. □

REMARK 2.11. A closer look at our method reveals that every approximation $\{f_n\}_{n\in N}$ of some $f \in \mathscr{BF}$ (not necessarily in $\mathscr{CBF}$) satisfying

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x > 0 \quad \text{and } f_n \in \mathscr{BF},$$

and, writing $\widehat{\gamma_n} = f_n/f$ (γ_n is a signed measure, the proof of Lemma 2.3 remains valid),

$$\gamma_n \to \delta_0 \quad \text{weakly in the sense of signed measures}$$

leads to the same operator, namely $\lim_{n\to\infty} A^{f_n} = A^f$.

3. An asymptotic result

We will now study a converse of (2.6). Consider the equality

$$\frac{1}{f(l)}(b+f(x))\widehat{\beta_l}(x) = 1 - e^{-x/l}, \quad x > 0, l \in N, \tag{3.1}$$

where $b > 0$ and $\widehat{\beta_l}$ is the (one-sided) Laplace transform of the (signed) measure

$$\beta_l = f(l)(\nu - \delta_{1/l} \star \nu); \tag{3.2}$$

recall that ν is the representing measure of

$$\frac{1}{b+f(x)} = \int_{[0,\infty)} e^{-tx}\, \nu(dt). \tag{3.3}$$

Assume moreover that ν has a *decreasing* density n with respect to Lebesgue measure on $[0, \infty)$—this is always the case when $f \in \mathscr{CBF}$ since then $1/(b + f(x))$ is a Stieltjes transform with representing density $n \in \mathscr{CM}$. Then β_l has a density

$$\frac{d\beta_l}{dx}(x) = f(l)\left(n(x)) - 1_{[l^{-1},\infty)}(x)n(x - l^{-1})\right), \quad x > 0. \tag{3.4}$$

Since n is decreasing, the total variation of β_l satisfies

$$\begin{aligned}
\|\beta_l\| &= \int_{[0,\infty)} \left|\frac{d\beta_l}{dx}(x)\right| dx \\
&= f(l)\left(\int_{[0,l^{-1})} n(x)\,dx + \int_{[l^{-1},\infty)} \left(n(x-l^{-1}) - n(x)\right) dx\right) \\
&= 2f(l)\int_{[0,l^{-1})} n(x)\,dx \le 2f(l)e\int_{[0,l^{-1})} e^{-xl}n(x)\,dx \le 2e\,\frac{f(l)}{b+f(l)} \le 2e.
\end{aligned}$$

Thus, the family of measures $\{\beta_l\}_{l\in N}$ has uniformly bounded total mass. This immediately gives the following result.

LEMMA 3.1. *Let* $\{T_t\}_{t\ge0}$, A, $\{\mu_t\}_{t\ge0}$ *be as in Assumption* 1.6, $f \in \mathscr{BF}$ *and* $\nu(dt) = n(t)1_{(0,\infty)}(t)\,dt$ *as above, and* β_l *as in* (3.1). *Then*

$$\sup_{l\in N}\left\|\int_{[0,\infty)} T_t u\, \beta_l(dt)\right\| \le 2e\,\|u\| \tag{3.5}$$

holds for all $u \in X$.

Exactly as in the proof of Theorem 2.8, we obtain the following result.

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PROPOSITION 3.2. *Let* $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ *be as in Assumption* 1.6, $f \in \mathcal{BF}$ *as in Lemma* 3.1, *and* β_l *given by* (3.1). *Then*

$$f(l)(T_{1/l}-1)\int_{[0,\infty)} T_t u\,\gamma_k(dt) = \left(\int_{(0,\infty)}(T_t-1)m_k(t)\,dt - b\right)\circ\int_{[0,\infty)} T_s u\,\beta_l(ds) \tag{3.6}$$

holds for all $k, l \in N$, *and* $u \in X$.

REMARK 3.3. One should observe that in equality (3.6) both γ_k and β_l will depend on $b > 0$.

PROOF. As in the proof of Proposition 2.6, see Remark 2.7, it suffices to check (3.6) for the semigroup e^{-tx} with $x > 0$ fixed:

$$\begin{aligned} -f(l)(1-e^{-x/l})\widehat{\gamma_k}(x) &= -f(l)(1-e^{-x/l})\frac{f_k(x)+b}{f(x)+b} \\ &= -(f_k(x)+b)f(l)\frac{1-e^{-x/l}}{f(x)+b} \\ &= -(f_k(x)+b)\,\widehat{\beta_l}(x). \end{aligned}$$

□

The combination of Lemma 3.1 and Proposition 3.2 yields the analogue of Theorem 2.8.

THEOREM 3.4. *Let* $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ *be as in Assumption* 1.6, $f \in \mathcal{CBF}$ *as in* (2.2), *and* $\{\beta_l\}_{l\in N}$, $\{\gamma_k\}_{k\in N}$ *as before. Then* $\int_{[0,\infty)} T_t u\,\beta_l(dt) \in D(A^f)$ *for all* $u \in X$ *and* $l \in N$. *Moreover*

$$f(l)(T_{1/l}-1)u = (A^f - b)\int_{[0,\infty)} T_s u\,\beta_l(ds), \quad u \in X, \tag{3.7}$$

and, if $u \in D(A^f)$, *also*

$$f(l)(T_{1/l}-1)u = \int_{[0,\infty)} T_s(A^f-b)u\,\beta_l(ds), \quad u \in D(A^f). \tag{3.8}$$

Note that the proof of (3.8) uses the closedness of the operator $A^f - b$.

COROLLARY 3.5. *With the assumptions of Theorem* 3.4 *we have*

$$\|(T_\epsilon - 1)u\| = O\left(\frac{1}{f(\epsilon^{-1})}\right) \quad as \quad \epsilon \to 0 \tag{3.9}$$

for all $u \in D(A^f)$.

PROOF. If $u \in D(A^f)$, (3.8) gives for $l^{-1} = \epsilon$,

$$\|T_\epsilon u - u\| \le \frac{1}{f(\epsilon^{-1})} \int_{[0,\infty)} |\beta_{1/\epsilon}|(ds) \left(b\|u\| + \|A^f u\|\right), \tag{3.10}$$

and the assertion follows from $\|\beta_t\| \le 2e$, see Lemma 3.1. □

The following corollary contains the converse of Lemma 2.1.

COROLLARY 3.6. *Under the assumptions of Theorem 3.4, the operator A^f is bounded if and only if either A is bounded or f is bounded or both are bounded.*

PROOF. The sufficiency has already been established (under less restrictive assumptions) in Lemma 2.1. In order to see the necessity, let A^f be a bounded operator, that is, $\|A^f u\| \le c\|u\|$ for some $c > 0$ and all $u \in X$, and A be an *unbounded* operator. Suppose $\lim_{x\to\infty} f(x) = \infty$. In this case, (3.10) gives

$$\|T_\epsilon u - u\| \le \frac{2e}{f(\epsilon^{-1})} \left(b\|u\| + \|A^f u\|\right) \le \frac{2e}{f(\epsilon^{-1})} (b + c)\|u\|$$

for all $u \in X$. Thus,

$$\sup_{0\ne u\in X} \frac{\|T_\epsilon u - u\|}{\|u\|} = \|T_\epsilon - 1\| \le \frac{C}{f(\epsilon^{-1})}, \quad \epsilon \in (0, 1),$$

with a constant C not depending on ϵ. Letting $\epsilon \to 0$ we find

$$\lim_{\epsilon\to 0} \|T_\epsilon - 1\| \le \lim_{\epsilon\to 0} \frac{C}{f(\epsilon^{-1})} = \lim_{l\to\infty} \frac{C}{f(l)} = 0,$$

that is, $\{T_t\}_{t\ge 0}$ is continuous in the uniform operator norm. Therefore, see [14, p. 2, Theorem 1.2], its generator A is bounded which contradicts our assumption. □

Our next corollary is a generalization of the well-known relation $D(A^\beta) \subset D(A^\alpha)$ if $0 < \alpha \le \beta \le 1$.

COROLLARY 3.7. *Let $\{T_t\}_{t\ge 0}$, A, $\{\mu_t\}_{t\ge 0}$ be as in Assumption 1.6, $f \in \mathscr{CBF}$ given by (2.2) and g be another, not necessarily complete, Bernstein function of the form (1.7) with representing measure μ^g and $a = b = 0$. If*

$$\int_{(0,\epsilon)} \frac{\mu^g(dt)}{f(t^{-1})} < \infty \qquad \textit{for some } \epsilon > 0, \tag{3.11}$$

then we have $D(A^g) \supset D(A^f)$.

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PROOF. If $u \in D(A^f)$, we get for any $\epsilon > 0$

$$\left\| \int_{(0,\infty)} (T_t u - u)\, \mu^g(dt) \right\| \le \int_{(0,\epsilon)} \|T_t u - u\|\, \mu^g(dt) + \int_{[\epsilon,\infty)} \mu^g(dt)\, \|u\|$$
$$\le C \int_{(0,\epsilon)} \frac{\mu^g(dt)}{f(t^{-1})} \left(\|A^f u\| + \|u\|\right)$$

with a constant C depending on g, ϵ, and b. Assumption (3.11) allows us to choose $\epsilon \in (0, 1)$ such that the above integral converges. Therefore, $u \in D(A^g)$, and the proof is complete. □

REMARK 3.8. It is easy to see that

$$\mu^g([\delta, \epsilon)) \le \frac{1}{(1 - e^{-1})} \int_{[\delta,\infty)} (1 - e^{-s/\delta}) \mu^g(ds) \le \frac{e}{e-1} g(\delta^{-1})$$

holds for any $0 < \delta < \epsilon$. Since

$$\lim_{\delta \to 0} \int_{[\delta,\epsilon)} \frac{\mu^g(dt)}{f(t^{-1})} = \int_{(0,\epsilon)} \frac{\mu^g(dt)}{f(t^{-1})},$$

whenever the integral on the right hand side exists, and, since by integration by parts

$$\int_{[\delta,\epsilon)} \frac{\mu^g(dt)}{f(t^{-1})} = \frac{\mu^g([\delta, \epsilon))}{f(\delta^{-1})} + \int_{[\delta,\epsilon)} \frac{t^{-2} f'(t^{-1})}{f^2(t^{-1})} \mu^g([t, \epsilon))\, dt$$
$$\le \frac{e}{e-1} \left(\frac{g(\delta^{-1})}{f(\delta^{-1})} + \int_{[\delta,\epsilon)} \frac{t^{-2} f'(t^{-1}) g(t^{-1})}{f^2(t^{-1})}\, dt \right)$$

for some fixed $\epsilon > 0$ and $\delta \to 0$, it is a sufficient condition for (3.11) to hold that

$$\tau \mapsto \frac{f'(\tau) g(\tau)}{f^2(\tau)}$$

is integrable in some neighborhood of $+\infty$ and that

$$\frac{g(\tau)}{f(\tau)} = O(1) \quad \text{as } \tau \to \infty.$$

The above remark contains an interesting special case. Since for Bernstein functions $f'(x)/f(x) \le 1/x$ holds (see, for example, [12]), both conditions in Remark 3.8 are met if $g(x)/f(x)$ decays like some power $x^{-\lambda}$. Let us state this criterion in the following corollary.

COROLLARY 3.9. *Let* $\{T_t\}_{t\ge0}$, A, $\{\mu_t\}_{t\ge0}$, $f \in \mathscr{CBF}$, *and* $g \in \mathscr{BF}$ *be as in Corollary* 3.7. *If for some* $\lambda > 0$ *the ratio* $g(x)/f(x) = O(x^{-\lambda})$ *as* $x \to \infty$, *then* $D(A^g) \supset D(A^f)$.

4. Towards a functional calculus

In our last section we will show that subordination in the sense of Bochner gives rise to a reasonable functional calculus for the generators of subordinate semigroups. In particular, we will prove some multiplication rule for these generators. First of all, however, let us discuss the relation to other functional calculi. In [2] and [19] it was shown that the above construction of A^f extends the Dunford-Taylor functional calculus [7, Chapter VII.3, VII.9] in the following sense: the resolvent of $(A-\epsilon)^f$, $\epsilon > 0$, is given by

$$(4.1) \qquad (\lambda - (A-\epsilon)^f)^{-1}u = (\lambda + f(\epsilon - A))^{-1}u, \quad \lambda > -f(-\omega),$$

where the right hand side is to be understood in the sense of the Dunford-Taylor functional calculus. Both sides of (4.1) converge as $\epsilon \to 0$, see [19, Theorem 4.3] or [2, Proposition 5.17; Theorem 4.2]. It is therefore justified to write $-f(-A) = A^f$ for $f \in \mathscr{CBF}$.

From now on we will, however, follow a somewhat different route which is closer to Dunford and Schwartz's definition of functions of an infinitesimal generator, see [7, Chapter VIII.2] and [10, Chapter XV.1,2]. In particular, our definition of A^f for bounded f, for example, for the approximations f_k of (2.3), matches the definition given in Hille and Phillips [10, equation (15.4.1)].

Here are some rules for generators of subordinate semigroups.

THEOREM 4.1. *Let $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ be as in Assumption* 1.6 *and $f, g \in \mathscr{CBF}$ with representations of type* (2.2). *Then we have*

(1) $\alpha f \in \mathscr{CBF} \quad (\alpha > 0)$ *and* $A^{\alpha f} = \alpha A^f$
(2) $f + g \in \mathscr{CBF}$ *and* $A^{f+g} = \overline{A^f + A^g}$
(3) $f \circ g \in \mathscr{CBF}$ *and* $A^{f\circ g} = \left(A^g\right)^f$
(4) *if* $a \geq 0$ *then* $A^{a+x+f} = -a + A + A^f$
(5) *if* $fg \in \mathscr{CBF}$ *then* $A^{fg} = -A^f \circ A^g = -A^g \circ A^f$
(5-bis) $g(x) = x/f(x), g \in \mathscr{CBF}$ *and* $A = -A^f \circ A^g = -A^g \circ A^f$

where the right-hand sides of (1)–(5-bis) *are equalities in the sense of closed operators with the usual domains for sums, compositions, and so on, of operators. Note, that we can always identify A^f, A^g, and A^{fg} with $-f(-A)$, $-g(-A)$, and $-(fg)(-A)$, respectively.*

PROOF. Clearly, $f, g \in \mathscr{CBF}$ implies that αf, $f+g$, $f\circ g \in \mathscr{CBF}$, see, for example, [2, 18]. Since $D(A) \subset D(A^f) \cap D(A^g)$, all of the above operators are densely defined. By the linearity of the definition of $\tilde{A}^f$, that is, of A^f (Definition 2.2), (1), (2), and (3) are immediate, (3) being a consequence of the transitivity of subordination, $(T_t^g)^f = T_t^{f\circ g}$, see [5, Theorem 4.4.3]. In order to prove (4), observe that A^f is an A–bounded operator in the sense that $\|A^f u\| \leq \epsilon\|Au\| + c_\epsilon\|u\|$, see, for example, the proof of Lemma 2.1 or [19, Theorem 5.1]. Thus, $(b + A + A^f, D(A))$ is a closed operator and the assertion follows from formula (2.1).

For (5) we proceed as in the proof of Theorem 2.8 (with $b = 0$) and note the following identity ($k, l, m \in \mathbf{N}$)

$$f_k(x) g_l(x) \widehat{\gamma}_m^{fg}(x) = (fg)_m(x) \widehat{\gamma}_l^g(x) \widehat{\gamma}_k^f(x) \tag{4.2}$$

where we used the notation of Lemma 2.3, that is,

$$\widehat{\gamma}_k^h(x) = \frac{h_k(x)}{h(x)} \in \mathscr{CM}, \quad x > 0,\ k \in \mathbf{N},$$

with $h = f, g, fg$ and h_k means the truncation as done in (2.3). Now (4.2) shows for all $u \in X$

$$A^{f_k} A^{g_l} \int_{[0,\infty)} T_t u\, \gamma_m^{fg}(dt) = -A^{(fg)_m} \int_{[0,\infty)} \int_{[0,\infty)} T_{t+s} u\, \gamma_l^g(dt)\, \gamma_k^f(ds). \tag{4.3}$$

If $u \in D(A^f \circ A^g) = \{u \in X : u \in D(A^g) \text{ and } A^g u \in D(A^f)\}$, we find by letting $l \to \infty$, $k \to \infty$, and *then* $m \to \infty$ in (4.3) that $u \in D(A^{fg})$, defined as in Definition 2.2, hence $A^f \circ A^g \subset -A^{fg}$ and, by symmetry, $A^g \circ A^f \subset -A^{fg}$.

If, conversely, $u \in D(A^{fg})$ (given by Definition 2.2 and Corollary 2.10), we first let $m \to \infty$ in (4.3) and afterwards $l \to \infty$ and $k \to \infty$. This shows that $u \in D(A^f \circ A^g)$ and therefore $-A^{fg} \subset A^f \circ A^g$ and $-A^{fg} \subset A^g \circ A^f$, respectively.

Assertion (5-bis) follows directly from (5) and Theorem 1.5. Note, that with our assumptions on f one has $\lim_{x\to\infty} g(x)/x = 0$ (that is, g has no linear part) but it might occur that $\lim_{x\to 0} g(x) = 1/f'(0) > 0$ (if $f'(0) < \infty$). Thus, g is up to a trivial part of the form (2.2) and (5) is indeed applicable (see also (4)). □

REMARK 4.2. (a) Theorem 4.1 shows, in particular, that for $f, g \in \mathscr{CBF}$ such that $fg \in \mathscr{CBF}$ always $D(A^f \circ A^g) = D(A^g \circ A^f) = D(A^{fg})$ holds true. This is but to say

$$u \in D(A^f) \text{ and } A^f u \in D(A^g) \quad \text{if and only if} \quad u \in D(A^g) \text{ and } A^g u \in D(A^f).$$

Therefore, it is possible to speak of A^f and A^g as *commuting* operators whenever $fg \in \mathscr{CBF}$.

(b) Another consequence of 4.1 (5) is that for $f \in \mathscr{CBF}$

$$-A^f = \left(A^{\sqrt{f}}\right)^2 \quad \text{hence} \quad -\sqrt{-A^f} = A^{\sqrt{f}}$$

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(c) The representing measure of fg can be explicitly calculated. If f has the representation

$$\frac{f(x)}{x} = \int_{(0,\infty)} \frac{\sigma^f(dt)}{t+x}, \qquad \frac{f_k(x)}{x} = \int_{(0,k)} \frac{\sigma^{f_k}(dt)}{t+x};$$

with $\sigma^{f_k} = 1_{(0,k)}(\cdot)\sigma^f$—see Theorem 1.5 (2), (2.2), (2.3)—and similarly $x^{-1}g(x)$ and $x^{-1}g_k(x)$, we can use a convolution-type theorem for the Stieltjes transform, see [20, (2),(3)], that implies

$$\sigma^{fg} = \int_{(0,\infty)} \frac{\sigma^g(dy)}{y-t}\sigma^f + \int_{(0,\infty)} \frac{\sigma^f(dy)}{y-t}\sigma^g.$$

Here, the integrals are to be understood as Cauchy principal values.

(d) It might be instructive to note that Theorem 4.1 covers the case of the Yosida approximation of an infinitesimal generator A. The *Yosida approximation* is the family

$$A_\lambda = \lambda A(\lambda - A)^{-1}, \quad \lambda > 0$$

of bounded linear operators that strongly approach A on $D(A)$ as $\lambda \to \infty$.

If we put $y(\lambda; x) = \lambda x(\lambda + x)^{-1}$, we find on X

$$A^{y(\lambda;\cdot)} = -y(\lambda; -A) = \lambda A(\lambda - A)^{-1} = A_\lambda$$

and similarly

$$\left(A^f\right)^{y(\lambda;\cdot)} = A^{y(\lambda; f(\cdot))} = \left(A^f\right)_\lambda$$

for any $f \in \mathscr{CBF}$.

The requirement that $fg \in \mathscr{CBF}$ for $f, g \in \mathscr{CBF}$ in (5) of Theorem 4.1 is quite restrictive. We can overcome this difficulty if we use Berg's result [1] that the cone of Stieltjes functions is logarithmically convex, that is to say that for all Stieltjes transforms Φ, Γ and any $0 \le \lambda \le 1$, $\Phi^\lambda \Gamma^{1-\lambda}$ is again a Stieltjes transform. Combining this with Theorem 1.5 (1), (2) we get

$$f^\lambda g^{1-\lambda} \in \mathscr{CBF} \quad \text{for all} \quad f, g \in \mathscr{CBF}, 0 \le \lambda \le 1. \tag{4.4}$$

This observation together with Theorem 4.1 enables us to write down a consistent formula for the operator $(fg)(-A)$ and all $f, g \in \mathscr{CBF}$—even if A^{fg} has no longer any meaning since it cannot be the generator of a subordinate semigroup if $fg \notin \mathscr{BF}$.

DEFINITION 4.3. For $f_1, f_2, \dots, f_N \in \mathscr{CBF}$, $N \in \mathbf{N}$, we set

$$(f_1 \cdot f_2 \cdots f_N)(-A) = \left[(-A^{f_1^{1/N}}) \circ (-A^{f_2^{1/N}}) \circ \cdots \circ (-A^{f_N^{1/N}})\right]^N \tag{4.5}$$

on its natural domain.

Let $\mathscr{A} = \{f_1 f_2 \cdots f_N : N \in \mathbb{N}, f_j \in \mathscr{CBF}, 0 \leq j \leq N\}$ denote the set of all finite products of $\mathscr{CBF}$-functions. We want to show that the above definition extends Theorem 4.1 (5) to the set $\mathscr{A}$. First, however, we have to check that (4.5) gives a well-defined operator. Let $F \in \mathscr{A}$ with representations $F = f_1 f_2 \cdots f_N$ and $F = g_1 g_2 \cdots g_M$, $M \geq N$, $f_j, g_j \in \mathscr{CBF}$. Extending (4.4) to N-fold products we see, by Theorem 4.1 (5)

$$\left[(-A^{f_1^{1/N}}) \circ (-A^{f_2^{1/N}}) \circ \cdots \circ (-A^{f_N^{1/N}})\right]^N = (-1)^{N^2} \left[A^{F^{1/N}}\right]^N = \left[-A^{F^{1/N}}\right]^N$$

(note that $(-1)^N = (-1)^{N^2}$) and an analogous formula for the other representation of F. In particular, we find $F^{1/N}, F^{1/M} \in \mathscr{CBF}$ and it remains to show that $[-A^{F^{1/N}}]^N = [-A^{F^{1/M}}]^M$. However,

$$\left[\left[-A^{F^{1/N}}\right]^N\right]^{1/M} = \left[-A^{F^{1/N}}\right]^{N/M} = -A^{f^{1/M}},$$

where the first equality is well-known for (arbitrary) powers of closed operators, while the second identity follows from Theorem 4.1 (3). (All equalities are to be understood in the sense of closed operators.) Thus, (4.5) is well-defined for $\mathscr{A}$ and the following corollary shows that it indeed extends Theorem 4.1 (5).

COROLLARY 4.4. *Let $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ be as in Assumption 1.6, $f, g \in \mathscr{CBF}$ with representations of type (2.2) and put $H = fg \in \mathscr{A}$. Then*

$$H(-A) = A^f \circ A^g = A^g \circ A^f \tag{4.6}$$

$$(xf)(-A) = A \circ A^f = A^f \circ A. \tag{4.7}$$

More generally, if $H = FG$ with $F, G \in \mathscr{CBF}$, then

$$H(-A) = F(-A) \circ G(-A) = G(-A) \circ F(-A). \tag{4.8}$$

The above equalities are to be understood in the sense of closed operators.

PROOF. Put $B = A^{\sqrt{fg}}$. By definition, $H(-A) = B^2$, and $B = A^{\sqrt{fg}} = A^{\sqrt{f}} \circ A^{\sqrt{g}} = A^{\sqrt{g}} \circ A^{\sqrt{f}}$ by Theorem 4.1 (5). Since

$$\begin{aligned} D(B^2) &= \left\{u \in X : u \in D(A^{\sqrt{f}\sqrt{g}}) \quad \text{and} \quad A^{\sqrt{f}\sqrt{g}}u \in D(A^{\sqrt{f}\sqrt{g}})\right\} \\ &= \{u \in X : u \in D(A^{\sqrt{g}}), A^{\sqrt{g}}u \in D(A^{\sqrt{f}}), A^{\sqrt{f}}A^{\sqrt{g}}u \in D(A^{\sqrt{g}}) \\ &\qquad \text{and } A^{\sqrt{g}}A^{\sqrt{f}}A^{\sqrt{g}}u \in D(A^{\sqrt{f}})\} \end{aligned}$$

we get from Remark 4.2 (1) that $D(B^2) = D(A^f \circ A^g) = D(A^g \circ A^f)$ and, again by Remark 4.2 (1), (2), that

$$B^2 = \left(A^{\sqrt{f}} A^{\sqrt{g}}\right) \circ \left(A^{\sqrt{f}} A^{\sqrt{g}}\right) = \left(A^{\sqrt{f}} A^{\sqrt{f}}\right) \circ \left(A^{\sqrt{g}} A^{\sqrt{g}}\right) = A^f \circ A^g = A^g \circ A^f.$$

This proves (4.6). Now (4.7) follows from (4.6) since we have $xf(x) = \sqrt{x}\sqrt{x}f(x)$ and $\sqrt{x} \in \mathscr{CBF}$ which has also a representation of type (2.2).

Finally, (4.6), (4.7), and Theorem 4.1 (5) prove (4.8) if F, G are in $\mathscr{CBF}$, but do not necessarily have the form (2.2). Assume now that $F \in \mathscr{CBF}$ and $G = g_1 g_2 \cdots g_M$ with $g_j \in \mathscr{CBF}$. Then $H = F \cdot g_1 g_2 \cdots g_M \in \mathscr{A}$ and

$$H(-A) = F(-A) \circ g_1(-A) \circ \cdots \circ g_M(-A) = F(-A) \circ G(-A)$$

which proves (4.8) first for $F \in \mathscr{CBF}$ and $G \in \mathscr{A}$ and, by iteration, for $F, G \in \mathscr{A}$. □

Let us point out the connection of Corollary 4.4 with results obtained by deLaubenfels [6] where automatic extensions of (bounded) functional calculi to wider classes of functions are considered. Our extension resembles Method II introduced there, see in particular [6, Examples 4.1, 4.2]. However, in our situation these constructions seem not directly applicable since neither $\mathscr{CBF}$ nor $\mathscr{A}$ are algebras. It is obvious that we cannot transfer our results to the algebra generated by $\mathscr{CBF}$ or $\mathscr{A}$ without losing all the nice properties, for example, information on closedness and so on.

We finally prove some convergence results for sequences of subordinate generators.

THEOREM 4.5. *Let $\{T_t\}_{t\geq 0}$, A, $\{\mu_t\}_{t\geq 0}$ be as in Assumption* 1.6 *and $\{f^{(n)}\}_{n\in N}$ be a sequence of complete Bernstein functions of the form* (2.2) *with pointwise limit f. If f has no linear part, that is, if* $\lim_{x\to\infty} x^{-1} f(x) = 0$, *then*

$$\text{strong-}\lim_{n\to\infty} A^{f^{(n)}} u = A^f u \tag{4.9}$$

for all $u \in D(A^f) \cap \bigcap_{n=1}^{\infty} D(A^{f^{(n)}})$. *(Note that always* $D(A) \subset D(A^f) \cap \bigcap_{n=1}^{\infty} D(A^{f^{(n)}})$.)

PROOF. As pointwise limit of complete Bernstein functions f is itself in $\mathscr{CBF}$. Define as in (2.3) for $k \in N$ the truncation $(f^{(n)})_k$ of $f^{(n)}$ with $n \in N$ being fixed. Since $f^{(n)}$ is of the form (2.2), some elementary calculations show $f^{(n)}(x) = \int_{(0,\infty)} x/(t(x + t))\rho^{(n)}(dt)$ and $(f^{(n)})_k(x) = \int_{(0,k)} x/(t(x+t))\rho^{(n)}(dt)$, see also Theorem 1.5 (2) and (4).

We claim that

$$\lim_{n,k\to\infty} (f^{(n)})_k(x) = \lim_{k\to\infty}\lim_{n\to\infty} (f^{(n)})_k(x) = \lim_{n\to\infty}\lim_{k\to\infty} (f^{(n)})_k(x) = f(x), \quad x > 0, \tag{4.10}$$

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holds true. In order to prove (4.10) we note

$$\left|f(x)-(f^{(n)})_k(x)\right| \le \left|f(x)-f^{(n)}(x)\right| + \left|f^{(n)}(x)-(f^{(n)})_k(x)\right|$$
$$= \left|f(x)-f^{(n)}(x)\right| + \int_{[k,\infty)} \frac{x}{t(x+t)}\,\rho^{(n)}(dt).$$

Moreover, for every $l > 1$

$$\int_{[k,\infty)} \frac{x}{t(x+t)}\,\rho^{(n)}(dt) \le \frac{x+k/l}{x+k}\int_{[k,\infty)} \frac{xl}{t(xl+t)}\,\rho^{(n)}(dt)$$
$$\le \frac{x+k/l}{x+k} f^{(n)}(xl)$$

which implies

$$\lim_{k\to\infty}\lim_{n\to\infty} \frac{x+k/l}{x+k} f^{(n)}(xl) = \lim_{n\to\infty}\lim_{k\to\infty} \frac{x+k/l}{x+k} f^{(n)}(xl) = \frac{f(xl)}{l}.$$

Since f has no linear part, $\lim_{l\to\infty} f(xl)l^{-1} = 0$, and (4.10) follows.

Using the results of Section 2 we get

$$\frac{(f^{(n)})_k}{f^{(n)}} = \widehat{\gamma_k^{(n)}} \qquad \text{and} \qquad \frac{f^{(n)}}{f} = \widehat{\gamma^{(n)}} \qquad k,n\in N$$

for suitable sub-probability measures $\gamma_k^{(n)}$ and $\gamma^{(n)}$. Hence,

$$\frac{(f^{(n)})_k}{f} = \frac{(f^{(n)})_k}{f^{(n)}}\,\frac{f^{(n)}}{f} = \left(\gamma_k^{(n)} \star \gamma^{(n)}\right)^{\frown}$$

with another sub-probability measure $\gamma_k^{(n)} \star \gamma^{(n)}$. Therefore, Remark 2.11 applies and shows that the double sequence

$$\lim_{k,n\to\infty} A^{(f^{(n)})_k} u = A^f u$$

converges strongly for all $u \in D(A^f)$.

For fixed $u \in D(A^f)$ and any $\epsilon > 0$ there is a number $N = N(\epsilon, u)$ such that

$$\left\|A^f u - A^{(f^{(n)})_k} u\right\| \le \epsilon \quad \text{for all} \quad n,k \ge N.$$

By the triangle inequality we get for $u \in D(A^f)$ and $n,k \ge N(\epsilon,u)$

$$\left\|A^f u - A^{f^{(n)}} u\right\| \le \left\|A^f u - A^{(f^{(n)})_k} u\right\| + \left\|A^{(f^{(n)})_k} u - A^{f^{(n)}} u\right\|$$
$$\le \epsilon + \left\|A^{(f^{(n)})_k} u - A^{f^{(n)}} u\right\|$$

If also $u \in D(A^{f^{(n)}})$ for all $n \ge N(u,\epsilon)$, we find that $\lim_{n\to\infty} \|A^f u - A^{f^{(n)}} u\| \le \epsilon$ for $k \to \infty$ and the assertion follows as $\epsilon \to 0$. □

The assumption $\lim_{x\to\infty} x^{-1} f(x) = 0$ in the statement of Theorem 4.5 is essential. For example, put $f^{(n)}(x) = nx(n+x)^{-1}$ which is a complete Bernstein function with limiting function $f(x) = x$ as $n \to \infty$ (note that $-f^{(n)}(-A)$ is just the Yosida approximation of A). Clearly, $\rho^{(n)} = n\delta_n$ and therefore

$$\lim_{n\to\infty} \lim_{k\to\infty} (f^{(n)})_k(x) = x \neq \lim_{k\to\infty} \lim_{n\to\infty} (f^{(n)})_k(x) = 0, \quad x > 0,$$

though still $\lim_{n\to\infty} A^{f^{(n)}} = A$ strongly on $D(A)$. The reason that the proof of Theorem 4.5 fails is that $\rho^{(n)}$ as $n \to \infty$ pushes too much mass too fast to ∞—this is the only way to produce linear growth in the limiting function if for the approximations $\lim_{x\to\infty} x^{-1} f^{(n)}(x) = 0$, $n \in \mathbf{N}$, holds. Therefore we have to adapt our truncation procedure, that is, we have to choose in $(f^{(n)})_k$ the truncation $k \in \mathbf{N}$ in dependence of $n \in \mathbf{N}$.

COROLLARY 4.6. *In the situation of Theorem* 4.5 *assume that the sequence* $f^{(n)}$ *has a limit* f *such that* $\lim_{x\to\infty} x^{-1} f(x) = b > 0$. *Then* $D(A^f) = D(A)$ *and for* $u \in D(A)$ *we have* $\lim_{n\to\infty} A^{f^{(n)}} u = A^f u$ *in the strong topology.*

PROOF. That $D(A^f) = D(A)$ if (and only if) $b > 0$ was, for example, shown in [19]. Since $\int_{[0,\infty)} (t(1+t))^{-1} \rho^{(n)}(dt) < \infty$ for every $n \in \mathbf{N}$, we can choose a $k = k(n) \in \mathbf{N}$ such that $\int_{[k(n),\infty)} t^{-2} \rho^{(n)}(dt) \le n^{-1}$. Thus,

$$\begin{aligned}\left|f(x) - (f^{(n)})_{k(n)}(x)\right| &\le \left|f(x) - f^{(n)}(x)\right| + \left|f^{(n)}(x) - (f^{(n)})_{k(n)}(x)\right| \\ &\le \left|f(x) - f^{(n)}(x)\right| + \frac{1}{n},\end{aligned}$$

that is, $\lim_{n\to\infty} (f^{(n)})_{k(n)}(x) = f(x)$ and with the same reasoning as in the proof of Theorem 4.5 we find for every $\epsilon > 0$ and $u \in D(A)$ a number $N(u,\epsilon)$ such that

$$\left\|A^f u - A^{(f^{(n)})_{k(n)}} u\right\| \le \epsilon \quad \text{for all} \quad n \ge N(u,\epsilon).$$

Since $\lim_{m\to\infty} A^{(f^{(n)})_m} u = A^{f^{(n)}} u$ for all $n \in \mathbf{N}$ and $u \in D(A)$, we may choose a natural number $M(u,\epsilon,n) \ge N(u,\epsilon)$ such that

$$\left\|A^{f^{(n)}} u - A^{(f^{(n)})_m} u\right\| + \left\|A^{(f^{(n)})_l} u - A^{(f^{(n)})_m} u\right\| \le 2\epsilon \quad \text{for all} \quad m, l \ge M(u,\epsilon,n).$$

Enlarging $k(n)$ such that $k(n) \ge M(u,\epsilon,n)$, if necessary, we find

$$\begin{aligned}&\left\|A^f u - A^{f^{(n)}} u\right\| \\ &\quad \le \left\|A^f u - A^{(f^{(n)})_{k(n)}} u\right\| + \left\|A^{(f^{(n)})_{k(n)}} u - A^{(f^{(n)})_m} u\right\| + \left\|A^{(f^{(n)})_m} u - A^{f^{(n)}} u\right\| \\ &\quad \le 3\epsilon\end{aligned}$$

if $n \ge N(u,\epsilon)$ and $k(n), m \ge M(u,\epsilon,n)$, and the claim follows as $n \to \infty$. □

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Note added in proof

All results of Section 2 and those of Section 4 that do not rely upon the log-convexity of the cone $\mathscr{CBF}$—it is used only in Corollary 4.4—hold also for $\mathscr{BF}$ and not only for the subclass $\mathscr{CBF}$. In order to see this, we remark that for any $g \in \mathscr{BF}$ with $g(x) = a + bx + \int_{(0,\infty)}(1 - e^{-sx})\,\mu(ds)$ we have

$$\frac{g(x)}{x} = \frac{a}{x} + b + \int_{(0,\infty)} \frac{1 - e^{-sx}}{x}\,\mu(ds) = \frac{a}{x} + b + \int_{(0,\infty)} e^{-rx}\,\mu([r,\infty))\,dr \in \mathscr{CM},$$

and thus, see [3, Exercise 9.10], $g \circ f/f \in \mathscr{CM}$ for all $f \in \mathscr{BF}$. We use now the Yosida approximation $y(n; x) = nx/(n + x)$, $n \in \mathbb{N}$, which is itself in $\mathscr{BF}$—see Remark 4.2(d) for details—and find

$$\lim_{n\to\infty} y(n; f(x)) = f(x) \qquad \text{and} \qquad \frac{y(n; f(x))}{f(x)} = \widehat{\gamma}_n(x) \in \mathscr{CM}$$

with positive sub-probability measures γ_n. By Lévy's continuity theorem, $\gamma_n \to \delta_0$ (the argument of Lemma 2.4, 2.5 applies), hence the conditions of Remark 2.11 are met (with obvious changes in the notation, for example, f_n of Section 2 has to be redefined as $f_n := y(n; f(\cdot))$ etc.).

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