A GENERALISATION OF DIRICHLET'S MULTIPLE INTEGRAL

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1. A previous note (**2**) showed how the integral of $f(\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n)$ over the interior of a simplex could be reduced to a contour integral. The same idea is applied here in Theorems 1 and 2 to give a generalisation of Dirichlet's multiple integral ((**1**), pp. 169-172). These results are then used in Theorem 3 to reduce an integral over all real n-dimensional space to a contour integral. In Theorem 4 an integral over the group of all 3×3 orthogonal matrices of determinant 1 is reduced to a contour integral. This result can be extended formally to the case of 4×4 matrices; beyond this it seems difficult to go.

2. In this paragraph theorems 1 and 2 are stated and proved in the case of three variables; the extension to the general case is then obvious.

Theorem 1. *Suppose* $f(w) = \sum_{n=0}^{\infty} a_n w^n$ *for* $|w| < R$, *and* $\max\{|\alpha|, |\beta|, |\gamma|\} < R$, *and* p, q, r *are positive and* $g(w)$ *is such that*

$$\iiint_T f(\alpha x + \beta y + \gamma z) x^{p-1} y^{q-1} z^{r-1} g(x+y+z) dx dy dz \tag{1}$$

exists, where T *is the region* $x \geqq 0$, $y \geqq 0$, $z \geqq 0$, $x+y+z \leqq 1$.

Then the value of (1) *is*

$$\frac{1}{2\pi i} \int_C \frac{F(w) dw}{(w-\alpha)^p (w-\beta)^q (w-\gamma)^r}$$

where

$$F(w) = \Gamma(p)\Gamma(q)\Gamma(r) \sum_{n=0}^{\infty} \frac{G(p+q+r+n)}{\Gamma(p+q+r+n)} n! a_n w^{p+q+r+n-1} \tag{2}$$

and

$$G(p+q+r+n) = \int_0^1 t^{p+q+r+n-1} g(t) dt \tag{3}$$

and C *is the circle* $|w| = \rho < R$, *enclosing* $w = \alpha, \beta, \gamma$.

Proof. By expanding $f(\alpha x + \beta y + \gamma z)$ and using the multinomial theorem on $(\alpha x + \beta y + \gamma z)^n$, (1) becomes

$$\sum_{n=0}^{\infty} n! a_n \sum_{i+j+k=n} \frac{\alpha^i \beta^j \gamma^k}{i! j! k!} \iiint_T x^{p+i-1} y^{q+j-1} z^{r+k-1} g(x+y+z) dx dy dz \tag{4}$$

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and, by Dirichlet's Integral, this is

$$\sum_{n=0}^{\infty} n!a_n \sum_{i+j+k=n} \frac{\alpha^i\beta^j\gamma^k}{i!j!k!} \frac{\Gamma(p+i)\Gamma(q+j)\Gamma(r+k)}{\Gamma(p+q+r+i+j+k)} \int_0^1 t^{p+q+r+i+j+k-1} g(t)dt.$$

Using principal values,

$$(1-w)^{-u} = \sum_{s=0}^{\infty} \frac{\Gamma(u+s)}{\Gamma(u)s!} w^s,$$

so

$$\sum_{i+j+k=n} \frac{\Gamma(p+i)\Gamma(q+j)\Gamma(r+k)}{\Gamma(p)\Gamma(q)\Gamma(r)i!j!k!} \alpha^i\beta^j\gamma^k \tag{5}$$

is the coefficient of z^n in the expansion of $(1-\alpha z)^{-p}(1-\beta z)^{-q}(1-\gamma z)^{-r}$, and so (5) is

$$\frac{1}{2\pi i}\int_{C^*} \frac{1}{(1-\alpha z)^p(1-\beta z)^q(1-\gamma z)^r} \frac{dz}{z^{n+1}},$$

where C^* is the circle

$$|z| = \rho < R^* = \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}\}.$$

Now let $w = z^{-1}$ and (4) becomes

$$\sum_{n=0}^{\infty} n!a_n \frac{G(p+q+r+n)}{\Gamma(p+q+r+n)} \frac{1}{2\pi i}\int_C \frac{\Gamma(p)\Gamma(q)\Gamma(r)w^{p+q+r+n-1}}{(w-\alpha)^p(w-\beta)^q(w-\gamma)^r} dw.$$

Theorem 2. *If $s>0$ and $p+q+r+s = k>1$, and*

$$F_{k-1}(w) = \frac{1}{\Gamma(k-1)}\int_0^w f(t)(w-t)^{k-2}dt \tag{6}$$

then

$$\iiint_T f(\alpha x+\beta y+\gamma z)x^{p-1}y^{q-1}z^{r-1}(1-x-y-z)^{s-1}dxdydz$$
$$= \frac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{2\pi i}\int_C \frac{F_{k-1}(w)dw}{(w-\alpha)^p(w-\beta)^q(w-\gamma)^r w^s}. \tag{7}$$

Proof. By (3), since now $g(w) = (1-w)^{s-1}$,

$$G(p+q+r+n) = \int_0^1 t^{p+q+r+n-1}(1-t)^{s-1}dt = \frac{\Gamma(p+q+r+n)\Gamma(s)}{\Gamma(p+q+r+s+n)},$$

and so by (2),

$$F(w) = \Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)\sum_{n=0}^{\infty} \frac{n!a_n w^{p+q+r+n-1}}{\Gamma(k+n)}$$
$$= \frac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{w^s\Gamma(k-1)}\sum_{n=0}^{\infty} a_n w^{k+n-1}\int_0^1 u^n(1-u)^{k-2}du$$
$$= \frac{\Gamma(p)\Gamma(q)\Gamma(r)\Gamma(s)}{w^s\Gamma(k-1)}\sum_{n=0}^{\infty} a_n \int_0^w t^n(w-t)^{k-2}dt \tag{8}$$

which is equivalent to (6).

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When k is an integer, $F_{k-1}(w)$ is, to within a polynomial $P_{k-2}(w)$ of degree $k-2$, the $k-1$ times repeated indefinite integral of $f(w)$. But for $\rho \geqq R$,

$$\int_{|w|=\rho} \frac{P_{k-2}(w)dw}{(w-\alpha)^p(w-\beta)^q(w-\gamma)^r w^s}$$

is, by deformation of contours, a constant with respect to ρ; it is also $O\left(\frac{\rho^{k-2}\rho}{\rho^k}\right)$ for large ρ, so the integral is in fact zero, and the $P_{k-2}(w)$ can be neglected in (7).

3. For the application in the next paragraph only the case $n = 3$ of the next result is required. However the form of the result is best seen when working with the general case.

Here and in § 4, $|M|$ denotes the determinant of the $n \times n$ matrix M. Using a dash for the transpose of a matrix, if x is a column vector and

$$x' = \{x_1, x_2, \ldots, x_n\},$$

we write $dx = \prod_{r=1}^{n} dx_r$.

Theorem 3. *Let S, S_1 be two $n \times n$ symmetric matrices, S being positive definite. Then if $\lambda > 0$, $k > \frac{n}{2}$ and μ are constants,*

$$\int f\left(\frac{\mu + x'S_1x}{\lambda + x'Sx}\right)\frac{dx}{(\lambda + x'Sx)^k} = \frac{\{\Gamma(\frac{1}{2})\}^n}{2\pi i}\Gamma\left(k - \frac{n}{2}\right)\int_C \frac{F_{k-1}(z)dz}{|zS - S_1|^{\frac{1}{2}}(\lambda z - \mu)^{k-\frac{n}{2}}}, \quad (9)$$

the multiple integral, assumed convergent, being taken over all real n-dimensional space and C being a contour inside $|z| = R$, enclosing $z = \mu/\lambda$ and all the roots of $|zS - S_1| = 0$.

Proof. Let $S_2 = \frac{1}{\lambda}S$, $S_3 = \frac{1}{\lambda}S_1$ and $\nu = \mu/\lambda$, then the multiple integral in (9) becomes

$$\frac{1}{\lambda^k}\int f\left(\frac{\nu + x'S_3x}{1 + x'S_2x}\right)\frac{dx}{(1 + x'S_2x)^k}.$$

There is a positive definite symmetric matrix S_4 such that $S_2 = S_4'S_4$, so let $\xi = S_4x$ and the Jacobian of this transformation is $\frac{\partial(\xi)}{\partial(x)} = |S_4| = |S_2|^{\frac{1}{2}}$ and the multiple integral becomes

$$\frac{1}{\lambda^k |S_2|^{\frac{1}{2}}}\int f\left(\frac{\nu + \xi' S_4'^{-1} S_3 S_4^{-1}\xi}{1 + \xi'\xi}\right)\frac{d\xi}{(1 + \xi'\xi)^k}. \quad (10)$$

Now there is an orthogonal matrix H such that $H'S_4'^{-1}S_3S_4^{-1}H = \Lambda$,

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where Λ is a diagonal matrix, with diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\xi = Hy$ and the absolute value of the Jacobian $\frac{\partial(\xi)}{\partial(y)}$ is 1, and (10) becomes

$$\frac{1}{\lambda^k \mid S_2 \mid^{\frac{1}{2}}} \int f\left(\frac{v+y'\Lambda y}{1+y'y}\right) \frac{dy}{(1+y'y)^k}. \tag{11}$$

Now

$$\frac{v+y'\Lambda y}{1+y'y} = \frac{v+\lambda_1 y_1^2+\lambda_2 y_2^2+\ldots+\lambda_n y_n^2}{1+y_1^2+y_2^2+\ldots+y_n^2}$$
$$= v + \frac{(\lambda_1-v)y_1^2+(\lambda_2-v)y_2^2+\ldots+(\lambda_n-v)y_n^2}{1+y_1^2+y_2^2+\ldots+y_n^2},$$

so let

$$z_r = \frac{y_r^2}{1+y_1^2+\ldots+y_n^2}, \quad 1 \leqq r \leqq n, \tag{12}$$

then

$$\frac{\partial(z_1, z_2, \ldots, z_n)}{\partial(y_1, y_2, \ldots, y_n)} = \frac{2^n y_1 y_2 \ldots y_n}{(1+y_1^2+\ldots+y_n^2)^{n+1}}$$

and

$$1-z_1-z_2-\ldots-z_n = \frac{1}{1+y_1^2+\ldots+y_n^2} \text{ and } z_1 z_2 \ldots z_n = \frac{(y_1 y_2 \ldots y_n)^2}{(1+y_1^2+\ldots+y_n^2)^n}.$$

The integral (11) now becomes, since the mapping (12) is 2^n to 1,

$$\frac{1}{\lambda^k \mid S_2 \mid^{\frac{1}{2}}} \int_{T_n} f(v+(\lambda_1-v)z_1+\ldots+(\lambda_n-v)z_n)$$
$$\times (z_1 z_2 \ldots z_n)^{-\frac{1}{2}}(1-z_1-z_2-\ldots-z_n)^{k-\frac{n}{2}-1} dz \tag{13}$$

where T_n is the region $z_r \geqq 0$ $(1 \leqq r \leqq n)$, $z_1+z_2+\ldots+z_n \leqq 1$.

By Theorem 2, (13) is

$$\frac{\{\Gamma(\frac{1}{2})\}^n \Gamma\left(k-\frac{n}{2}\right)}{2\pi i \lambda^k \mid S_2 \mid^{\frac{1}{2}}} \int_{C_1} \frac{F_{k-1}(v+w)dw}{\{(w-\lambda_1+v)(w-\lambda_2+v)\ldots(w-\lambda_n+v)\}^{\frac{1}{2}} w^{k-\frac{n}{2}}}. \tag{14}$$

Now let $z = v+w$, and since $\mid zI - S_4'^{-1} S_3 S_4^{-1} \mid = (z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n)$, it follows that $\mid zS - S_1 \mid = (z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n) \mid S_2 \mid \lambda^n$, which gives the result.

4. Let $\mathscr{H}_n$ denote the compact topological group of all $n \times n$ orthogonal matrices H of determinant $+1$, and let dH be the left and right invariant measure on this group. If $f(z)$ is a regular function and $\sigma(M)$ denotes the trace of the matrix M, the integral to be evaluated in this paragraph is of the type $\int_{\mathscr{H}_n} f(\sigma(AH))dH$, where A is a constant matrix. Now there are $H_1, H_2 \in \mathscr{H}_n$ such that $H_1 A H_2 = \Lambda$, a diagonal matrix and since $\sigma(AB) = \sigma(BA)$ the integral reduces, by the invariance of the measure, to $\int_{\mathscr{H}_n} f(\sigma(\Lambda H))dH$.

This integral can be transformed into an ordinary multiple integral by using Cayley's parametrisation $H = (I-\Sigma)(I+\Sigma)^{-1} = 2(1+\Sigma)^{-1}-I$, where Σ is a skew symmetric matrix. It is known ((3), pp. 149-150) that the Jacobian $\partial(H)/\partial(\Sigma)$ of this change of variables is

$$\frac{2^{\frac{1}{2}n(n-1)}}{|I+\Sigma|^{n-1}}.$$

Thus the integral becomes

$$2^{\frac{1}{2}n(n-1)}\int f(2\sigma(\Lambda(I+\Sigma)^{-1})-\sigma(\Lambda))\frac{d\Sigma}{|I+\Sigma|^{n-1}},$$

where $d\Sigma = \prod_{i=1}^{N} d\sigma_i$, the σ_i being the $\frac{1}{2}n(n-1) = N$ elements of Σ and the integral being taken over all real N-dimensional space.

Theorem 4. *If Λ has diagonal elements α, β, γ and $F(z)$ is the indefinite integral of $f(z)$, regular for $|z|<R$, then*

$$\int_{\mathscr{H}_3} f(\sigma(\Lambda H))dH = \frac{v(\mathscr{H}_3)}{2\pi i}\int_C \frac{F(z)dz}{\{(z-\alpha+\beta+\gamma)(z+\alpha-\beta+\gamma)(z+\alpha+\beta-\gamma)(z-\alpha-\beta-\gamma)\}^{\frac{1}{2}}} \quad (15)$$

where $v(\mathscr{H}_3)$ is the Euclidian volume of $\mathscr{H}_3$ and C is a contour in $|z|<R$ enclosing all the zeros of the expression in $\{\ \}$.

Proof. If $\Sigma = \begin{pmatrix} 0, & \zeta, & \eta \\ -\zeta, & 0, & \xi \\ -\eta, & -\xi, & 0 \end{pmatrix}$ the integral on the left hand side of (15) becomes

$$2^{\frac{3}{2}}\int f\left(\frac{(\alpha+\beta+\gamma)+(\alpha-\beta-\gamma)\xi^2+(\beta-\gamma-\alpha)\eta^2+(\gamma-\alpha-\beta)\zeta^2}{1+\xi^2+\eta^2+\zeta^2}\right)\frac{d\xi d\eta d\zeta}{(1+\xi^2+\eta^2+\zeta^2)^2}$$

and now Theorem 3 gives the result, since ((3), p. 146), $2^{\frac{3}{2}}\pi^2 = v(\mathscr{H}_3)$.

REFERENCES

(1) J. Edwards, *A Treatise on the Integral Calculus*, Vol. II (London, 1922).

(2) H. Jack, An integral over the interior of a simplex, *Proc. Edin. Math. Soc.* (2), **13** (1962), 167-171.

(3) F. W. Ponting and H. S. A. Potter, The volume of orthogonal and unitary space, *Quart. J. Math.* (Oxford), **20** (1949), 146-154.

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