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STOCHASTIC STABILITY OF ANOSOV DIFFEOMORPHISMS

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§ 0. Introduction

R. Bowen [1] introduced the notion of pseudo-orbit for a homeomorphism f of a metric space X as follows: A (double) sequence $\{x_i\}_{i \in Z}$ of points x_i in X is called a δ-pseudo-orbit of f iff

$$d(fx_i, x_{i+1}) \leq \delta$$

for every $i \in \boldsymbol{Z}$, where d denotes the metric in X. We say f is stochastically stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_i\}_{i \in Z}$ of f is ε-traced by some $x \in X$, i.e.,

$$d(f^i x, x_i) \leq \varepsilon$$

for every $i \in \boldsymbol{Z}$. He proved in [1] that if a compact hyperbolic set Λ for a diffeomorphism f of a compact manifold M has local product structure then the restriction $f|\Lambda$ of f to Λ is stochastically stable, using stable and unstable manifolds.

In this paper we prove first that an Anosov diffeomorphism f of a compact manifold M is topologically stable, in the set of all continuous maps of M into M, in a sense (Theorem 1). Next, making use of Theorem 1 we give another proof for Bowen's result, in the case of f an Anosov diffeomorphism (Theorem 2). The idea of this paper is inspired by a result of A. Morimoto [2], which says that a topologically stable homeomorphism f of a manifold M with $\dim M \geq 3$ is stochastically stable. The method of the proof follows that of P. Walters [3].

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§ 1. Preparatory lemmas

M will always denote a compact C^∞ manifold without boundary.

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DEFINITION 1. A C^1 diffeomorphism f of M is called an Anosov diffeomorphism if there exist a Riemannian metric $\|\cdot\|$ on M and constants $C > 0$, $0 < \lambda < 1$ such that the tangent bundle of M can be written as the Whitney sum of two continuous subbundles, $TM = E^s \oplus E^u$, and the following conditions are satisfied:

$$Tf(E^\sigma) = E^\sigma \qquad (\sigma = s, u) . \tag{1.1}$$

$$\begin{aligned} \|Tf^n(v)\| &\leq C\lambda^n \|v\| , & v \in E^s,\ n \geq 0 , \\ \|Tf^{-n}(v)\| &\leq C\lambda^n \|v\| , & v \in E^u,\ n \geq 0 . \end{aligned} \tag{1.2}$$

f will always denote an Anosov diffeomorphism of M. We can find a Riemannian metric for which we can take $C = 1$, and fix it (cf. [3]). Let $\mathfrak{X}(M)$ denote the Banach space of all continuous vector fields with the norm

$$\|v\| = \sup_{x \in M} \|v(x)\| , \qquad v \in \mathfrak{X}(M) .$$

Let $\mathfrak{X}^\sigma(M)$ denote the subspace of all $v \in \mathfrak{X}(M)$ with $v(x) \in E^\sigma_x$ for every $x \in M$ $(\sigma = s, u)$. Clearly $\mathfrak{X}(M) = \mathfrak{X}^s(M) \oplus \mathfrak{X}^u(M)$ (direct sum). We define a linear operator $f_\#: \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$f_\#(v) = Tf \circ v \circ f^{-1} , \qquad v \in \mathfrak{X}(M) .$$

Let $d(\ ,\ )$ denote the metric on M induced by $\|\cdot\|$, and for each $x \in M$ $\exp_x$: $TM_x \to M$ denote the exponential map with respect to $\|\cdot\|$. Let Map (M) denote the metric space of all continuous maps of M into M with the metric

$$d(\phi, \psi) = \sup_{x \in M} d(\phi x, \psi x) , \qquad \phi, \psi \in \mathrm{Map}\,(M) .$$

For $\delta > 0$ we put $\mathrm{Map}\,(M, \delta) = \{\phi \in \mathrm{Map}\,(M) : d(\phi, \mathrm{id}) \leq \delta\}$, and $\sum_\delta = \{(x, y) \in M \times M : d(x, y) \leq \delta\}$.

The following lemma is due to P. Walters [3].

LEMMA 1. *There exist $\delta_1 > 0$ and $\tau_1 > 0$ satisfying the following conditions:*

(1.3) *For every $(x, y) \in \sum_{\delta_1}$ there exists a linear isomorphism $L_{(x,y)}: TM_x \to TM_y$ such that $L_{(x,y)}(E^\sigma_x) = E^\sigma_y$ $(\sigma = s, u)$, and $L_{(x,y)}$ is continuous with respect to $(x, y) \in \sum_{\delta_1}$.*

(1.4) *For every $(x, y) \in \sum_{\delta_1}$ there exists a continuous map $\gamma_{(x,y)}: TM_x(\tau_1) \to TM_y$ such that*

$$\exp_x(v) = \exp_y(L_{(x,y)}(v) + \gamma_{(x,y)}(v)) , \qquad v \in TM_x(\tau_1)$$

and $\gamma_{(x,y)}$ *is continuous with respect to* $(x, y) \in \sum_{\delta_1}$, *where* $TM_x(\tau_1) = \{v \in TM_x : \|v\| \le \tau_1\}$.

(1.5) $x = \exp_y(\gamma_{(x,y)}(0))$, $\qquad (x, y) \in \sum_{\delta_1}$.

(1.6) $\|L_{(x,y)}\|$ *and* $\|(L_{(x,y)})^{-1}\|$ *converge uniformly to* 1 *as* $d(x, y) \to 0$.

(1.7) *For every* $(x, y) \in \sum_{\delta_1}$ *there exists* $K(x, y) \ge 0$ *such that*

$$\|\gamma_{(x,y)}(v) - \gamma_{(x,y)}(v')\| \le K(x, y) \|v - v'\| , \qquad v, v' \in TM_x(\tau_1)$$

and $K(x, y)$ *converges uniformly to* 0 *as* $d(x, y) \to 0$.

Proof. See Lemma 1 [3].

DEFINITION 2. For $\phi \in \mathrm{Map}(M, \delta_1)$ we define continuous linear maps $J_\phi, R_\phi : \mathfrak{X}(M) \to \mathfrak{X}(M)$, a continuous map $\gamma_\phi : \mathfrak{X}(M)(\tau_1) \to \mathfrak{X}(M)$, and a constant $K(\phi) \ge 0$ as follows: For $v \in \mathfrak{X}(M)$ and $x \in M$

$$J_\phi(v)(x) = L_{(\phi x, x)}(v(\phi x)) ,$$
$$R_\phi(v)(x) = (L_{(x, \phi x)})^{-1}(v(\phi x)) .$$

For $v \in \mathfrak{X}(M)(\tau_1)$ and $x \in M$

$$\gamma_\phi(v)(x) = \gamma_{(\phi x, x)}(v(\phi x)) ,$$

where $\mathfrak{X}(M)(\tau_1) = \{v \in \mathfrak{X}(M) : \|v\| \le \tau_1\}$.

$$K(\phi) = \sup_{x \in M} K(\phi x, x) .$$

By Lemma 1 we have the following lemma:

LEMMA 2. *For* $\phi \in \mathrm{Map}(M, \delta_1)$, $v, v' \in \mathfrak{X}(M)(\tau_1)$ *and* $x \in M$

(1.8) $$J_\phi(\mathfrak{X}^\sigma(M)) \subset \mathfrak{X}^\sigma(M),\ R_\phi(\mathfrak{X}^\sigma(M)) \subset \mathfrak{X}^\sigma(M) \qquad (\sigma = s, u) ,$$

(1.9) $$\exp_{\phi x} v(\phi x) = \exp_x(J_\phi(v) + \gamma_\phi(v))(x) ,$$

(1.10) $$\exp_x \gamma_\phi(0) = \phi(x) ,$$

(1.11) $$\|\gamma_\phi(v) - \gamma_\phi(v')\| \le K(\phi) \|v - v'\| ,$$
$$K(\phi) \longrightarrow 0 \quad as \quad d(\phi, \mathrm{id}) \longrightarrow 0 ,$$

(1.12) $$\|J_\phi\|, \|R_\phi\| \longrightarrow 1 \quad as \quad d(\phi, \mathrm{id}) \longrightarrow 0 .$$

LEMMA 3. *If* $\phi, \psi \in \mathrm{Map}(M, \delta_1)$ *and a subset* S *of* M *satisfy*

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$$\psi\phi(x) = x$$

for every $x \in S$, *then*

(1.13) $$R_\psi J_\phi(v)(\phi x) = v(\phi x) ,$$

(1.14) $$J_\phi R_\psi(v)(x) = v(x)$$

for every $x \in S$ *and* $v \in \mathfrak{X}(M)$.

Proof. By Definition 2 we have

$$\begin{aligned} J_\phi R_\psi(v)(x) &= L_{(\phi x, x)}(R_\psi(v)(\phi x)) \\ &= L_{(\phi x, x)}(L_{(\phi x, \psi\phi x)})^{-1}(v(\psi\phi x)) \\ &= v(x) , \end{aligned}$$

which proves (1.14). Similarly, we have

$$\begin{aligned} R_\psi J_\phi(v)(\phi x) &= (L_{(\phi x, \psi\phi x)})^{-1}(J_\phi(v)(\psi\phi x)) \\ &= (L_{(\phi x, x)})^{-1} L_{(\phi x, x)}(v(\phi x)) \\ &= v(\phi x) , \end{aligned}$$

which proves (1.13).

LEMMA 4. *There exists* $\tau_2 > 0$ *satisfying the following conditions: For every* $v \in \mathfrak{X}(M)(\tau_2)$ *there exists* $s(v) \in \mathfrak{X}(M)$ *such that*

(1.15) $$f \exp_{f^{-1}x} v(f^{-1}x) = \exp_x (f_\#(v) + s(v))(x) , \qquad x \in M, \quad s(0) = 0 ,$$

(1.16) $$\|s(v) - s(v')\| \le C(\tau_2) \|v - v'\|$$

for every $v, v' \in \mathfrak{X}(M)(\tau_2)$, *where* $C(\tau_2) \to 0$ as $\tau_2 \to 0$.

Proof. See Lemma 2 [3].

LEMMA 5. *There exist constants* $0 < \delta_2 < \delta_1$ *and* $\alpha > 0$ *satisfying the following conditions: For every* $\phi, \psi \in \mathrm{Map}\,(M, \delta_2)$ *there exist a constant* $\mu(\phi, \psi) > 0$ *and a continuous linear map* $P = P_{\phi,\psi}: \mathfrak{X}(M) \to \mathfrak{X}(M)$ *such that if a subset* S *of* M *satisfies*

$$\psi\phi(x) = x$$

for every $x \in S$, *then*

(1.17) $$(I - R_\psi f_\#)P(v)(\phi x) = v(\phi x)$$

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for every $x \in S$, *and*

(1.18) $$\|P\| \leq \frac{\alpha}{1-\mu(\phi, \psi)\lambda}\,,$$
$$\mu(\phi, \psi) \longrightarrow 1 \text{ as } d(\phi, \mathrm{id}),\ d(\psi, \mathrm{id}) \longrightarrow 0\,.$$

Proof. There exists $\alpha > 0$ such that

(1.19) $$\|v_s\| + \|v_u\| \leq \alpha\,\|v_s + v_u\|$$

for every $v_\sigma \in \mathfrak{X}^\sigma(M)$ $(\sigma = s, u)$. For $\phi, \psi \in \mathrm{Map}\,(M, \delta_1)$ we put

(1.20) $$\mu(\phi, \psi) = \mathrm{Max}\,\{\|J_\phi\|, \|R_\psi\|\}\,.$$

Then, by (1.12) there exists $0 < \delta_2 \leq \delta_1$ and λ_1 such that

(1.21) $$\mu(\phi, \psi)\lambda < \lambda_1 < 1$$

for every $\phi, \psi \in \mathrm{Map}\,(M, \delta_2)$.

By (1.1) and (1.8) we can define as follows: $f^\sigma_\# = f_\# \,|\, \mathfrak{X}^\sigma(M)$, $J^\sigma_\phi = J_\phi \,|\, \mathfrak{X}^\sigma(M)$ and $R^\sigma_\psi = R_\psi \,|\, \mathfrak{X}^\sigma(M)$ $(\sigma = s, u)$. By (1.2), (1.20) and (1.21) we have

$$\|R^s_\psi f^s_\#\| \leq \|R^s_\psi\|\,\|f^s_\#\| \leq \mu(\phi, \psi)\lambda < 1\,.$$

Therefore, the Neumann series $\sum_{n=0}^\infty (R^s_\psi f^s_\#)^n$ is convergent. Putting $P_s = \sum_{n=0}^\infty (R^s_\psi f^s_\#)^n$ we have

(1.22) $$\|P_s\| \leq \frac{1}{1-\mu(\phi, \psi)\lambda}\,.$$

Similarly, since $\|(f^u_\#)^{-1}J^u_\phi\| \leq \mu(\phi, \psi)\lambda < 1$ the Neumann series $\sum_{n=1}^\infty ((f^u_\#)^{-1}J^u_\phi)^n$ is convergent. Putting $P_u = -\sum_{n=1}^\infty ((f^u_\#)^{-1}J^u_\phi)^n$ we have

(1.23) $$\|P_u\| \leq \frac{1}{1-\mu(\phi, \psi)\lambda}\,.$$

Now we put $P = P_s + P_u$. By (1.19), (1.22) and (1.23) we get

$$\|P\| \leq \alpha\,\mathrm{Max}\,\{\|P_s\|, \|P_u\|\} \leq \frac{\alpha}{1-\mu(\phi, \psi)\lambda}\,,$$

which proves (1.18). Next, we shall prove (1.17). By (1.13) and (1.14) we have

$$\begin{aligned}
&(I - R^u_\psi f^u_\#)P_u(v)(\phi x) \\
&\quad = P_u(v)(\phi x) + R^u_\psi f^u_\# (f^u_\#)^{-1} J^u_\phi \left[\sum_{n=0}^\infty ((f^u_\#)^{-1}J^u_\phi)^n(v)\right](\phi x) \\
&\quad = P_u(v)(\phi x) + \sum_{n=0}^\infty ((f^u_\#)^{-1}J^u_\phi)^n(v)(\phi x) \\
&\quad = v(\phi x)
\end{aligned}$$

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for $v \in \mathfrak{X}^u(M)$ and $x \in S$. Clearly, $(I - R_\psi^s f_\sharp^s)P_s = I$.

Thus, we have proved (1.17).

§2. Proof of Theorem 1

THEOREM 1. *An Anosov diffeomorphism f of M is topologically stable in the following sense: For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ satisfying the following conditions: If $g, \tilde{g} \in \mathrm{Map}(M)$ with $d(f, g)$, $d(f\tilde{g}, \mathrm{id}) \leq \delta$ and a subset S of M satisfy*

$$\tilde{g}g(x) = x$$

for every $x \in S$, then there exists $h \in \mathrm{Map}(M)$ such that

$$hg(x) = fh(x) \tag{2.1}$$

for every $x \in S$, and

$$d(h, \mathrm{id}) \leq \varepsilon\ . \tag{2.2}$$

Proof. First, take $\varepsilon_0 \leq \mathrm{Min}\{\tau_1, \tau_2, \varepsilon\}$ so small that for every ϕ, $\psi \in \mathrm{Map}(M, \delta_2)$

$$\frac{\alpha\mu(\phi, \psi)}{1 - \mu(\phi, \psi)\lambda} C(\varepsilon_0) \leq \frac{1}{4}\ . \tag{2.3}$$

This is possible since $C(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$. Next, take $0 < \delta \leq \delta_2$ so small that for every $\phi, \psi \in \mathrm{Map}(M, \delta)$

$$\frac{\alpha\mu(\phi, \psi)}{1 - \mu(\phi, \psi)\lambda} \delta \leq \frac{1}{2}\varepsilon_0 \tag{2.4}$$

and

$$\frac{\alpha\mu(\phi, \psi)}{1 - \mu(\phi, \psi)\lambda} K(\phi) \leq \frac{1}{4}\ . \tag{2.5}$$

This is possible since $K(\phi) \to 0$ as $d(\phi, \mathrm{id}) \to 0$.

For $\phi, \psi \in \mathrm{Map}(M, \delta)$ we define a continuous map $\Phi : \mathfrak{X}(M)(\varepsilon_0) \to \mathfrak{X}(M)$ by

$$\Phi(v) = P_{\phi,\psi} R_\psi (s(v) - \gamma_\phi(v))\ , \qquad v \in \mathfrak{X}(M)(\varepsilon_0)\ .$$

To find a fixed point of Φ we shall first show that the Lipschitz constant of $\Phi \leq \frac{1}{2}$. Take two elements $v, v' \in \mathfrak{X}(M)(\varepsilon_0)$. By (1.11), (1.16), (1.18), (1.20), (2.3) and (2.5) we have

$$
\begin{aligned}
\|\Phi(v) - \Phi(v')\| & \\
&\leq \|P\|\,\|R_\psi\|\,(\|s(v) - s(v')\| + \|\gamma_\phi(v) - \gamma_\phi(v')\|) \\
&\leq \frac{\alpha\mu(\phi,\psi)}{1-\mu(\phi,\psi)\lambda}(C(\varepsilon_0)\,\|v - v'\| + K(\phi)\,\|v - v'\|) \\
&\leq (\tfrac{1}{4} + \tfrac{1}{4})\,\|v - v'\| = \tfrac{1}{2}\,\|v - v'\| .
\end{aligned}
$$

Next, we shall show $\Phi(\mathfrak{X}(M)(\varepsilon_0)) \subset \mathfrak{X}(M)(\varepsilon_0)$. By (1.10), (1.15), (1.18), (1.20) and (2.4) we have

$$
\begin{aligned}
\|\Phi(v)\| &\leq \|\Phi(0)\| + \|\Phi(v) - \Phi(0)\| \\
&\leq \|P\|\,\|R_\psi\|\,\delta + \tfrac{1}{2}\,\|v\| \\
&\leq \frac{\alpha\mu(\phi,\psi)}{1-\mu(\phi,\psi)\lambda}\delta + \frac{1}{2}\varepsilon_0 \\
&\leq \tfrac{1}{2}\varepsilon_0 + \tfrac{1}{2}\varepsilon_0 = \varepsilon_0
\end{aligned}
$$

for $v \in \mathfrak{X}(M)(\varepsilon_0)$. Thus, Φ is a contraction of a complete metric space $\mathfrak{X}(M)(\varepsilon_0)$. Therefore, Φ has a unique fixed point $v_0 = v_0(\phi, \psi) \in \mathfrak{X}(M)(\varepsilon_0)$, i.e.

$$
v_0 = P_{\phi,\psi}R_\psi(s(v_0) - \gamma_\phi(v_0)) . \tag{2.6}
$$

We put $h\,(= h_{\phi,\psi}) = \exp v_0$.

Now assume that $g, \tilde{g} \in \mathrm{Map}\,(M)$ with $d(f, g)$, $d(f\tilde{g}, \mathrm{id}) \leq \delta$ and a subset S of M satisfy that $\tilde{g}g(x) = x$ for every $x \in S$. Putting $\phi = gf^{-1}$ and $\psi = f\tilde{g}$ we see that $\phi, \psi \in \mathrm{Map}\,(M, \delta)$ and $\psi\phi(fx) = f(x)$ for every $x \in S$. By Definition 2, (1.14), (1.17) and (2.6) we obtain

$$
\begin{aligned}
J_\phi(v_0)(fx) &- f_\sharp(v_0)(fx) \\
&= J_\phi(v_0)(fx) - J_\phi R_\psi f_\sharp(v_0)(fx) \\
&= J_\phi(I - R_\psi f_\sharp)(v_0)(fx) \\
&= J_\phi(I - R_\psi f_\sharp)PR_\psi(s(v_0) - \gamma_\phi(v_0))(fx) \\
&= L_{(\phi fx, fx)}[(I - R_\psi f_\sharp)PR_\psi(s(v_0) - \gamma_\phi(v_0))(\phi fx)] \\
&= L_{(\phi fx, fx)}[R_\psi(s(v_0) - \gamma_\phi(v_0))(\phi fx)] \\
&= L_{(\phi fx, fx)}(L_{(\phi fx, \psi\phi fx)})^{-1}((s(v_0) - \gamma_\phi(v_0))(\psi\phi fx)) \\
&= s(v_0)(fx) - \gamma_\phi(v_0)(fx)
\end{aligned}
$$

for every $x \in S$. Thus we have

$$
(J_\phi(v_0) + \gamma_\phi(v_0))(fx) = (f_\sharp(v_0) + s(v_0))(fx) \tag{2.7}
$$

for every $x \in S$. By (1.9), (1.15) and (2.7), for every $x \in S$ we have

$$\begin{aligned} hg(x) &= \exp_{\phi f x} v_0(\phi f x) \\ &= \exp_{fx} (J_\phi(v_0) + \gamma_\phi(v_0))(fx) \\ &= \exp_{fx} (f_\#(v_0) + s(v_0))(fx) \\ &= f \exp_{f^{-1}fx} v_0(f^{-1}fx) \\ &= fh(x) , \end{aligned}$$

which proves (2.1). Clearly, $d(h, \mathrm{id}) = \|v_0\| \leq \varepsilon_0 \leq \varepsilon$, which proves (2.2). This completes the proof of Theorem 1.

Remark. Let $g \in \mathrm{Map}(M)$ be a homeomorphism of M with $d(f, g) \leq \delta$. Clearly, we see that $d(fg^{-1}, \mathrm{id}) \leq \delta$ and $g^{-1}g(x) = x$ for every $x \in M$. By Theorem 1 there exists $h \in \mathrm{Map}(M, \varepsilon)$ such that

$$hg(x) = fh(x)$$

for every $x \in M$. Thus, Theorem 1 is a generalization of P. Walters' result (Theorem 1 [3]), except the uniqueness of the semiconjugacy h with $d(h, \mathrm{id}) \leq \varepsilon$.

§ 3. Proof of Theorem 2

THEOREM 2. *An Anosov diffeomorphism f of M is stochastically stable.*

Proof. For $\varepsilon > 0$ we put $\delta_0 = \delta(\varepsilon/2)$, where $\delta(\varepsilon/2)$ is as in Theorem 1, and $\delta = \delta_0/3$. For every δ-pseudo-orbit $\{x_i\}_{i \in Z}$ of f, we shall find $x \in M$ such that

$$d(f^i x, x_i) \leq \varepsilon , \qquad i \in \boldsymbol{Z} . \tag{3.1}$$

CLAIM 1. *For every positive integer k and δ-pseudo-orbit $\{x_i\}_{i \in Z}$ of f, there exists $z \in M$ such that*

$$d(f^i z, x_i) \leq \varepsilon , \qquad i = 0, 1, \cdots, k . \tag{3.2}$$

Proof. There exists a $(\frac{2}{3}\delta_0)$-pseudo-orbit $\{x'_i\}_{i \in Z}$ such that

$$\begin{aligned} &d(x'_i, x_i) \leq \varepsilon/2 , \qquad && i = 0, 1, \cdots, k , \\ &x'_i \neq x'_j , && 0 \leq i \neq j \leq k + 1 . \end{aligned} \tag{3.3}$$

Since $f(x'_i) \neq f(x'_j)$ $(0 \leq i \neq j \leq k + 1)$ and $d(fx'_i, x'_{i+1}) \leq \frac{2}{3}\delta_0$, we can find $\phi, \psi \in \mathrm{Map}(M, \delta_0)$ such that

$$\phi f(x'_i) = x'_{i+1}, \ \psi(x'_{i+1}) = f(x'_i) , \qquad i = 0, 1, \cdots, k .$$

Put $S = \{x'_0, \cdots, x'_k\}$, $g = \phi f$ and $\tilde{g} = f^{-1}\psi$. Then we see that $d(f, g) = d(\phi, \mathrm{id})$, $d(f\tilde{g}, \mathrm{id}) = d(\psi, \mathrm{id}) \leq \delta_0$, and $\tilde{g}g(x'_i) = f^{-1}\psi\phi f(x'_i) = x'_i$, $i = 0, 1, \cdots, k$. By Theorem 1, there exists $h \in \mathrm{Map}(M, \varepsilon/2)$ such that $hg(x'_i) = fh(x'_i)$, for $i = 0, 1, \cdots, k$. Therefore, we have

$$f^i h(x'_0) = h(x'_i)\ , \qquad i = 0, 1, \cdots, k\ . \tag{3.4}$$

Putting $z = h(x'_0)$, by (3.3) and (3.4) we obtain

$$\begin{aligned} d(f^i z, x_i) &\leq d(f^i h(x'_0), x'_i) + d(x'_i, x_i) \\ &\leq d(h(x'_i), x'_i) + \varepsilon/2 \\ &\leq d(h, \mathrm{id}) + \varepsilon/2 \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon\ , \end{aligned}$$

which proves (3.2).

CLAIM 2. *Let $\{x_i\}_{i \in Z}$ be a δ-pseudo-orbit of f. For every positive integer k there exists $z = z_k \in M$ such that*

$$d(f^i z, x_i) \leq \varepsilon\ , \qquad |i| \leq k\ . \tag{3.5}$$

Proof. Take a positive integer k and fix it. Putting $y_i = x_{-k+i}$ we see that $\{y_i\}_{i \in Z}$ is a δ-pseudo-orbit. By Claim 1 there exists $z' \in M$ such that $d(f^i z', y_i) \leq \varepsilon$, for $i = 0, 1, \cdots, 2k$. Putting $z = f^k(z')$ we get $d(f^i z, x_i) = d(f^{i+k} z', y_{i+k}) \leq \varepsilon$, $|i| \leq k$, which proves (3.5).

By the compactness of M we can find a subsequence $\{z_{k_\nu}\}$ of $\{z_k\}$ such that $\lim_{\nu \to \infty} z_{k_\nu} = x$ for some $x \in M$. Take $i \in Z$ and fix it. By (3.5) we have that $d(f^i z_{k_\nu}, x_i) \leq \varepsilon$ for every ν with $|i| \leq k_\nu$. Therefore we obtain $d(f^i x, x_i) = \lim_{\nu \to \infty} d(f^i z_{k_\nu}, x_i) \leq \varepsilon$, which proves (3.1).

This completes the proof of Theorem 2.

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