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SOME CONSEQUENCES OF LAŠNEV'S THEOREM IN SHAPE THEORY

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ABSTRACT. In this paper we use the Lašnev Theorem in order to give some properties of a class of metrizable spaces having compact metric shape.

Introduction and basic notations. In this brief note we deal with metrizable spaces which have the same shape, in the sense of Fox [5] like in [1] and [2], as a compact metrizable space. No special constructions are made; on the contrary, the results in this paper are immediate consequences of the results in [2] and the Lašnev's Theorem, but they maintain the initial geometrical character of shape theory. A more exhaustive research paper on spaces having compact metric shape can be found in [7].

Let us recall some basic concepts from [1] and [2]: If X is a metrizable space, then the space of quasicomponents of X (denoted by ΔX) is the space whose elements are the quasicomponents of X and its topology is such that the natural projection $p:X \to \Delta X$ is a quotient map. If in addition p is a closed map, then we say that the decomposition of X into quasicomponents is upper semicontinuous (denoted by $X \in USDQ$).

As usual $\square(X)$ denotes the space of components of X.

We say that a metrizable space belongs to the class S_0 ($X \in S_0$) if the two following conditions are satisfied:

(I) $X \in USDQ$.

(II) (covering) $\dim(\Delta X) = 0$.

As we have pointed out in [1] and [2], the class S_0 behaves well in Fox shape theory in the sense that some results from the compact case can be transferred to the S_0 case. On the other hand, the following relations hold: (all spaces considered are metrizable)

(I) If $\dim(X) = 0$, then $X \in S_0$.

(II) If X is a locally compact space with compact components, then $X \in S_0$.

(III) In the realm of locally compact spaces the two following statements

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are equivalent:
(a) $X \in USDQ$
(b) $X \in S_0$.
(IV) Locally connected spaces and $AWNR$-spaces are in the class S_0.
(V) The class S_0 is closed under mutational retractions (see Lemma 3.3 in [2]).
(VI) If $X \in S_0$, then every quasicomponent is connected, i.e. $\square(X) = \Delta X$ (see Proposition 1.4 in [1]).

In order to end this introduction, we are going to state Lašnev's result which we shall use in this paper:

THEOREM (*see* [6] *and* [4]). *Let X be a metrizable space and* $f:X \to Y$ *a closed continuous function onto Y, then*

$$Y = Y_0 \cup \left(\bigcup_{n=1}^{\infty} Y_n \right),$$

where $f^{-1}(y)$ *is compact for every* $y \in Y_0$ *and* Y_n *is closed and discrete in Y for* $n \geqq 1$.

1. **Concerning spaces in the class S_0 with compact metric shape.** We begin this section with the following fact:

PROPOSITION 1. *Let* $X \in USDQ$ *be a metrizable space without compact quasicomponents, then* $X \in S_0$.

PROOF. By the hypothesis we have that $p:X \to \Delta X$ is a closed map and from Lašnev's Theorem it follows that $\Delta X = \cup_{n=1}^{\infty} C_n$, where C_n is closed and discrete for every $n \in N$. Finally from the Countable Sum Theorem for dimension, it follows that $\dim(\Delta X) = 0$ and consequently $X \in S_0$. In particular every quasicomponent of X is connected.

EXAMPLE 2. There exist spaces satisfying the hypothesis of Proposition 1 and such that the corresponding space of components is not discrete. For example, let us consider

$$X = \bigcup_{n=1}^{\infty} (-1/2, 1/2) \times \{1/n\} \cup \cup(-1, 1) \times \{0\}$$

as a subspace of R^2.

As a direct consequence of Proposition 1 in this paper and Corollary 2.8 in [2], we have:

COROLLARY 3. *Let X, Y be two spaces satisfying the hypothesis of Proposition 1. If in addition* $Sh(X) = Sh(Y)$, *then there exists a homeomorphism* Λ *of* $\square(X)$

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onto $\square(Y)$ *such that for every* $H \subset \square(X)$ *with* $H = F \cap A$, *where* F *is open and* A *is closed in* $\square(X)$, *the equality* $Sh(p^{-1}(H)) = Sh(q^{-1}(\Lambda(H)))$ *holds. Where* $p:X \to \square(X)$, $q:Y \to \square(Y)$ *are the corresponding projections.*

Let us prove now what we consider as the most significant result in this paper.

PROPOSITION 4. *Let* $X \in S_0$ *be a metrizable space with compact metric shape, then every component of* X *has compact boundary in* X, *the cardinal of the set of all non compact components of* X *is at most* $\aleph_0$ *and every component has the shape of a metric continuum.*

PROOF. Let Y be a compact metric space such that $Sh(X) = Sh(Y)$. From Corollary 1.5 in [2] it follows that there exists a homeomorphism Λ of $\square(X)$ onto $\square(Y)$ such that $Sh(X_0) = Sh(\Lambda(X_0))$ for every component X_0 of X and consequently every component of X has the shape of a metric continuum. On the other hand, since $\square(Y)$ is a compact metric space we have that $\square(X)$ is a compact metric space and since $p:X \to \square(X)$ is a closed map we have, see for example Theorem 3.1 in [3], that the boundary of every component is a compact subset of X. Finally let C be the subset of all non compact components of X. Using the Lašnev's Theorem we have that

$$\square(X) = F_0 \cup \left(\bigcup_{n=1}^{\infty} F_n\right)$$

where $p^{-1}(f)$ is compact for every $f \in F_0$ and F_n is closed and discrete in $\square(X)$ for every $n \in N$. From the compactness of $\square(X)$ it follows that F_n is a finite subset for every $n \in N$. On the other hand $C \subset \cup_{n=1}^{\infty} F_n$ and consequently

$$\mathrm{Card}(C) \leqq \mathrm{Card}\left(\bigcup_{n=1}^{\infty} F_n\right) \leqq \aleph_0.$$

COROLLARY 5. *Let* $X \in USDQ$ *be a metrizable space without compact quasi-components. If* X *has compact metric shape then* $\mathrm{Card}(\square(X)) \leqq \aleph_0$.

REMARK 6. (I) The hypothesis $X \in USDQ$ is essential in Corollary 5. For example the space $Z = R \times C$ (where R is the real line and C is the cantor set) has the shape of C.

(II) The space X described in Example 2 satisfies the hypothesis of Corollary 5.

In order to end, we have the following fact:

PROPOSITION 7. *Let* X *be a metrizable space, then the two following statements are equivalent*:

(I) X *is compact.*

(II) $X \in S_0$, *all components of* X *are compact spaces and* X *has compact metric shape.*

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References

1. M. Alonso Moron, *Upper semicontinuous decompositions and movability in metric spaces*. To appear in Bull. Acad. Polon. Sci.

2. ———, *On the problem of components in shape theory for metrizable spaces*. Preprint.

3. D. Burke, *Closed mapping*, *Surveys in General Topology*, Academic Press (1980), pp. 1-32.

4. J. Chaber, *Generalizations of Lasnev's Theorem*, Fund. Math. **119** (1983), pp. 85-91.

5. R. H. Fox, *On shape*, Fund. Math. **74** (1973), pp. 47-71.

6. N. Lašnev, *Continuous decompositions, and closed mappings of metric spaces*, Soviet Math. Dokl. **6** (1965), pp. 1504-1506.

7. T. Watanabe, *On spaces which have the shape of compact metric spaces*, Fund. Math. **104** (1979), pp. 1-11.

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