GROUPS ASSOCIATED WITH CERTAIN LOCI IN [5]

A. F. HORADAM

1. Introduction. If x_i $(i = 1, 2, \ldots, 6)$ are homogeneous co-ordinates in [5] (the complex projective space of five dimensions), then the equation

$$(1.1) \qquad x_1 x_2 x_3 + x_4 x_5 x_6 = 0$$

represents the well-known **(11)** *Perazzo cubic primal* P_4^3 of order 3 and dimension 4. With it is associated the *Segre cubic threefold* S_3^3 **(12)**; specifically, S_3^3 is the section of P_4^3 by a tangent [4]. For example, the equations of the S_3^3 at $(111 -1 -1 -1)$ are

$$(1.2) \qquad x_1 x_2 x_3 + x_4 x_5 x_6 = 0, \qquad x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0,$$

which, by means of an appropriate substitution **(2)**, may be reduced to the symmetrical form

$$(1.2)' \qquad \sum_{i=1}^{6} y_i^3 = 0, \qquad \sum_{i=1}^{6} y_i = 0.$$

In this article we introduce U_3^9, a threefold locus of order 9 in [5], whose equations are

$$(1.3) \quad x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = 0, \qquad x_1 x_2 x_3 + x_4 x_5 x_6 = 0,$$

so that U_3^9 is the intersection of P_4^3 and another cubic primal. For simplicity, U_3^9 will be referred to as U, or a U-locus. From **(6)** it is known that U may be obtained as a [5]-section of the locus L_4^{45} (or, more simply, L) of order 45 and dimension 4 in [8], which is left invariant under the operations of a certain group, CT, the *Clifford transform group* of order $51{,}840 \times 81 = 2^7.3^8.5$. (Thus L is related to the system of 27 lines on a cubic surface whose group has order $51{,}840 \div 2$.) Properties of L and of the various *Clifford groups* (as well as their related projective groups) have been developed by Beverley Bolt, Room, and Wall **(3)**, by Beverley Bolt **(4)**, and by the author **(6–10)**.

The aim of this article is first to describe in brief some of the geometrical properties of U, and then to determine the order and nature of the subgroup CT_U of CT that leaves U invariant. Our main result is:

THEOREM. CT_U *has order* $432 \times 81 = 2^4.3^7$ *and index* 120 *in* CT.

Organization of our argument proceeds as follows. In §2, certain reference material essential to the development is reproduced; §3 tells something of the geometrical features of U; and §4 reveals that there are 120 loci similar to U—hence the conclusion that CT_U has index 120 in CT. Although §3 is not necessary

Received September 21, 1964. This paper was presented to the Amer. Math. Soc. on April 20, 1964.

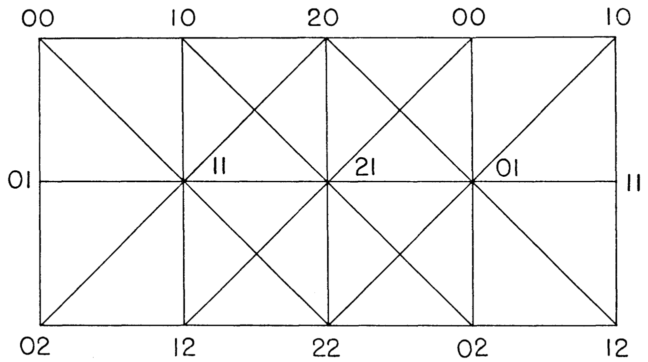
for the group-theoretic work, it is felt that the basic properties of U are worth recording.

2. Reference material. While the notation and contents of (**6**–**10**) are assumed known, we collect for convenience the facts that are actually needed.

Let $x_{00}, x_{10}, x_{20}, x_{01}, x_{11}, x_{21}, x_{02}, x_{12}, x_{22}$ be the ordered set of general homogeneous co-ordinates in [8] over the field of complex numbers. Then (**6**) the locus L is determined by the five linearly independent primals

$$(2.1)\quad \begin{cases} \theta \equiv \sum_{i,j=0}^{2} x_{ij}^{3} = 0, \\ \phi_1 \equiv x_{00}\,x_{10}\,x_{20} + x_{01}\,x_{11}\,x_{21} + x_{02}\,x_{12}\,x_{22} = 0, \\ \phi_2 \equiv x_{00}\,x_{01}\,x_{02} + x_{10}\,x_{11}\,x_{12} + x_{20}\,x_{21}\,x_{22} = 0, \\ \phi_3 \equiv x_{00}\,x_{11}\,x_{22} + x_{01}\,x_{12}\,x_{20} + x_{02}\,x_{10}\,x_{21} = 0, \\ \phi_4 \equiv x_{00}\,x_{12}\,x_{21} + x_{01}\,x_{10}\,x_{22} + x_{02}\,x_{11}\,x_{20} = 0. \end{cases}$$

Co-ordinates x_{ij} may be arranged in the accompanying array (which we call the *Clebsch diagram*) wherein the symbols ij are the suffixes of x_{ij}. Originally Clebsch (**5**) used this diagram to illustrate the collinearity of the inflexions of a plane cubic curve.



A [2]-section of L (by, say, the primes $x_{i1} = x_{i2} = 0$ $(i = 0, 1, 2)$) yields a canonical cubic curve whose nine inflexions determine the *Jacobian configuration* that is built up from the 12 *critic centres* $(1\cdot\cdot)$, $(\cdot1\cdot)$, $(\cdot\cdot1)$, $(1\epsilon^a\epsilon^b)$, where $a, b = 0, 1, 2$, $\epsilon = \exp(2\pi i/3)$, dots indicate zeros, and zero co-ordinates x_{i1}, x_{i2} are omitted. A plane specifically containing the Jacobian configuration is called a *Jacobian plane.*

From the critic centres are formed the four inflexion triangles T_i $(i = 1, 2, 3, 4)$ whose sides have line-co-ordinates

$$(2.2)\quad \begin{cases} T_1\colon (1\cdot\cdot),\ (\cdot1\cdot),\ (\cdot\cdot1); & T_2\colon (111),\ (1\epsilon\epsilon^2),\ (1\epsilon^2\epsilon); \\ T_3\colon (1\epsilon^2\epsilon^2),\ (11\epsilon),\ (1\epsilon1); & T_4\colon (1\epsilon\epsilon),\ (11\epsilon^2),\ (1\epsilon^21) \end{cases}$$

(which, if interpreted as point-co-ordinates, give the critic centres). These sets of triads, which form a characteristic feature of the subsequent development, will be called T_i-triads.

Generators of CT are known **(6)** to be

$$D = \operatorname{diag}(11\epsilon^2\, 11\epsilon^2\, 11\epsilon^2)$$

and Q, whose form is given in **(6)**. Subsequently it transpires that, while we require D, we do not need Q but rather

$$(2.4)\qquad Q^2 = -k\begin{bmatrix} 1\cdot\cdot & \cdot\epsilon^2\cdot & \cdot\cdot\epsilon^2 \\ 1\cdot\cdot & \cdot 1\cdot & \cdot\cdot\epsilon \\ 1\cdot\cdot & \cdot\epsilon\cdot & \cdot\cdot 1 \\ \cdot 1\cdot & \cdot\cdot\epsilon^2 & \epsilon^2\cdot\cdot \\ \cdot 1\cdot & \cdot\cdot 1 & \epsilon\cdot\cdot : \\ \cdot 1\cdot & \cdot\cdot\epsilon & 1\cdot\cdot \\ \cdot\cdot 1 & \epsilon^2\cdot\cdot & \cdot\epsilon^2\cdot \\ \cdot\cdot 1 & 1\cdot\cdot & \cdot\epsilon\cdot \\ \cdot\cdot 1 & \epsilon\cdot\cdot & \cdot 1\cdot \end{bmatrix}, \qquad Q^3 = k\begin{bmatrix} 111 & \cdots & \cdots \\ \cdots & \cdots & 111 \\ \cdots & 111 & \cdots \\ \cdots & \epsilon\epsilon^2 1 & \cdots \\ \epsilon\epsilon^2 1 & \cdots & \cdots \\ \cdots & \cdots & \epsilon\epsilon^2 1 \\ \cdots & \cdots & \epsilon 1\epsilon^2 \\ \cdots & \epsilon 1\epsilon^2 & \cdots \\ \epsilon 1\epsilon^2 & \cdots & \cdots \end{bmatrix},$$

where

$$(2.5)\qquad k = (-3)^{-\frac{1}{2}} = (1 + 2\epsilon)^{-1}.$$

Also, we shall require the permutation matrices j, w, w^2 corresponding to the permutations

$$(2.6)\qquad j:\ (132\ 465\ 798),$$

$$(2.7)\qquad w:\ (123\ 564\ 978),\ w^2 = (123\ 645\ 897),$$

so that

$$(2.8)\qquad j^2 = w^3 = I, \qquad w^{-1} = jwj,$$

where I is the 9×9 unit matrix. No confusion should exist between j and $J = Q^5$ **(6)**, but we note in passing that $jJ = Jj$ with $(jJ)^2 = (Jj)^2 = I$.

3. Some geometry of U**.** Take the section of L (2.1) by the triaxial [5] determined by the three primes

$$(3.1)\qquad x_{i0} = 0 \qquad (i = 0, 1, 2),$$

obtaining the equations of U (cf. (1.3))

$$(3.2)\qquad \begin{cases} \theta \equiv x_{01}{}^3 + x_{11}{}^3 + x_{21}{}^3 + x_{02}{}^3 + x_{12}{}^3 + x_{22}{}^3 = 0, \\ \phi \equiv x_{01}\, x_{11}\, x_{21} + x_{02}\, x_{12}\, x_{22} = 0. \end{cases}$$

Evidently this [5] lies on the primals ϕ_2, ϕ_3, ϕ_4 (2.1) since their equations are satisfied by the three primes.

On ϕ lie nine double lines **(1; 11)** which form the sides and diagonals of a skew hexagon. But θ can contain no double elements, as may be verified. However, θ cuts each of these double lines in three points, so that U has $9 \times 3 = 27$ double points arising from the double lines.

Clearly, there are three other primals similar to ϕ, namely,

$$(3.3)\qquad \Phi_i \equiv \theta - 3\epsilon^i\phi = 0 \qquad (i = 0, 1, 2).$$

https://doi.org/10.4153/CJM-1966-015-0 Published online by Cambridge University Press

Consequently, by (3.2) and (3.3) the aggregate of double points on U is $27 \times 4 = 108$, whose co-ordinates may readily be found following the technique of Baker **(1)**.

On U lie 54 Jacobian planes

$$x_{01} + \alpha x = 0, \qquad x_{11} + \beta y = 0, \qquad x_{21} + \gamma z = 0, \tag{3.4}$$

where $\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1$ and xyz is a permutation of $x_{02}\, x_{12}\, x_{22}$. Half of these planes occur as sections of solids on L, and the other half exist as planes on L. (Under the triaxial section (3.1), only one third of the 81 solids on L **(6)** becomes planes; the remainder become inflexions of cubic curves on the sides of two particular triangles in the simplex of reference in [8].) Moreover, the Φ_i (3.3) do not contribute any fresh planes to U, that is, the totality of planes on U is 54.

Many incidence properties relating to the planes and double elements of U are here suppressed in the interests of brevity. Perhaps, though, it is worth recording briefly a verification of a known feature of (1.1) and an associated $S_3{}^3$. For instance, the sextic cone of planes $K_2{}^6$ **(12)** for the $S_3{}^3$ at $(111\ {-1}\ {-1}\ {-1})$ (cf. (1.2)) consists of the six Jacobian planes of (3.4) for which $\alpha = \beta = \gamma = 1$ and xyz are the six possible permutations of $x_{02}\, x_{12}\, x_{22}$. Each (Jacobian) plane of U lies on $(108 \times 6)/54 = 12$ Segre loci, since there exists one Segre locus for each double point.

4. Further loci like U. From the Clebsch diagram (§2), we observe that there are 11 other loci like U, each locus being determined by three primes whose equations are the vanishing of the co-ordinates in the three rows, three columns, and six diagonals of the Clebsch diagram. Notice that for each set of three co-ordinates,

$$\sum \text{(first subscripts)} = \sum \text{(second subscripts)} \equiv 0 \pmod 3.$$

Renumbering the binary symbols 00, 10, 20, 01, 11, 21, 02, 12, 22 in the natural order 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively, we may thus distinguish the 12 loci by means of subscripts, thus: U_{123}, U_{456}, U_{789}, U_{147}, U_{258}, U_{369}, U_{159}, U_{267}, U_{348}, U_{168}, U_{249}, U_{357}. Of course, the U of (3.2) is U_{123}.

How many loci like U are there altogether?

Consider the pencil of primals

$$\lambda\theta + \sum_{i=1}^{4} \mu_i\, \phi_i = 0 \tag{4.1}$$

which pass through L. Select from the pencil (4.1) the five primals

$$\begin{cases} \theta' \equiv \theta + 6\phi_3 = 0, \\ \phi'_1 \equiv \theta - 3\phi_3 = 0, \\ \phi'_2 \equiv \phi_1 + \epsilon^2\phi_2 + \epsilon\phi_4 = 0, \\ \phi'_3 \equiv \phi_1 + \phi_2 + \phi_4 = 0, \\ \phi'_4 \equiv \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_4 = 0. \end{cases} \tag{4.2}$$

https://doi.org/10.4153/CJM-1966-015-0 Published online by Cambridge University Press

(Note the T_2-triad (2.2) of coefficients occurring as coefficients of ϕ_1, ϕ_2, ϕ_4.)

Let

$$x^{\tau} = (x_{00}, x_{10}, x_{20}, x_{01}, x_{11}, x_{21}, x_{02}, x_{12}, x_{22}) \tag{4.3}$$

be the transpose of the column vector of the co-ordinates in [8].

Similarly, let

$$X^{\tau} = (X_{00}, X_{10}, X_{20}, X_{01}, X_{02}, X_{21}, X_{02}, X_{12}, X_{22}) \tag{4.4}$$

be the transpose of the column vector of another set of co-ordinates in [8].

Apply the transformation (2.3), (2.4), (4.3), (4.4)

$$X = Q^2Dx \tag{4.5}$$

to (4.2), whence, after considerable algebraic manipulation and simplification, we derive

$$\begin{cases} \theta' \equiv \sum_{i,j=0}^{2} X_{ij}{}^3 = 0, \\ \phi'_1 \equiv X_{00}X_{10}X_{20} + X_{01}X_{11}X_{21} + X_{02}X_{12}X_{22} = 0, \\ \phi'_2 \equiv X_{00}X_{01}X_{02} + X_{10}X_{11}X_{12} + X_{20}X_{21}X_{22} = 0, \\ \phi'_3 \equiv X_{00}X_{11}X_{22} + X_{01}X_{12}X_{20} + X_{02}X_{10}X_{21} = 0, \\ \phi'_4 \equiv X_{00}X_{12}X_{21} + X_{01}X_{10}X_{22} + X_{02}X_{11}X_{20} = 0, \end{cases} \tag{4.6}$$

ignoring scalar factors k, $k\epsilon^2$ (cf. (2.5)) wherever they occur. This means that the equations of the primals (4.2) of the pencil through L have exactly the formal appearance of the equations of the primals (2.1) determining L. Consequently, a Clebsch diagram (§2) exists for $X_{00}, X_{10}, \ldots, X_{22}$ and the previous results may be paralleled. Accordingly, we obtain the nine new loci U'_{147}, U'_{258}, U'_{369}, U'_{159}, U'_{267}, U'_{348}, U'_{168}, U'_{249}, U'_{357}. For instance, U'_{147} arises from the section of L by the [5] $X_{00} = X_{01} = X_{02} = 0$. Observe that there are no new loci U'_{123}, U'_{456}, U'_{789} since, for example, the section by

$$X_{00} = X_{10} = X_{20} = 0$$

merely reproduces the section by $x_{00} = x_{11} = x_{22} = 0$. That is,

$$U'_{123} \equiv U_{159} \text{ (=a section of } \phi_3).$$

Likewise $U'_{456} \equiv U_{267}$, $U'_{789} \equiv U_{348}$. Prime-co-ordinates of the triads of primes determining the nine new loci are, allowing for division by ϵ^2 where necessary, either T_2-triads or are based on T_2-triads (2.2).

Similarly to (4.5), we may show that Q^2 and Q^2D^2 are transformation matrices producing further U-loci. In other words, 27 U-loci are associated with the matrices Q^2D^t ($t = 0, 1, 2$). Specifically, Q^2 (or Q^2D^2) refers to primals of (4.1) whose forms differ from those of (4.2) only by having a factor ϵ (or ϵ^2) associated with ϕ_3 and by having the T_4- (or T_3-)triads associated with ϕ_1, ϕ_2, ϕ_4. Loci U_{159}, U_{267}, U_{348} are repeated in each case.

The remaining U-loci, numbering $27 \times 3 = 81$, are generated in an exactly analogous manner. Apart from a (possibly) scalar factor involving one or

https://doi.org/10.4153/CJM-1966-015-0 Published online by Cambridge University Press

more of k, ϵ, and ϵ^2, the sets of five primals corresponding to (4.2) are:

$$(4.7)\qquad \begin{cases} \theta'' \equiv \theta + 6\phi_1 = 0, \\ \phi''_3 \equiv \theta - 3\phi_1 = 0, \\ \phi''_1 \equiv \phi_2 + \phi_3 + \phi_4 = 0, \\ \phi''_2 \equiv \phi_2 + \epsilon^2\phi_3 + \epsilon\phi_4 = 0, \\ \phi''_4 \equiv \phi_2 + \epsilon\phi_3 + \epsilon^2\phi_4 = 0; \end{cases}$$

$$(4.8)\qquad \begin{cases} \theta''' \equiv \theta + 6\phi_2 = 0, \\ \phi'''_1 \equiv \theta - 3\phi_2 = 0, \\ \phi'''_2 \equiv \phi_1 + \epsilon\phi_3 + \epsilon^2\phi_4 = 0, \\ \phi'''_3 \equiv \phi_1 + \phi_3 + \phi_4 = 0, \\ \phi'''_4 \equiv \phi_1 + \epsilon^2\phi_3 + \epsilon\phi_4 = 0; \end{cases}$$

$$(4.9)\qquad \begin{cases} \theta'''' \equiv \theta + 6\phi_4 = 0, \\ \phi''''_1 \equiv \theta - 3\phi_4 = 0, \\ \phi''''_2 \equiv \phi_1 + \epsilon^2\phi_2 + \epsilon\phi_3 = 0, \\ \phi''''_3 \equiv \phi_1 + \phi_2 + \phi_3 = 0, \\ \phi''''_4 \equiv \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_3 = 0. \end{cases}$$

Following our procedure in (4.2)–(4.6) *et seq.*, we determine the relevant transformation matrices to be Q^3D^t, $Q^2D^tw^2$, and Q^2D^tj respectively ($t = 0, 1, 2$) (cf. (2.3), (2.4), (2.6), (2.7), (2.8))—in particular, Q^3 for (4.7), Q^2Dw^2 for (4.8), and $Q^2\mathrm{D}j$ for (4.9).

There can be no other loci like U, since all the possibilities have been exhausted.

It is worth emphasizing that the generator Q of CT does not itself produce any of the 120 U-loci. Ignoring scalar factors, we find that with Q as transformation matrix the loci corresponding to (4.2) are

$$(4.10)\qquad \begin{cases} \theta + 6(\phi_1 + \phi_2 + \phi_3 + \phi_4), \\ \theta - 3(\phi_1 - 2\phi_2 + \phi_3 + \phi_4), \\ \theta - 3(\phi_1 + \phi_2 - 2\phi_3 + \phi_4), \\ \theta - 3(-2\phi_1 + \phi_2 + \phi_3 + \phi_4), \\ \theta - 3(\phi_1 + \phi_2 + \phi_3 - 2\phi_4), \end{cases}$$

respectively, though the primes corresponding to those derived from (4.6) reduce to primes through the basic primes of the Clebsch pattern (§2).

Thus, the 120 U-loci occur in four sets of 30 (based on (4.2), (4.7), (4.8), (4.9)), each set of 30 consisting of the 27 indicated above together with the corresponding three from the original set of 12.

Geometrically, the transformations of the group CT must permute the 120 U-loci among themselves. This means that CT_U has index 120 in CT and therefore is of order

$$\frac{51{,}840 \times 81}{120} = \frac{2^7.3^8.5}{2^3.3.5} = 2^4.3^7 = 34{,}992.$$

Accordingly, the theorem of §1 is proved.

5. Concluding remarks. Associated with each U there is a corresponding group CT_U leaving it invariant. All the 34,992 elements of $CT_{U_{123}}$ have been enumerated explicitly by the author and incorporated into a single simple matrix expression involving j (2.6), w (2.7), $J = Q^5 = Q^2Q^3$ (2.4) and other elements of CT, among which are certain *Clifford matrices* **(6)**. Most of the 34,992 matrices represent monomial transformations; indeed, three quarters of them do. Because the details of this enumeration are lengthy and depend on unpublished material, they are excluded from this treatment.

Many other aspects of the work are open to detailed investigation, such as the geometry associated with (say) $CT_{U_{123}}$, the explicit enumeration of the elements of other CT_U, and the transformations by which the elements of one CT_U become the elements of an isomorphic group. Some of the basic material of these avenues of approach has been obtained.

Finally, it may be noted that the theorem can be attacked from other directions, for example, by using the methods of **(3)**. It has been suggested to the author that a fruitful technique might be to observe, in conjunction with the Clebsch diagram (§2), that the x_{ij} can represent the nine points of a 13-point affine geometry over GF(3), and to use the fact that the group of affine collineations has known order $432 = 2^4.3^3$. Perhaps other devices commend themselves to the attention of the interested reader.

References

1. H. W. Baker, *Segre's ten nodal cubic primal in space of four dimensions and del Pezzo's surface in five dimensions*, J. Lond. Math. Soc., *6* (1931), 176–185.
2. ——— *Principles of Geometry*, vol. 4 (Cambridge, 1950), pp. 148, 151.
3. Beverley Bolt, T. G. Room, and G. E. Wall, *On the Clifford collineation, transform and similarity groups I, II*, J. Aust. Math. Soc., *2* (1961), 60–96.
4. Beverley Bolt, *On the Clifford collineation, transform and similarity groups III. Generators and involutions*, J. Aust. Math. Soc., *2* (1962), 334–344.
5. A. Clebsch, *Leçons sur la géométrie* (Paris, 1880), pp. 232–233.
6. A. F. Horadam, *A locus in* [*8*] *invariant under a group of order 51840 × 81*, Quart. J. Math. (Oxford), (2) *8* (1957), 241–259.
7. ——— *Projection of an invariant locus in* [*8*] *from a solid lying on it*, Quart. J. Math. (Oxford), (2) *9* (1958), 81–87.
8. ——— *Involutions associated with the Burkhardt configuration in* [*4*], Can. J. Math. *11* (1959), 18–33.
9. ——— *Clifford groups in the plane*, Quart. J. Math. (Oxford), (2) *10* (1959), 294–295.
10. ——— *Clifford matrices and the Hessian group*, Rend. del Circ. Mat. di Palermo (2), *10* (1961), 347–352.
11. U. Perazzo, *Sopra una forma con 9 rette doppie dello spazio a cinque dimensioni*, Atti Reale Acc. Sci. Torino, *36* (1900–1), 891–916.
12. C. Segre, *Sulle varieta cubiche dello spazio a quattro dimensioni e su certi sistemi di rette e certe superficie dello spazio ordinario*, Mem. Reale Acc. della Scienze Torino, (2) *39* (1889), 3–48.

University of New England,
Armidale, Australia, and
The University,
Leeds, England

https://doi.org/10.4153/CJM-1966-015-0 Published online by Cambridge University Press