A REMARK ON THE ORTHOGONALITY RELATIONS IN THE REPRESENTATION THEORY OF FINITE GROUPS

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Let G be a finite group of order g, and

$$t \longrightarrow (a_{ij}^{(\mu)}(t)) \qquad (\mu = 1, 2, \ldots, k)$$

be an absolutely irreducible representation of degree f_μ over a field of characteristic zero. As is well known, by using Schur's lemma **(1)**, we can prove the following orthogonality relations for the coefficients $a_{ij}^{(\mu)}(t)$:

$$\sum_{t \in G} a_{ij}^{(\mu)}(t)\, a_{kl}^{(\nu)}(t^{-1}) = \delta_{\mu\nu}\delta_{il}\delta_{jk}\frac{g}{f_\mu}. \tag{1}$$

It is easy to conclude from (1) the following orthogonality relations for characters:

$$\sum_{t \in G} \chi^{(\mu)}(t)\chi^{(\nu)}(t^{-1}) = \delta_{\mu\nu} g \tag{2}$$

$$\sum_{\mu=1}^{k} \chi^{(\mu)}(t)\chi^{(\mu)}(s^{-1}) = \bar{\delta}_{t,s} n(t) \tag{3}$$

where

$$\chi^{(\mu)}(t) = \sum_i a_{ii}^{(\mu)}(t),$$

and $\bar{\delta}_{t,s}$ is 1 or 0 according as t and s are conjugate in G or not, and $n(t)$ is the order of the normalizer of t.

In this short note, we remark that we can conclude (1) from (3) or from a special case of (3):

$$\sum_{\mu=1}^{k} f_\mu \chi^{(\mu)}(t) = \delta_{1,t} g. \tag{3'}$$

Let us now asume (3′). Setting $t = 1$ in (3′) we have

$$g = \sum_\mu f_\mu^2.$$

Therefore the number of $(\mu; i, j)$ such that $1 \leqslant \mu \leqslant k$ and $1 \leqslant i, j \leqslant f_\mu$ is g. Let A be the matrix of degree g with the row index t, column index $(\mu; i, j)$ and $(t, (\mu; i, j))$-element $a_{i,j}^{(\mu)}(t)$.

Let B be the matrix with row index $(\mu; i, j)$, column index t and $((\mu; i, j), t)$-element

$$\frac{f_\mu}{g} \cdot a_{ji}^{(\mu)}(t^{-1}).$$

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The (t, s)-element in AB is

$$\sum_{\mu, i, j} a_{i,j}^{(\mu)}(t) \cdot \frac{f_\mu}{g} a_{ji}^{(\mu)}(s^{-1})$$
$$= \sum_{\mu, i} \frac{f_\mu}{g} a_{ii}^{(\mu)}(ts^{-1}) = \frac{1}{g} \cdot \sum_{\mu} f_\mu \cdot \chi^{(\mu)}(ts^{-1}) = \delta_{t,s}.$$

This shows that $AB = E$, and hence $BA = E$.

Since the $((\mu; i, j), (\nu; k, l))$-element of BA is

$$\frac{f_\mu}{g} \sum_{t \in G} a_{ij}^{(\mu)}(t)\, a_{kl}^{(\nu)}(t^{-1}),$$

we have (1).

Reference

1. I. Schur, *Neue Begründung der Theorie der Gruppencharactere*, Sitzungsber. Preuss. Akad. d. Wiss. (1905), 406–32.

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