THE a-POINTS OF FABER POLYNOMIALS FOR A SPECIAL FUNCTION†

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1. Introduction. Let $f(\zeta)$ be a power series of the form

$$\zeta + a_0 + a_1/\zeta + \dots, \tag{1}$$

where $\limsup |a_n|^{1/n} < \infty$. The Faber polynomials $\{f_n(\zeta)\}$ $(n = 0, 1, 2, \dots)$ are the polynomial parts of the formal expansion of $(f(\zeta))^n$ about $\zeta = \infty$. Series (1) defines an analytic element of an analytic function which we designate as $w = f(\zeta)$. Since $f'(\zeta) \neq 0$ at $\zeta = \infty$, the analytic element is univalent in some neighborhood of infinity; thus the inverse of this element is uniquely determined in some neighborhood of $w = \infty$, and has a Laurent expansion of the form

$$w + b_0 + b_1/w + \dots, \tag{2}$$

where $\limsup |b_n|^{1/n} = \rho < \infty$. Let $\zeta = g(w)$ be this single-valued function defined by (2) in $|w| > \rho$. No analytic continuation of $g(w)$ will be considered.

Let $\Delta(\zeta)$ and $\Delta_a(\zeta)$ $(a \neq 0)$ be the derived sets, in the ζ-plane, of the zeros of $f_n(\zeta)$ and $f_n(\zeta) - a$, respectively. These sets can be described by means of certain sets in the ζ-plane whose definitions follow:

DEFINITION. *A point ζ_1, in the ζ-plane, is said to belong to the set c_1 if $g(w) - \zeta_1 = 0$ has a solution w_1 in $|w| > \rho$ such that $g'(w_1) \neq 0$, $g(w_2) \neq \zeta_1$ for $|w_2| \geqq |w_1|$, $w_2 \neq w_1$. A point ζ_1, in the ζ-plane, is said to belong to the set s_1 if ζ_1 is in c_1 and the corresponding solution, of greatest modulus, for $g(w) - \zeta_1 = 0$ is of modulus greater than* 1.

Ullman [4] proved the following theorem concerning $\Delta(\zeta)$.

THEOREM 1. (*a*) *$\Delta(\zeta)$ lies in the complement of c_1 and* (*b*) *$\Delta(\zeta)$ contains every boundary point of c_1*.

In [1] the author extended Ullman's results to $\Delta_a(\zeta)$:

THEOREM 2. (*a*) *$\Delta_a(\zeta)$ lies in the complement of s_1 and* (*b*) *$\Delta_a(\zeta)$ contains every boundary point of s_1*.

Theorem 2 indicates an interesting difference between the cases $\rho > 1$ and $\rho < 1$. It shows that $a = 0$ is a special case when $\rho > 1$, while it is an exceptional case when $\rho < 1$.

The object of this paper is the location of $\Delta(\zeta)$ and $\Delta_a(\zeta)$ for a special function, namely

$$w = f(\zeta) = \zeta e^{1/(\lambda\zeta)} = \zeta + 1/\lambda + 1/(2\lambda^2\zeta) + \dots, \tag{3}$$

where λ is an arbitrary positive number. In §3 the following theorem concerning the location of $\Delta(\zeta)$ and $\Delta_a(\zeta)$ is established. We state the theorem relative to the z-plane, where $z = 1/\lambda\zeta$.

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THEOREM 3. (*a*) $\Delta(z)$ *is the set* $\Gamma = \{z \mid |ze^{1-z}| = 1, |z| \leqq 1\}$. (*b*) *For* $\lambda < e$, $\Delta_a(z)$ *is the set* Γ *as in part* (*a*), *while for* $\lambda \geqq e$ *it is the set* $\Gamma_1 = \{z \mid |\lambda ze^{-z}| = 1, |z| \leqq 1\}$.

Finally, in §4 an asymptotic distribution of the a-points along Γ and Γ_1 is established.

2. Discussion of results. The methods used in proving Theorems 1 and 2 are hard to apply for the special function (3). Instead we employ methods used by Szegö [3], which lend themselves naturally to this case.

To obtain the exterior mapping radius ρ associated with (3), we use Bürmann–Lagrange series (see for example [2]) and get

$$g(w) = w - \sum_{0}^{\infty} \frac{n^n w^{-n}}{(n+1)!\, \lambda^{n+1}}.$$

Thus

$$\rho = e/\lambda. \tag{4}$$

In order to determine the sets c_1 and s_1 for the special function, we need to discuss the mapping

$$\tau = z\, e^{1-z}. \tag{5}$$

The level curve $|z\, e^{1-z}| = 1$ is symmetrical with respect to the x-axis and consists of two parts:

$$\left.\begin{aligned} \Gamma &= \{z \mid |z\, e^{1-z}| = 1, |z| \leqq 1\}, \\ \Gamma' &= \{z \mid |z\, e^{1-z}| = 1, |z| \geqq 1\}. \end{aligned}\right\} \tag{6}$$

From the polar equations of (5), one can easily see that Γ is a simple closed curve intersecting the x-axis at -0.278 and 1. The second part Γ' intersects the x-axis at 1 alone; thus the level curve has a double point at $z = 1$ and makes the angles $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ with the x-axis there. Let I, II and III be the domains interior of Γ, to the right of Γ' and bounded by Γ and Γ', respectively. Using the polar equations of (5), one can easily show that (5) maps I in a one-to-one manner onto $|\tau| < 1$, and maps II in a similar manner onto the infinite Riemann surface which has been constructed with a cut along the negative x-axis, for which $|\tau| < 1$. Domain III is mapped by (5) in a similar manner onto the above Riemann surface for which $|\tau| > 1$.

Since no analytic continuation is considered for $\zeta = g(w)$, the inverse of the special function, the set c_1 is easily seen through the transformations $\zeta = 1/(\lambda z)$, $\tau = e/(\lambda w)$ as the set I in the z-plane. Similarly s_1 becomes, in the z-plane, the part of I corresponding to $|w| \geqq 1$ or $|\tau| \leqq e/\lambda$. Hence s_1 is the interior of Γ_1 (See (9) below). Thus to establish Theorems 1 and 2 for the special function (3) is equivalent to proving Theorem 3.

3. Location of $\Delta(z)$ and $\Delta_a(z)$. Let

$$s_n(z) = \sum_{0}^{n} (nz)^p/p! \;(n = 1, 2, \ldots) \quad \text{and} \quad g_n(z) = 1 - e^{-nz} s_n(z).$$

Szegö used the following lemma to show, among other things, *that the derived set of the zeros of* $s_n(z)$ *is the curve* Γ *given by* (6).

LEMMA 1. *For* $z \neq 1$,

(*a*) $g_n(z) = (1/\sqrt{(2\pi n)})(z e^{1-z})^n(z/(1-z))(1+\varepsilon_n(z))$ *for z in* I, III *or on* Γ.

(*b*) $g_n(z) = 1+(1/\sqrt{(2\pi n)})(z e^{1-z})^n(z/(1-z))(1+\varepsilon_n'(z))$ *for z in* II, III *or on* Γ'.

In (*a*) *and* (*b*) $\lim \varepsilon_n(z) = \lim \varepsilon_n'(z) = 0$ *uniformly in every finite region which is located entirely in the corresponding regions of* (*a*) *and* (*b*) *and does not include* $z = 1$.

Since $\zeta^n e^{n/\lambda\zeta} = \zeta^n(1+n/\lambda\zeta+\ldots+n^n/n!\lambda^n\zeta^n+\ldots)$, the Faber polynomials associated with (3) are given by

$$f_n(\zeta) = \zeta^n\{1+n/(\lambda\zeta)+\ldots+n^n/(n!\,\lambda^n\zeta^n)\}.$$

From $\zeta = 1/(\lambda z)$,

$$f_n(\zeta) = f_n(1/\lambda z) = (1+nz+n^2z^2/2!+\ldots+n^nz^n/n!)/\lambda^n z^n = s_n(z)/\lambda^n z^n.$$

Thus the zeros of $f_n(\zeta)$ in the ζ-plane are those of $s_n(z)$ in the z-plane. It follows then that $\Delta(z)$ is the curve Γ, *which is part* (*a*) *of Theorem* 3.

Let $q_n(z) = f_n(\zeta)-a = s_n(z)/\lambda^n z^n - a$. Substitution yields

$$g_n(z) = 1-e^{-nz}s_n(z) = 1-e^{-nz}\lambda^n z^n(a+q_n(z)).$$

It is now clear that $\Delta_a(z)$ is the derived set of the solutions of $g_n(z) = 1-a\,e^{-nz}\lambda^n z^n$. Set

$$G_n(z) = g_n(z)+a\,e^{-nz}\,\lambda^n z^n. \tag{7}$$

The set $\Delta_a(z)$ becomes the derived set of the solutions of $G_n(z) = 1$. We need the following lemma in order to locate $\Delta_a(z)$ when $\rho = e/\lambda > 1$ (See (4)).

LEMMA 2. *For* $\rho = e/\lambda > 1$, $z \neq 1$ *we have*

(*a*) $G_n(z) = (1/\sqrt{(2\pi n)})(z e^{1-z})^n(z/(1-z))(1+E_n(z))$,

for z in I, III *or on* Γ.

(*b*) $G_n(z) = 1+(1/\sqrt{(2\pi n)})(z e^{1-z})^n(z/(1-z))(1+E_n'(z))$,

for z in II, III *or on* Γ'.

$E_n(z)$ *and* $E_n'(z)$ *have the same limit behavior as* $\varepsilon_n(z)$, $\varepsilon_n'(z)$ *in Lemma* 1.

The above lemma can be proved easily from Lemma 1. In fact Lemma 2 gives the same representations for $G_n(z)$ as Lemma 1 for $g_n(z)$. Thus it yields the same conclusion, namely that $\Delta_a(z)$ is Γ, $\rho > 1$, *which is the first part of* (*b*) *of Theorem* 3.

Consider

$$\tau' = \lambda z\,e^{-z}. \tag{8}$$

For $\rho = e/\lambda \leqq 1$, the level curve $|\tau'| = 1$ consists of two curves:

$$\left.\begin{aligned} \Gamma_1 &= \{z \mid |\lambda z\,e^{-z}| = 1, |z| \leqq 1\}, \\ \Gamma_1' &= \{z \mid |\lambda z\,e^{-z}| = 1, |z| \geqq 1\}. \end{aligned}\right\} \tag{9}$$

Denote the interior of Γ_1 by I′, the domain left of Γ_1' by II′, and the domain bounded by Γ_1 and Γ_1' by III′. Note that I′ ⊆ I, II′ ⊆ II, III′ ⊆ III. We shall prove the following lemma.

LEMMA 3. *For $\rho = e/\lambda \leqq 1$, $z \neq 1$ we have*

(*a*) $G_n(z) = a(\lambda z e^{-z})^n(1+\eta_n(z))$,

for z in I, III *or on* Γ.

(*b*) $G_n(z) = 1+a(\lambda z e^{-z})^n(1+\eta_n'(z))$,

for z in II, III *or on* Γ'.

$\eta_n(z)$ and $\eta_n'(z)$ have the same limit behavior as the corresponding functions in Lemmas 1 *and* 2.

The above lemma is a direct consequence of Lemma 1. For instance, to prove part (*a*), one can use (7) and part (*a*) of Lemma 1 to get

$$G_n(z) = a e^{-nz}\lambda^n z^n + (1/\sqrt{(2\pi n)})(z e^{-z})^n (z/(1-z))(1+\varepsilon_n(z))$$
$$= a(\lambda z e^{-z})^n[1+(e/\lambda)^n(1/a\sqrt{(2\pi n)})(z/(1-z))](1+\varepsilon_n(z)).$$

Since $e/\lambda \leqq 1$, the expression in the square brackets will approach 1 uniformly. Thus part (*a*) is proved.

From Lemma 3, we have

$$\lim G_n(z) = \begin{cases} 0 \text{ for } z \text{ in I}', \\ 1 \text{ for } z \text{ in II}', \\ \infty \text{ for } z \text{ in III}', \end{cases}$$

uniformly in every region which is entirely located in I′, II′ and III′, respectively. Consequently, for large n, $G_n(z) \neq 1$ in I′ or in III′. As for z in II′ and Γ_1', part (*b*) of Lemma 3 shows that $\lim(G_n(z)-1)/a(\lambda z e^{-z})^n = 1$. Thus for n sufficiently large, a theorem due to Hurwitz yields that $G_n(z)-1 \neq 0$ in II′, III′ or on Γ_1'. The only possible location of $\Delta_a(z)$ then is Γ_1. However, that $\Delta_a(z)$ occupies every point of Γ_1 is a consequence of Theorem 4 below. *This completes the second part of part* (*b*) *of Theorem* 3.

4. An asymptotic distribution of the zeros and the a-points of $f_n(\zeta)$. Using Lemma 1, Szegö not only proved that the derived set of the zeros of $s_n(z)$ and $s_n(z)-a$ is identical to Γ, but also that its elements are positioned along any arc of Γ in such a way that the distribution along the arc is asymptotically equal to the change in $(1/2\pi)(\arg(z e^{1-z})^n)$ along the arc. We shall call such a distribution *uniform*. Obviously the distribution of the zeros of $f_n(\zeta)$ along Γ in the z-plane is uniform. Also, since Lemma 2 is the same as Lemma 1, the distribution of the a-points of $f_n(\zeta)$ along Γ, when $\rho > 1$, is uniform. As for the distribution of the a-points of $f_n(\zeta)$ for $\rho \leqq 1$, we shall show that it is uniform along Γ_1 in the z-plane.

Let $0 < r < 1 < R$, $0 < \theta_1 < \theta_2 < 2\pi$. Consider the region in the τ'-plane bounded by

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two line segments and two circular arcs whose vertices are $re^{i\theta_1}$, $Re^{i\theta_1}$, $Re^{i\theta_2}$, and $re^{i\theta_2}$. Let D be the region in the z-plane whose image in the τ'-plane under (8) is the above region.

THEOREM 4. *Let r, R, θ_1, θ_2 be chosen as before. For sufficiently large n, let $N(r,R,\theta_1,\theta_2)$ be the number of zeros of $G_n(z)-1$ in D when $\rho \leqq 1$. Then*

$$N(r,R,\theta_1,\theta_2) = n(\theta_2-\theta_1)/2\pi + O(1). \tag{10}$$

Proof. For every n, we associate with D two regions D_n^-, D_n^+, such that $D_n^- \subset D \subset D_n^+$. In order to construct D_n^-, for example, replace the right-hand boundary of D by another curve whose image under (8) consists of two line segments and one circular arc connecting the following points in the positive direction: $re^{i(\theta_1+\beta/n)}$, $\frac{1}{2}(1+R)e^{i(\theta_1+\beta/n)}$, $\frac{1}{2}(1+R)e^{i\theta_1}$, $Re^{i\theta_1}$, and such that

$$n\theta_1+\beta \equiv -\alpha+\pi \,(\mathrm{mod}\, 2\pi), \text{ where } \alpha = \arg a \quad \text{and} \quad 0 \leqq \beta < 2\pi.$$

Replace the left-hand boundary of D by a similar interior curve. Thus $D_n^- \subset D$. The replacement of the right and left boundary parts of D by two exterior curves constructed in a way similar to the above is D_n^+. Thus $D \subset D_n^+$. Let N_n^-, N_n^+ be the number of zeros of $G_n(z)-1$ in D_n^-, D_n^+, respectively. We shall show that

$$N_n^- = n(\theta_2-\theta_1)/2\pi + O(1),$$

and

$$N_n^+ = n(\theta_2-\theta_1)/2\pi + O(1).$$

Since $N_n^- \leqq N_n \leqq N_n^+$, Theorem 4 will then be proved. We shall show the above for D_n^-; D_n^+ is handled similarly.

Let A, B, C, D, E, F, G, H, be the points on the boundary of D_n^- which are the images under (8) of the points $re^{i(\theta_2-\beta/n)}$, $re^{i(\theta_1+\beta/n)}$), $\frac{1}{2}(1+R)e^{i(\theta_1+\beta/n)}$, $\frac{1}{2}(1+R)e^{i\theta_1}$, $Re^{i\theta_1}$, $Re^{i\theta_2}$, $\frac{1}{2}(1+R)e^{i\theta_2}$, $\frac{1}{2}(1+R)e^{i(\theta_2-\beta/n)}$, respectively. Let

$$F(z) = a(\lambda z e^{-z})^n. \tag{11}$$

In what follows n is chosen large enough to satisfy the different statements mentioned below. From (8) and (11) it follows that

$$|(F(z)-1)/F(z)| > 1/|a|r^n - 1 > \tfrac{1}{2}$$

for z on AB, while

$$|(F(z)-1)/F(z)| > 1 - 1/|a|R^n > \tfrac{1}{2}$$

for z on EF. Since the curve BC is mapped by (11) onto a line segment joining $-|a|r^n$ and $-|a|((1+R)/2)^n$, $F(z)$ is closer to the origin than to $(1,0)$ when z traverses BC. Hence $|(F(z)-1)/F(z)| > 1$ for z on BC. For z on CD or DE, $|(F(z)-1)/F(z)| > 1-2^n/|a|(1+R)^n > \frac{1}{2}$. In short, $|(F(z)-1)/F(z)| > \frac{1}{2}$ whenever z is on the curve $ABCDEF$. Similarly, the above inequality holds on the rest of the boundary of D_n^-. Thus

$$|F(z)-1| > \tfrac{1}{2}|F(z)|, \tag{12}$$

for z on the boundary of D_n^- and for sufficiently large n.

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From Lemma 3, part (a), one obtains

$$G_n(z)-1 = F(z)-1+F(z)\eta_n(z). \tag{13}$$

Since $\eta_n(z)\to 0$, $|\eta_n(z)| < \frac{1}{2}$ for z on the boundary of D_n^-. From this and (12) it follows that

$$|F(z)-1| > \tfrac{1}{2}|F(z)| > |F(z)\eta_n(z)|,$$

for z on the boundary of D_n^-. Rouché's theorem yields that $F(z)-1$ and $F(z)-1+F(z)\eta_n(z)$ have the same number of zeros in D_n^-. It follows from (13) that the number of zeros of $G_n(z)-1$ is the same as the number of zeros of $a(\lambda z e^{-z})^n-1$ in D_n^-. Note that the change of the argument of $a(\lambda z e^{-z})^n-1$ as z traverses the boundary of D_n^- is determined by the change of the argument as z traverses the arc EF except for an additive term which remains bounded for sufficiently large n. Using the argument principle, we get

$$N_n^- = n(\theta_2-\theta_1)/2\pi+O(1).$$

Similarly $N_n^+ = n(\theta_2-\theta_1)/2\pi+O(1)$ and (10) follows.

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