AVERAGING OPERATORS ON THE RING OF CONTINUOUS FUNCTIONS ON A COMPACT SPACE[1]

BARRON BRAINERD[2]

(received 26 June 1963, revised 11 March 1964)

1. Introduction

In this note we answer the following question: Given $C(X)$ the lattice-ordered ring of real continuous functions on the compact Hausdorff space X and T an averaging operator on $C(X)$, under what circumstances can X be decomposed into a topological product $\mathfrak{A}\times\mathfrak{L}$ such that $\mathfrak{L}$ supports a measure m and $Tf = h$ where

$$h(\alpha, \beta) = \int_{\mathfrak{L}} f(\alpha, \xi)dm(\xi)?$$

By an *averaging operator* we mean a linear transformation T on $C(X)$ such that:

1. T is positive, that is, if $f > 0$ ($f(x) \geqq 0$ for all $x \in X$ and $f(x) > 0$ for some $x \in X$), then $Tf > 0$.
2. $T(fTg) = (Tf)(Tg)$.
3. $T1 = 1$ where $1(x) = 1$ for all $x \in X$.

If X is the unit square $[0, 1]\times[0, 1]$, $f(x_1, x_2) \in C(X)$, then the linear operator T given by the relation,

$$\{Tf\}(x_1, x_2) = \int_0^1 f(x_1, \xi)d\xi,$$

is clearly an averaging operator. Our problem in effect is to characterize those pairs (X, T) where X is a compact Hausdorff space and T an averaging operator such that X can be decomposed into a product and T can be written as an integration over one factor of that product.

Averaging operators have been the object of considerable attention. A recent survey of the literature is given by G. Birkhoff in [1]. In [2] the author discusses the above question when the role of $C(X)$ is played by a

[1] This paper was written while the author was a Visiting Fellow in the Institute of Advanced Studies, Australian National University.

[2] The author wishes to express his gratitude to the referee whose suggestions greatly simplified the proof of the results.

bounded F-ring. Bounded F-rings are one of the more natural generalizations of L^∞-spaces.

2. When X is known to be a product space

Let X be a topological product $\mathfrak{A} \times \mathfrak{B}$ of compact Hausdorff spaces $\mathfrak{A}$ and $\mathfrak{B}$. Let D be the subalgebra of $C(X)$ consisting of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on $\alpha \in \mathfrak{A}$, that is $f(\alpha, \beta) = f_1(\alpha)$ for all $\alpha \in \mathfrak{A}$ and $\beta \in \mathfrak{B}$ where f_1 is a continuous function on $\mathfrak{A}$. Let E be the subalgebra consisting of all functions f which depend only on β.

THEOREM 1. *A necessary and sufficient set of conditions for the existence of a normalized Borel measure*[3] *m on $\mathfrak{B}$ such that*

$$(Tf)(\alpha, \beta) = \int_{\mathfrak{B}} f(\alpha, \xi) dm(\xi)$$

for all $f \in C(X)$, $\alpha \in \mathfrak{A}$, $\beta \in \mathfrak{B}$ is the following:

(i) *for all $f \in E$, Tf should be a constant function,*

(ii) *for all $f \in D$, $g \in E$, $T(fg) = f \cdot T(g)$.*

PROOF. (A) The necessity is trivial.

(B) Sufficiency. For each continuous function φ on $\mathfrak{B}$, let φ^* be the continuous function on X given by

$$\varphi^*(\alpha, \beta) = \varphi(\beta).$$

Define $U(\varphi)$ to be the value of the constant function $T\varphi^*$. Then, since T is a positive linear operator on X, U is a positive Radon measure and hence by the Riesz representation theorem, there exists a normalized Borel measure m on $\mathfrak{B}$ such that

$$U(\varphi) = \int_{\mathfrak{B}} \varphi(\xi) dm(\xi)$$

for every continuous function φ on $\mathfrak{B}$.

For all $f \in C(X)$, let

$$(Vf)(\alpha, \beta) = \int_{\mathfrak{B}} f(\alpha, \xi) dm(\xi).$$

Since for fixed α, $f(\alpha, \beta)$ is continuous in β and hence a continuous function on $\mathfrak{B}$, it follows that $Vf(\alpha, \beta)$ is a well defined function on X. Since m is a normalized Borel measure, it follows that $Vf \in C(X)$ and V is a bounded linear operator.

Now for $f \in D$ and $g \in E$,

[3] A normalized Borel measure on X is a non-negative countably additive measure μ on the Borel sets of X for which $\mu(X) = 1$.

https://doi.org/10.1017/S144678870002406X Published online by Cambridge University Press

$$\{T(f\cdot g)\}(\alpha,\beta) = f(\alpha,\beta)\cdot\{(Tg)(\alpha,\beta)\} \qquad \text{(by (ii))}$$
$$= f(\alpha,\beta)\int_{\mathfrak{Q}} g(\alpha,\xi)dm(\xi)$$

because g does not depend on α. Since $f(\alpha,\beta)$ is independent of β

$$T(f\cdot g)(\alpha,\beta) = \int_{\mathfrak{Q}} f(\alpha,\beta)g(\alpha,\xi)dm(\xi)$$
$$= \{V(f\cdot g)\}(\alpha,\beta).$$

Thus $T(f\cdot g) = V(f\cdot g)$ for all $f\in D$, $g\in E$. However, the subalgebra of $C(X)$ generated by $D\cup E$ is dense in $C(X)$ and by linearity T and V coincide on this subalgebra. Therefore $T = V$ on $C(X)$.

3. The main result

In order to prove our main result (Theorem 3) we employ certain results of MacDowell [3]. MacDowell makes the following definitions:

(i) A subalgebra A of $C(X)$ is *analytic* if it contains the constant functions.

(ii) Two subalgebras A_1 and A_2 are *additively related* if for $f\in A_1$ and $g\in A_2$ either $||f+g|| = ||f||+||g||$ or $||f-g|| = ||f||+||g||$.[4]

We define A_1 and A_2 to be *multiplicatively related* if for $f\in A_1$ and $g\in A_2$,

$$||fg|| = ||f||\,||g||.$$

It is a matter of direct verification that subalgebras A_1 and A_2 are additively related if and only if they are multiplicatively related.

THEOREM 2. *Let X be an arbitrary compact Hausdorff space. If A is a subalgebra of $C(X)$ such that for all $g\in A$, Tg is a constant function, then TC and A are multiplicatively (and hence additively) related.*

PROOF. First note that TC is closed under multiplication, because if $f_1, f_2\in TC$ and $f_1 = Th_1$, $f_2 = Th_2$, then

$$T(h_1Th_2) = Th_1Th_2 = f_1f_2.$$

Suppose TC and A are not multiplicatively related. Then there exists $f\in TC$ and $g\in A$ such that

$$||fg|| < ||f||\,||g||. \tag{1}$$

Thus $|f(x)| < 1$ when $|g(x)| = 1$. Therefore there is an $x\in X$ for which

$$\{f(x)\}^2\cdot\{g(x)\}^2 < \{g(x)\}^2,$$

and so

[4] By $||f||$ we mean the usual sup norm: $||f|| = \sup_{x\in X}|f(x)|$.

https://doi.org/10.1017/S144678870002406X Published online by Cambridge University Press

(2) $$0 \leqq T(f^2g^2) < T(g^2)$$

because T is a positive operator. Now since $g^2 \in A$, $T(g^2)$ is constant and

(3) $$||T(f^2g^2)|| < T(g^2).$$

On the other hand $f^2 \in TC$ and so $f^2 = Th$ for some $h \in C(X)$. Thus

$$T(f^2g^2) = T(g^2)f^2,$$

and

$$||T(f^2g^2)|| = T(g^2)\ ||f^2|| = T(g^2)$$

which contradicts Equation (3). Thus TC and A are multiplicatively related.

LEMMA. *Let $X = \mathfrak{A} \times \mathfrak{L}$ be a topological product of compact Hausdorff spaces $\mathfrak{A}$, $\mathfrak{L}$. Let D consist of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on α and let E consist of those $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on β. If A, B are analytic subalgebras of $C(X)$ such that $A \subset D$ and $B \subset E$ and if the subalgebra generated by $A \cup B$ is dense in $C(X)$, then A is dense in D and B is dense in E.*

PROOF. Follows from the Stone-Weierstrass theorem.

Now we come to prove our main result and answer the question posed in the Introduction.

THEOREM 4. *Let T be an averaging operator on $C(X)$ where X is a compact Hausdorff space and let A be an analytic subalgebra of $C(X)$ such that for all $g \in A$, Tg is a constant function. If $TC \cup A$ separates points of X, then X can be decomposed into a topological product $\mathfrak{A} \times \mathfrak{L}$ of compact Hausdorff spaces and there is a normalized Borel measure m on $\mathfrak{L}$ such that*

(i) *for every $f \in C(X)$*

(4) $$(Tf)(\alpha, \beta) = \int_{\mathfrak{L}} f(\alpha, \xi)dm(\xi)$$

and

(ii) *A is contained in the subalgebra E of $C(X)$ consisting of all $f \in C(X)$ such that $f(\alpha, \beta)$ depends only on β.*

PROOF. TC is clearly an analytic subalgebra of $C(X)$. By Theorem 2, TC and A are additively related. Since $TC \cup A$ separates points of X, the algebra generated by it is dense in C. Therefore by [3] T14, X is a product space $\mathfrak{A} \times \mathfrak{L}$. Let D be the subalgebra of C consisting of all $f \in C$ such that $f(\alpha, \beta)$ depends only on α and let E be the subalgebra of C consisting of $f \in C$ such that $f(\alpha, \beta)$ depends only on β.

The spaces $\mathfrak{A}$, $\mathfrak{L}$ are evidently compact. Since they have for their points certain closed subsets of C and for their topologies the usual topologies for spaces of closed sets, they are Hausdorff.

In his statement of T14, MacDowell does not say that

$$TC \subset D, \quad A \subset E, \tag{5}$$

but these relations can easily be deduced from the proof of his T13 in the following manner. To use his notation, put $F_1 = TC$, $F_2 = A$, $\mathfrak{S}_1 = \mathfrak{A}$, $\mathfrak{S}_2 = \mathfrak{L}$. In his proof ([3]T12) a homeomorphism φ of X onto a space $\mathfrak{S}$ is constructed by letting

$$\varphi(x) = \{f \in C \mid |f(x)| = ||f||\}.$$

The points of $\mathfrak{S}$ are there subsets of C. A homeomorphism F of $\mathfrak{S}$ onto $\mathfrak{S}_1 \times \mathfrak{S}_2$ is then constructed by putting

$$F(S) = (S \cap F_1, S \cap F_2).$$

If $h \in F_1$ then $h\varphi^{-1}F^{-1}(\mu_1, \mu_2)$ depends only on U_1, because if

$$x_0 = \varphi^{-1}F^{-1}(\mu_1, \mu_2), \quad x_0' = \varphi^{-1}F^{-1}(\mu_1, \mu_2')$$

then for

$$S_0 = \{f \in C \mid |f(x_0)| = ||f||\}$$
$$S_0' = \{f \in C \mid |f(x_0')| = ||f||\}$$

we have

$$\mu_1 = S_0 \cap F_1 = S_0' \cap F_1,$$

and so for every $f \in F_1$ either

(a) $$|f(x_0)| = ||f|| \quad \text{and} \quad |f(x_0')| = ||f||$$

or

(b) $$|f(x_0)| < ||f|| \quad \text{and} \quad |f(x_0')| < ||f||.$$

Now, using $h \in F_1$, construct the non-negative function

$$g = ||\{h-h(x_0)\}^2|| - \{h-h(x_0)\}^2$$

in F_1. It is clear that

$$0 \leqq g(x) \leqq ||\{h-h(x_0)\}^2||$$

for all $x \in X$ and that

$$g(x_0) = ||\{h-h(x_0)\}^2||.$$

Therefore

$$||g|| = g(x_0),$$

and from (a) it follows that $g(x_0') = ||g||$. Thus

$$\{h(x_0')-h(x_0)\}^2 = 0 \quad \text{so} \quad h(x_0) = h(x_0').$$

https://doi.org/10.1017/S144678870002406X Published online by Cambridge University Press

Since for each $h \in TC = F_1$,

$$(h\varphi^{-1}F^{-1})(\mu_1, \mu_2)$$

depends only on μ_1, it follows that $TC \subset D$. Similarly $A \subset E$. Therefore statement (5) is valid.

By the Lemma, TC is dense in D, and A is dense in E. Hence every $f \in D$ is a limit of functions in TC and every $g \in E$ is a limit of functions in A. However, for $h \in A$, Th is constant, and so Tg is a constant for each $g \in E$. Similarly if $f \in D$ and $g \in E$, then by the density of TC in D

$$T(fg) = fTg.$$

Thus the conditions for the application of Theorem 1 are satisfied and so there exists a normalized Borel measure m on $\mathfrak{L}$ for which Equation (4) is valid.

Remarks. (1) It is clear that any operator on $C(X)$ of the form given in Equation (4) is an averaging operator. Then TC is the set of functions which depend only on α, and A can be taken as the set of functions depending only on β. In this case $TC \cup A$ generates an algebra dense in $C(X)$.

(2) For an averaging operator T, it is possible that the only subalgebra A of $C(X)$ for which $g \in A$ implies Tg is constant is exactly the algebra of constant functions. This occurs when $X = \{1, 2, 3\}$ and $Tf = g$ where

$$g(1) = g(2) = \frac{f(1)+f(2)}{2} \quad \text{and} \quad g(3) = f(3).$$

(3) It is also possible that an analytic subalgebra A of $C(X)$ exists which is maximal relative to the condition that for each $g \in A$, Tg is constant while $TC \cup A$ does not separate points in X. See, for example, Example 11.1 in [2].

References

[1] Birkhoff, G., Averaging Operators, Symposium in Lattice Theory. Amer. Math. Soc. 1960.
[2] Brainerd, B., On the structure of averaging operators, J. Math. Analysis and Applications, 5 (1962), 347—377.
[3] MacDowell, R., Banach spaces and algebras of continuous functions, Proc. Amer. Math. Soc. 6 (1955), 67—78.

Department of Mathematics
University of Toronto
Toronto, Canada.

https://doi.org/10.1017/S144678870002406X Published online by Cambridge University Press