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p-Adic Dynamics and Sullivan's No Wandering Domains Theorem

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Abstract. In this paper we study dynamics on the Fatou set of a rational function $\phi \in \overline{\mathbb{Q}}_p(z)$. Using a notion of 'components' of the Fatou set defined by Benedetto, we state and prove an analogue of Sullivan's No Wandering Domains Theorem for p-adic rational functions which have no wild recurrent Julia critical points.

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0. Introduction

Let $p \in \mathbb{Z}$ be a fixed prime number. Let $\mathbb{Q}_p$ denote the field of p-adic rational numbers, $\overline{\mathbb{Q}}_p$ its algebraic closure, and $\mathbb{C}_p$ the metric completion of $\overline{\mathbb{Q}}_p$, with absolute value denoted by $|\cdot|$. We consider the dynamics of a morphism ϕ, defined over $\overline{\mathbb{Q}}_p$, from the projective line $\mathbb{P}^1(\mathbb{C}_p)$ to itself; thus, ϕ is a rational function with coefficients in some finite extension K of $\mathbb{Q}_p$. As in the complex theory of dynamical systems, we define the Fatou set $\mathcal{F}$ of ϕ to be the set of all points in the projective line having a neighborhood on which the family of iterates $\{\phi^n\}$ is equicontinuous (see, for example, [6, 17]). Here, ϕ^n denotes the n-fold composition of ϕ with itself. The notions of neighborhood and equicontinuity are defined by the spherical metric on $\mathbb{P}^1(\mathbb{C}_p)$ determined by the non-Archimedean metric on $\mathbb{C}_p$. The Fatou set is clearly open; its complement, the Julia set $\mathcal{J}$, is therefore closed. Both sets are preserved under the application of ϕ.

The study of p-adic dynamical systems arises in Diophantine geometry in the construction of canonical local heights, used for counting rational points on algebraic varieties over a number field, as in [5]. Hsia ([10, 11]) has proven some basic properties of non-Archimedean Julia sets and described their relation to weak Néron models. In addition, p-adic fields have arisen in physics in the theory of superstrings, prompting questions about their dynamics (see, for example, [1, 22, 24, 25]). Other studies of non-Archimedean dynamics in the neighborhood of a periodic point and of the counting of periodic points over global fields using local (non-Archimedean) fields appear in [9, 13–15, 18–21, 26].

When studying a complex dynamical system, it is convenient to break the Fatou set into its connected components and study the action of ϕ on the set of components. Sullivan proved in [23] that there are no wandering components of a complex Fatou set; in other words, every component eventually maps to a periodic component. (A periodic component is one which maps to itself after finitely many iterations.) Thus, the study of dynamics on a component was effectively reduced to the study of dynamics on a fixed component, since any periodic component of period n is a fixed component of ϕ^n.

The D-components defined in [2, 3] make a similar theory possible in the non-Archimedean setting. In this paper, we prove a No Wandering Domains Theorem for p-adic rational functions without wild recurrent critical points in the Julia set. We also prove that such maps have only finitely many periodic cycles of components over any given finite extension of $\mathbb{Q}_p$. The methods are comparable to those of Mañé ([16]) and others ([4 ,7], for instance) from the complex case; however, while the complex methods must assume no recurrent Julia critical points at all, our methods allow tame recurrent Julia critical points for p-adic maps.

1. Terminology and Theorems

We begin by recalling some basic terminology from the theory of dynamical systems. Let X be a set, and let ϕ be a function mapping X to itself. We say $x \in X$ is *fixed* if $\phi(x) = x$; more generally, we say x is *periodic* (of period n) if $\phi^n(x) = x$ for some $n \geqslant 1$. We say x is *pre-periodic* if $\phi^m(x)$ is periodic for some $m \geqslant 0$.

If X is a metric space, we say that a point $x \in X$ is *recurrent* if x is contained in the closure of its (forward) orbit

$$\{\phi^n(x) : n \geqslant 1\}.$$

Equivalently, a point x is recurrent if it is contained in $\omega(x)$, the ω-limit set of the sequence $\{\phi^n(x)\}$. (Recall that the ω-limit set of a sequence $\{a_n\}$ is the intersection, over all $N \geqslant 1$, of the closures of the sets $\{a_n\}_{n \geqslant N}$.) We will often abuse language and say that x *accumulates at* y if $y \in \omega(x)$. Thus, we could define a recurrent point to be a point that accumulates at itself.

Given a prime integer p, recall that the p-adic absolute value on $\mathbb{Q}$ is defined by setting $|0| = 0$, and

$$\left|\frac{m}{n} p^e\right| = \frac{1}{p^e},$$

where m and n are nonzero relatively prime integers not divisible by p. Thus, numbers with numerators divisible by p are 'small', while numbers with denominators divisible by p are 'big'. The completion of $\mathbb{Q}$ with respect to this absolute value is denoted $\mathbb{Q}_p$; it should be viewed as the p-adic analogue of $\mathbb{R}$. The absolute value $|\cdot|$ extends in unique fashion to $\mathbb{Q}_p$, all finite extensions of $\mathbb{Q}_p$, and the fields $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$, defined in the introduction. Unlike the extension $\mathbb{C}$ of $\mathbb{R}$, the algebraic closure of $\mathbb{Q}_p$ is an infinite extension of $\mathbb{Q}_p$. Furthermore, $\mathbb{Q}_p$ and its finite extensions are locally compact, while $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$ are not.

We recall that a *non-Archimedean* metric space X is a metric space satisfying the ultrametric (strengthened) triangle inequality

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\} \quad \text{for all } x, y, z \in X.$$

We will study dynamics of rational functions defined over p-adic fields (i.e., $\mathbb{Q}_p$ and its extensions), which are non-Archimedean. In particular, such functions act on the p-adic projective line $\mathbb{P}^1(\mathbb{C}_p)$. We will view $\mathbb{C}_p$ as a subset of $\mathbb{P}^1(\mathbb{C}_p)$ by considering $\mathbb{P}^1(\mathbb{C}_p)$ to be the union $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$.

Even though $\mathbb{Q}_p$ and its extensions have characteristic zero, interesting phenomena occur when the ramification degree of a map at some point is divisible by p. We therefore make the following definition.

DEFINITION 1.1. Let $\phi \in \mathbb{C}_p(z)$ be a rational function. We say a critical point $x \in \mathbb{P}^1(\mathbb{C}_p)$ of ϕ is *wild* if the index of ramification of ϕ at x is divisible by p. If x is not wild, we say x is tame.

For example, if $\phi(z) = z^{np}$ for some positive integer n, then 0 and ∞ are wild critical points. However, if $\phi(z) = z^p + z^{p+1}$, then 0 is wild (since the lowest term in the Taylor expansion there has exponent p, which is divisible by p), while ∞ is tame (since the critical point has index $p+1$). Furthermore, there is another critical point at $-p/(p+1)$, with index 2; this point is wild if $p = 2$ and tame otherwise.

It should be pointed out that the above definition abuses standard terminology slightly. Usually, a critical point of a map (or ramified place of an extension) is called ‘wild’ if the index of ramification is divisible by the characteristic of the base field. In our case, $\mathbb{Q}_p$ and its extensions have characteristic zero. Nonetheless, our terminology is not unjustified, because local power series may be ‘reduced’ to maps over fields of characteristic p, where the terms ‘wild’ and ‘tame’ are applicable. The details of this reduction process are not relevant to our investigations, and we will not discuss them here.

If X is a metric space, $x \in X$, and $r > 0$, we shall denote by $D_r(x)$ and $\overline{D}_r(x)$, respectively, the open and closed disks of radius r centered at x. For the purposes of this paper, we will not consider singletons or the null set to be disks; by this convention, then, we are justified in requiring r to be positive. Note that if X is non-Archimedean, then all disks are both open and closed as topological sets. Furthermore, any point of a disk is a center, and if two disks intersect, then one is contained in the other.

If Y_1 and Y_2 are two subsets of $\mathbb{C}_p$, we will denote by $\operatorname{dist}(Y_1, Y_2)$ the distance between Y_1 and Y_2; that is,

$$\operatorname{dist}(Y_1, Y_2) = \inf\{|y_1 - y_2| : y_i \in Y_i\}.$$

If one or both of the sets is a singleton $\{a\}$ or $\{b\}$, we may abuse notation and write

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dist(a, Y_2), dist(Y_1, b), or dist(a, b) when our meaning is clear. If $D \subset \mathbb{C}_p$ is a disk (open or closed), we will denote by rad(D) the radius of D; the radius is always a well-defined positive real number. In fact,

$$\mathrm{rad}(D) = \sup\{|x - y| : x, y \in D\}.$$

We will also be interested in 'disks' in $\mathbb{P}^1(\mathbb{C}_p)$, and so we recall the following definition from [3].

DEFINITION 1.2. A *closed* $\mathbb{P}^1(\mathbb{C}_p)$*-disk* is a closed disk $\overline{D}_r(x)$, *or* the complement of an open disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus D_r(x)$, for some $x \in \mathbb{C}_p$ and $r > 0$. Similarly, an open $\mathbb{P}^1(\mathbb{C}_p)$*-disk* is an open disk $D_r(x)$, *or* the complement of a closed disk, $\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_r(x)$, for some $x \in \mathbb{C}_p$ and $r > 0$.

As noted in [8], the set of $\mathbb{P}^1(\mathbb{C}_p)$-disks coincides with the set of images of disks in $\mathbb{C}_p$ under automorphisms of $\mathbb{P}^1(\mathbb{C}_p)$.

The topology on a non-Archimedean field is too strong for connected components to be useful in dynamics; all components would be singletons. We therefore recall the definition of D-components from [3].

DEFINITION 1.3. Let X be a topological space with a set $\mathcal{D}$ of distinguished subsets. Let $U \subset X$ be an open subset, and let x be any point of U. We define the *D-component* of U containing x to be the set of all $y \in U$ with the following property: there exists a finite sequence of distinguished subsets

$$D_1, \dots, D_n \in \mathcal{D},$$

with $x \in D_1$ and $y \in D_n$, such that for any $i = 1, \dots, n-1$,

$$D_i \cap D_{i+1} \neq \emptyset.$$

If X is a metric space, we will choose $\mathcal{D}$ to be the set of all disks (of positive radius) in X. By this definition, it is easy to verify that the D-components of an open subset U of $\mathbb{C}$ are precisely the connected components of U (see [3]). On the other hand, if $X = \mathbb{C}_p$, then the D-component of an open subset U containing a given point $x \in U$ is simply the largest disk $D_r(x)$ or $\overline{D}_r(x)$ centered at x and contained in U.

However, we will usually consider $X = \mathbb{P}^1(\mathbb{C}_p)$. In that case, we choose $\mathcal{D}$ to be the set of all $\mathbb{P}^1(\mathbb{C}_p)$-disks; we recall the following simple result from [3].

PROPOSITION 1.1. *Let U be an open subset of $\mathbb{P}^1(\mathbb{C}_p)$, and let $x \in U$. If $U = \mathbb{P}^1(\mathbb{C}_p)$, or if U is the complement of a single point of $\mathbb{P}^1(\mathbb{C}_p)$, then the D-component of U containing x is U. Otherwise, the D-component is the largest $\mathbb{P}^1(\mathbb{C}_p)$-disk containing x and contained in U (i.e., it is the union of all such $\mathbb{P}^1(\mathbb{C}_p)$-disks, and that union is itself a $\mathbb{P}^1(\mathbb{C}_p)$-disk).*

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Given a rational function $\phi \in \mathbb{C}_p(z)$ with Fatou set $\mathcal{F} \subset \mathbb{P}^1(\mathbb{C}_p)$, the image $\phi(U)$ of any D-component U of $\mathcal{F}$ is contained in another D-component of $\mathcal{F}$. In particular, ϕ induces an action Φ on the set of D-components of $\mathcal{F}$, by

$$\Phi(U) = \text{ the D-component containing } \phi(U).$$

Thus, we may classify D-components as periodic, pre-periodic, or wandering under this action.

We note that there are other viable notions of components in the p-adic setting. For instance, we could invoke rigid analysis and define the 'analytic component' of U containing a given point x to be the union of all connected affinoids containing x and contained in U. However, analytic components are always at least as large as D-components, and so No Wandering Domains statements are weaker for analytic components. On the other hand, as shown in [2], all but finitely many iterates of a wandering analytic component must be disks. A similar statement is true for periodic components containing points in a given finite extension of $\mathbb{Q}_p$. Thus, results like the theorems below are equivalent to their analogues for analytic components. We will therefore restrict our attention to D-components, and we leave the subject of analytic components for a future paper. The interested reader may refer to the author's thesis ([2]) for more information on both analytic and D-components.

We are now prepared to state our main theorems.

THEOREM 1.2 (No Wandering Domains). *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ have no recurrent wild critical points in its Julia set. Then the Fatou set of ϕ has no wandering D-components.*

THEOREM 1.3. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ have no recurrent wild critical points in its Julia set. Then the Fatou set of ϕ has only finitely many periodic D-components which contain points of K.*

The result of Theorem 1.3 cannot be extended to infinite extensions like $K = \mathbb{C}_p$. For example, if p is an odd prime, and

$$\phi(z) = \frac{z^3 + (1+p)z^2}{z+1} = z^2 + \frac{pz^2}{z+1},$$

then ϕ has infinitely many periodic D-components (and in fact, infinitely many analytic components), even though all of its critical points are contained in the Fatou set. (See [2] for a detailed analysis of this function.) However, over any given finite extension of $\mathbb{Q}_p$, there are only finitely many periodic D-components, as dictated by the theorem.

Sullivan's proof of the complex No Wandering Domains Theorem (see [23]) is completely general; it uses the theory of quasi-conformal maps to generate too many functions in the moduli space of all rational maps of a given degree. Such a theory is

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not currently available in the p-adic setting; however, it still seems likely that the p-adic No Wandering Domains Theorem should hold in full generality:

CONJECTURE. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a rational function. Then ϕ has no wandering D-components. Furthermore, the Fatou set of ϕ has only finitely many periodic D-components containing points of K.*

While Theorem 1.2 is not as strong as the Conjecture, it is very strong in its own right. Generically, one would expect a critical point to lie in the Fatou set; after all, a map contracts very strongly in a small neighborhood of a critical point, making it more likely to produce equicontinuity. Of course, one can force a critical point to be in the Julia set by mapping it to a repelling periodic point. As an example, for any prime p, the map

$$\phi(z) = \frac{1}{p}(z^3 - z^2) + 1$$

takes the critical point 0 to the repelling fixed point 1; thus, 0 lies in the Julia set. There can even be wandering critical points in the Julia set. If we choose $p = 2$, then the map

$$\phi(z) = \frac{31}{4}(z^3 - z^2) + 1$$

has a critical point at 2/3 which can be shown to be both wandering and Julia (see [2]) in $\mathbb{C}_2$. However, in both of these examples, the critical points are not recurrent.

In fact, it is not currently known whether there exist maps with recurrent wild critical points in the Julia set. Many complex maps have recurrent Julia critical points; for example, any complex rational function with a Siegel disk has such a point. Although the arguments which prove the existence of complex recurrent critical points break down in the p-adic setting, we expect that maps with such points do exist. For instance, with $p = 2$, the map

$$\psi(z) = \frac{1}{2}(z^3 + z^2) - 33$$

may have a recurrent wild Julia critical point at 0. If $v_2(\cdot)$ denotes the 2-adic valuation on $\mathbb{C}_p$ (where $v_2(2) = 1$), then calculations made using PARI/GP show that, for instance:

$$v_2\big(\psi^{39}(0)\big) = 9, \qquad v_2\big(\psi^{2204}(0)\big) = 12,$$

$$v_2\big(\psi^{2836}(0)\big) = 13, \qquad v_2\big(\psi^{24210}(0)\big) = 16.$$

Thus, it seems plausible that 0 could be a recurrent critical point; since its ramification index is 2, that would make it a recurrent wild Julia critical point. However, it is unclear how one might prove such a statement.

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Even if maps with no recurrent wild Julia critical points do not make up all p-adic rational functions, they form a larger set than p-adic hyperbolic maps ([3]), which are p-adic maps without Julia critical points. For example, if $p > 2$, the set of p-adic rational functions of a given degree d without recurrent wild Julia critical points is dense in the moduli space of all p-adic rational functions of degree d. This statement is true for the simple reason that if $p > 2$, wild critical points have high multiplicity, and therefore some small perturbation of the function breaks any wild critical point into several critical points of lower multiplicity. Such a density statement is not currently known for hyperbolic p-adic maps. It is not even known for hyperbolic complex maps. Thus, Theorems 1.2 and 1.3 apply to a very large and generic class of p-adic maps, even if they may fail to be completely general.

2. Mapping Properties

In [3], a weaker version of Theorem 1.2 was proven using mapping properties of p-adic power series away from critical points. In this section we will state and prove certain mapping properties of p-adic power series, both away from and near critical points. Most of the results of this section can be proven using Newton polygons to determine information about roots of power series; for more information on Newton polygons and p-adic functions in general, we refer the reader to [12].

In particular, we will use the following statement, which is easy to verify via Newton polygons. Let $a \in \mathbb{C}_p$, and let $f \in \mathbb{C}_p[[z-a]]$ be a power series convergent on an open (resp., closed) disk V of radius r centered at a. If we write $f(z) = \sum_{i=0}^{\infty} c_i(z-a)^i$, with $c_i \in \mathbb{C}_p$, and if $\max_{i \geqslant 1} |c_i| r^i < \infty$, then the image $f(V)$ is an open (resp., closed) disk of radius $s = \max_{i \geqslant 1} |c_i| r^i$.

PROPOSITION 2.1. *Let $V = \overline{D}_r(a)$ be a closed disk in $\mathbb{C}_p$, and let*

$$f(z) = \sum_{i=0}^{\infty} c_i(z-a)^i, \quad c_i \in \mathbb{C}_p$$

be convergent on V. Then f is one-to-one on V if and only if for all $i > 1$, $|c_i| r^i < |c_1| r$. In this case, $|f'(z)| = |c_1|$ for all $z \in V$, and $\mathrm{rad}(f(V)) = |c_1| r$; furthermore, for any $x, y \in V$,

$$|f(x) - f(y)| = |c_1||x-y|.$$

We omit the proof of Proposition 2.1, which is a straightforward exercise in non-Archimedean power series and Newton polygons.

The reader should be cautioned that a power series may lack critical points on a disk but fail to be one-to-one; this situation is in sharp contrast with the complex setting, where an onto analytic function from a disk to a disk with no critical points is automatically one-to-one. Proposition 2.2 will help us understand the action of p-adic power series on disks near critical points; such disks are often mapped multiply-to-one without containing critical points themselves.

PROPOSITION 2.2. *Let $V = \overline{D}_r(a)$ be a closed disk in $\mathbb{C}_p$, and let*

$$f(z) = c_0 + \sum_{i=d}^{\infty} c_i(z-a)^i, \quad c_i \in \mathbb{C}_p$$

be convergent on V, with $d \geqslant 1$ and $c_d \neq 0$. Suppose that for all $i > d$, $|c_i|r^i < |d!c_d|r^d$. Let $b \in V$, $\sigma = |b-a|$, and $0 < \rho \leqslant \sigma$. Then

$$\operatorname{rad}\big(f\big(\overline{D}_\rho(b)\big)\big) = |dc_d|\sigma^d \max_{e=0,\ldots,v(d)} \left\{ \left|p^{-e}\right| \left(\frac{\rho}{\sigma}\right)^{p^e} \right\}.$$

Proposition 2.2 may be proven by re-centering the power series at b and then carefully computing the absolute values of the resulting coefficients. The condition $|c_i|r^i < |d!c_d|r^d$ is needed to ensure that those absolute values may be computed precisely, using ultrametricity. (Note that this condition may be guaranteed for any given convergent power series by choosing r small enough.) The full proof (which may be found in [2]) is straightforward but somewhat lengthy, and we omit it.

The statement of Proposition 2.2 is perhaps a little too complicated for easy application. The next two corollaries, which follow immediately from the proposition, will prove more useful for our purposes.

COROLLARY 2.3. *Let V, f, d, b, ρ, and σ be as in Proposition 2.2, and suppose p does not divide d. Then*

$$\operatorname{rad}\big(f\big(\overline{D}_\rho(b)\big)\big) = |c_d|\rho\sigma^{d-1}.$$

It should be noted that, for a map satisfying the hypotheses of Corollary 2.3, the radius of the image of the larger disk is

$$\operatorname{rad}\big(f\big(\overline{D}_\sigma(b)\big)\big) = |c_d|\sigma^d,$$

and so the ratio of the radii of the two image disks is the same as the original ratio of radii, ρ/σ. In spite of the fact that the map is not one-to-one, then, the relative sizes of the two disks are not changed by the map.

COROLLARY 2.4. *Let V, f, d, b, ρ, and σ be as in Proposition 2.2, and let $\alpha \in \mathbb{R}$ be the value $\alpha = |p|^{(p-1)^{-1}} < 1$. Suppose $\rho/\sigma \leqslant \alpha$. Then*

$$\operatorname{rad}\big(f\big(\overline{D}_\rho(b)\big)\big) = |dc_d|\rho\sigma^{d-1}.$$

Furthermore, if $\rho/\sigma < \alpha$, then f is one-to-one on $\overline{D}_\rho(b)$.

Corollary 2.4 will be used in much the same way as Corollary 2.3, except that it can be used at points a where Corollary 2.3 may not apply. In this case, the radius of the image of the larger disk is

$$\mathrm{rad}\big(f(\overline{D}_\sigma(b))\big) = |c_d|\sigma^d,$$

and so the ratio of the radii of the two image disks is $|d|(\rho/\sigma)$. While this ratio may be smaller than the original ratio of radii, we have at least some control over it.

3. Main Lemma

The statement and proof of our main lemma will be somewhat technical; we therefore propose the following two definitions for ease of language.

DEFINITION 3.1. Let $\phi \in \mathbb{C}_p(z)$ be a rational function. We say ϕ is *normalized* if ∞ is a non-repelling fixed point of ϕ, and $\phi(\mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)) \subseteq \mathbb{P}^1(\mathbb{C}_p) \setminus \overline{D}_1(0)$.

By [3], any $\phi \in \mathbb{C}_p(z)$ has a nonrepelling fixed point. By a change of coordinates, we can move this point to ∞. Then, by another change of the form $z \mapsto cz$, the second condition of the above definition will also hold. Therefore, any rational function is conjugate to a normalized function; and if the original function was defined over $\overline{\mathbb{Q}}_p$, we can guarantee that the normalized version is as well.

DEFINITION 3.2. Let $\phi \in \mathbb{C}_p(z)$ be a rational map with Fatou set $\mathcal{F}$ and Julia set $\mathcal{J}$, and let Φ denote the action of ϕ on D-components of $\mathcal{F}$. Let $x \in \mathcal{J}$ with $x \neq \infty$. Let $K \subset \mathbb{C}_p$ be a complete extension of $\mathbb{Q}_p$. Given a real number $\epsilon > 0$, we say that x has property $P(\epsilon, K)$ if there exist positive real numbers $M, r > 0$ (which depend on ϕ, K, and ϵ) such that the following condition holds:

For any D-component U of $\mathcal{F}$ with $U \subset \overline{D}_r(x), U \cap K \neq \emptyset$, and

$$\frac{\mathrm{rad(U)}}{\mathrm{dist}(U, x)} \geqslant \epsilon,$$

there is a nonnegative integer k such that $\mathrm{rad}\big(\Phi^k(U)\big) \geqslant M$.

For the purpose of the above definition, we will consider the radius of a D-component containing ∞ to be infinite.

The idea of Definition 3.2 is that if a D-component containing a K-point is large relative to its distance from a Julia point with property P, then some iterate of the D-component is large in a global sense. Thus, if we can prove that points of the Julia set have property P, and if we can prove that there are D-components which are large relative to their distance from such Julia points, then we automatically produce a large D-component. This idea will be crucial in the proof of our main lemma, which we are now prepared to state.

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MAIN LEMMA. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be a normalized rational map with Fatou set $\mathcal{F}$ and Julia set $\mathcal{J}$. Assume that $\mathcal{J}$ contains no recurrent wild critical points of ϕ. Let Φ denote the action of ϕ on the set of D-components of $\mathcal{F}$. Then there exists a positive constant $M > 0$ (depending only on ϕ and K) with the following property:*

If U is a D-component of $\mathcal{F}$ with $U \cap K \neq \emptyset$, then there is some integer $k \geqslant 0$ such that $\mathrm{rad}\big(\Phi^k(U)\big) \geqslant M$.

As in Definition 3.2, we will consider the D-component containing ∞ to have infinite radius.

The proof of the Main Lemma requires a series of technical lemmas. In these lemmas, we prove that successively more points of the Julia set have property P. The proofs of these lemmas are somewhat involved, and the reader may prefer to skip to Section 4 to see their use in proving the Main Lemma.

LEMMA 3.1. *Let K be a finite extension of $\mathbb{Q}_p$, let $\phi \in K(z)$ be a normalized rational function with Julia set $\mathcal{J}$, and let $x \in \mathcal{J}$. Suppose there is some integer $N \geqslant 0$ such that for any $\epsilon > 0$, $\phi^N(x)$ has property $P(\epsilon, K)$. Then for any $\epsilon > 0$, x has property $P(\epsilon, K)$.*

Proof. Expand ϕ^N as a power series

$$\phi^N(z) = c_0 + \sum_{i=d}^{\infty} c_i(z-x)^i$$

centered at x, with $c_d \neq 0$. Pick $s > 0$ so that the series converges on $\overline{D}_s(x)$, and $|c_i|s^i < |d!c_d|s^d$ for any $i > d$. By hypothesis, given $\epsilon > 0$, $\phi^N(x)$ has property $P(|d|\epsilon, K)$. Let r be the radius around $\phi^N(x)$ in Definition 3.2, and let M be the corresponding lower bound. Decrease s if necessary so that $\phi^N(\overline{D}_s(x)) \subseteq \overline{D}_r(\phi^N(x))$.

Let $\mathcal{F}$ be the Fatou set of ϕ, and let $\mathcal{U}_K$ be the set of all D-components of $\mathcal{F}$ containing points of K. By Proposition 2.2, it follows that if $U \in \mathcal{U}_K$ such that $U \subset \overline{D}_s(x)$, then

$$\mathrm{rad}\big(\Phi^N(U)\big) \geqslant \mathrm{rad}\big(\phi^N(U)\big) \geqslant |dc_d|\mathrm{dist}(U,x)^{d-1}\mathrm{rad}(U),$$

and

$$\mathrm{dist}\big(\Phi^N(U), \phi^N(x)\big) = |c_d|\mathrm{dist}(U,x)^d.$$

Therefore, if

$$\frac{\mathrm{rad(U)}}{\mathrm{dist}(U,x)} \geqslant \epsilon,$$

then

$$\frac{\mathrm{rad}(\Phi^N(\mathrm{U}))}{\mathrm{dist}\big(\Phi^N(U), \phi^N(x)\big)} \geqslant |d|\frac{\mathrm{rad(U)}}{\mathrm{dist}(U,x)} = |d|\epsilon.$$

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Since $\phi^N(x)$ has property $P(|d|\epsilon, K)$ with lower bound M, it follows that some iterate of $\Phi^N(U)$ has radius at least M, and we are done. $\square$

Before stating the next lemma, we need the following notation. Given $\phi \in \mathbb{C}_p(z)$ with Julia set $\mathcal{J}$, define

$$C_{\mathcal{J}} = \{y \in \mathcal{J} : \phi'(y) = 0\}$$

to be the set of all Julia critical points, and let $S_0 = \emptyset$ and $T_0 = C_{\mathcal{J}}$. Then, define S_i and T_i inductively for $i \geqslant 1$ by

$$S_i = \{C_{\mathcal{J}} \text{ points not accumulating at any wild } T_{i-1} \text{ points}\},$$

$$T_i = C_{\mathcal{J}} \setminus S_i.$$

LEMMA 3.2. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ be normalized. Let $x \in S_i$ for some $i \geqslant 0$. Then for any $\epsilon > 0$, x has property $P(\epsilon, K)$.*

Proof. We will proceed by induction on i. The statement is vacuous for $i = 0$; for positive i, assume that it is known for $i - 1$, and we will prove it for i.

Pick $x \in S_i$. Pick $N \geqslant 0$ such that there are no critical points in the set $\{\phi^n(x) : n \geqslant N\}$. Such an N must exist; otherwise, since there are only finitely many critical points, some iterate of x would be a periodic critical point and hence Fatou. By Lemma 3.1, it suffices to show that $\phi^N(x)$ has property $P(\epsilon, K)$ for any positive ϵ. Thus, we may assume that x has no critical points in its forward orbit.

Pick $\epsilon > 0$; we can assume that $\epsilon \in |K^*|$ and $\epsilon < 1$. Let $\mathcal{F}$ be the Fatou set and $\mathcal{J}$ the Julia set of ϕ. Let $\alpha = |p|^{(p-1)^{-1}} < 1$. Let $C_t \subset C_{\mathcal{J}}$ denote the set of tame Julia critical points. Extend K if necessary to contain $C_{\mathcal{J}}$, and also so that $\alpha \in |K^*|$. Let π be a uniformizer of K. Note that $x \in K$. Let $\mathcal{U}_K$ denote the set of all D-components of $\mathcal{F}$ which contain points of K.

We will now cover $\mathcal{J} \cap K$ with a finite set of disks. For any $z_0 \in \mathcal{J} \cap K$, there is some $s > 0$ such that $\phi(z)|_{\overline{D}_s(z_0)}$ is of the form $c_0 + \sum_{i=d}^{\infty} c_i(z - z_0)^i$, where $d \geqslant 1$, $c_d \neq 0$, and $|c_i|s^{i-d} < |d!c_d|$ for all $i > d$. Cover $\mathcal{J} \cap K$ by such disks and take a finite subcover. Let R be the minimum radius of the disks in the subcover; we may assume that $R \leqslant 1$. Let W be the union of all closed disks of radius R centered at points of $\mathcal{J} \cap K$.

Pick $r > 0$ such that for any critical point a at which x does not accumulate,

$$\left|\phi^n(x) - a\right| > \frac{r}{\alpha} \tag{1}$$

for all $n \geqslant 0$. (Note that inequality (1) implies that all accumulation points of x must also be at least distance r/α from such critical points.) Decrease r if necessary so that $r < R$. Let M be the minimum of the lower bounds required in the definition of

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property $P(\alpha\epsilon, K)$ for each of the (finitely many) points in S_{i-1}. Decrease M if necessary so that $0 < M \leqslant r\epsilon$.

Pick $U \in \mathcal{U}_K$ with $U \subset \overline{D}_r(x)$ and

$$\frac{\mathrm{rad(U)}}{\mathrm{dist}(U, x)} \geqslant \epsilon.$$

Pick $b \in U \cap K$. Let ρ_0 be the largest value in $|K^*|$ such that $\overline{D}_{\rho_0}(b) \subseteq U$, and let $\sigma_0 = \mathrm{dist}(U, x)$. Note that $\sigma_0 = |b - x| \in |K^*|$. For $k \geqslant 1$, define ρ_k and σ_k inductively, as follows. Given ρ_{k-1}, let

$$\rho_k = \mathrm{rad}\big(\phi\big(\overline{D}_{\rho_{k-1}}\big(\phi^{k-1}(b)\big)\big)\big) \leqslant \mathrm{rad}(\Phi^k(U)).$$

Given σ_{k-1}, if $\mathrm{dist}(\phi^{k-1}(b), C_t) \geqslant \sigma_{k-1}$, let

$$\sigma_k = \mathrm{rad}\big(\phi\big(\overline{D}_{\sigma_{k-1}}\big(\phi^{k-1}(b)\big)\big)\big).$$

Otherwise, if $y \in C_t$ with $|\phi^{k-1}(b) - y| = \mathrm{dist}(\phi^{k-1}(b), C_t) < \sigma_{k-1}$, let

$$\sigma'_{k-1} = |\phi^{k-1}(b) - y|$$

and

$$\sigma_k = \mathrm{rad}\Big(\phi\Big(\overline{D}_{\sigma'_{k-1}}\big(\phi^{k-1}(b)\big)\Big)\Big).$$

Note that for any $k \geqslant 0$, $\overline{D}_{\sigma_k}(\phi^k(b))$ contains a point of $\mathcal{J} \cap K$. This is because $\overline{D}_{\sigma_0}(b)$ contains such a point (namely x), and therefore all of its forward iterates do as well. In addition, when we shrink σ_k to σ'_k, we do so because $\overline{D}_{\sigma'_k}(\phi^k(b))$ contains $y \in \mathcal{J} \cap K$; hence, the new disk and all its iterates contain $\mathcal{J} \cap K$ points.

Let e_k denote the ratio ρ_k/σ_k. By our choice of U, note that $\epsilon \leqslant e_0 < 1$. Also note that $\rho_k, \sigma_k, e_k \in |K^*|$. We will now apply the following claim inductively.

CLAIM 3.1. *If* $\sigma_k \leqslant r$ *and* $S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b)) = \emptyset$, *then*

(1) *if* $\mathrm{dist}(\phi^k(b), C_t) \geqslant \sigma_k$, *then* $e_{k+1} = e_k$.
(2) *otherwise*, $e_{k+1} \geqslant |\pi^{-1}|e_k$.

Assume the claim is true for a moment. $\overline{D}_{\sigma_0}(b)$ intersects the Julia set (at x), so its iterates have arbitrarily large radii. Using the claim repeatedly, we see that at some step k, one of three obstacles arises: $\sigma_k \geqslant r$, or there is some $x' \in S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b))$, or some $y \in C_t$ is close to $\phi^k(b)$. In the first case, $\rho_k \geqslant r\epsilon \geqslant M$ (since $e_k \geqslant \epsilon$), and so $\Phi^k(U)$ has radius at least M, and we are done. In the second case, $\Phi^k(U)$ satisfies

$$\frac{\mathrm{rad}\big(\Phi^k(U)\big)}{\mathrm{dist}\big(\Phi^k(U), x'\big)} \geqslant \alpha\epsilon,$$

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(again, because $e_k \geqslant \epsilon$), and by the inductive hypothesis, some iterate of $\Phi^k(U)$ has radius at least M. In the third case, we note that $e_{k+1} > e_k$, and that $\overline{D}_{\sigma'_k}(\phi^k(b))$ intersects the Julia set (at y); therefore, its iterates must eventually have large radii. Thus, we can start our process again by iterating $\overline{D}_{\sigma'_k}(\phi^k(b))$.

Provided the radii stay smaller than r and the iterates stay away from S_{i-1} points, we can continue this process indefinitely. At each stage, we either produce an iterate of U with radius at least M (and the process stops), or we increase e_k. However, $e_k < 1$, since the disk of radius σ_k contains Julia points, and the disk of radius ρ_k does not. Furthermore, when e_k increases, it increases by a factor of at least $|\pi^{-1}|$; thus, it can only increase a bounded number of times. Thus, at some stage, we must produce an iterate of U with radius at least M. To prove the lemma, then, it suffices to prove the claim.

Fix $k \geqslant 0$, and suppose $\sigma_k \leqslant r$ and $S_{i-1} \cap \overline{D}_{\sigma_k/\alpha}(\phi^k(b)) = \emptyset$. Let

$$V_\sigma = \overline{D}_{\sigma_k}(\phi^k(b)) \quad \text{and} \quad V_\rho = \overline{D}_{\rho_k}(\phi^k(b)).$$

As we saw above, V_σ contains some point z of $\mathcal{J} \cap K$; because $\sigma_k \leqslant r < R$, we have $V_\sigma \subset \overline{D}_R(z)$. By our choice of R, we know that $\overline{D}_R(z)$ contains at most one critical point; and if there is a critical point, it must be in $\mathcal{J} \cap K$.

If there is no critical point in $\overline{D}_R(z)$, then by Proposition 2.1, our choice of R guarantees that ϕ is one-to-one on $\overline{D}_R(z)$ and hence on V_σ; thus, the ratio of radii of $\phi(V_\sigma)$ to $\phi(V_\rho)$ is the same as that of V_σ to V_ρ, and we are done.

If there is a wild critical point $a \in \overline{D}_R(z)$, then it must be outside $\overline{D}_{\sigma_k/\alpha}(\phi^k(b))$. This is because x does not accumulate at any wild points besides those in S_{i-1}; and by our definition of r, the ratio of σ_k to the distance between b and a is less than α. The reader may object that, by our choices of $\{\sigma_j\}$, we cannot assume that some iterate of x lies in $\overline{D}_{\sigma_k/\alpha}(\phi^k(b))$. However, if at some point we decreased σ_i to σ'_i, the resulting disk contained a critical point $y \in C_t$ which was within r of an iterate of x. By our choice of r, y must be an accumulation point of the iterates of x, and therefore some iterate of x must be nearby.

Thus, in the case of a wild critical point $a \in \overline{D}_R(z)$, we can apply Corollary 2.4 to the power series expansion of ϕ about a. We then see that ϕ is one-to-one on V_σ and therefore preserves the ratio of the radii of V_σ and V_ρ. By Corollary 2.3, the ratio is also preserved if there is a tame critical point in $\overline{D}_R(z)$ which is not in V_σ.

The only case that remains to be considered is that V_σ contains a tame critical point y. As before, y must in fact be an accumulation point of x.

If $\operatorname{dist}(\phi^k(b), y) = \sigma_k$, then applying ϕ to V_σ and V_ρ, we see by Corollary 2.3, that

$$e_{k+1} = \frac{\rho_{k+1}}{\sigma_{k+1}} = \frac{\operatorname{rad}\big(\phi\big(\overline{D}_{\rho_k}(\phi^k(b))\big)\big)}{\operatorname{rad}\big(\phi\big(\overline{D}_{\sigma_k}(\phi^k(b))\big)\big)} = \frac{\rho_k}{\sigma_k} = e_k.$$

On the other hand, if $\sigma'_k = \operatorname{dist}(\phi^k(b), y) < \sigma_k$, we can apply Corollary 2.3 to

$\overline{D}_{\sigma'_k}(\phi^k(b))$ to show that

$$e_{k+1} = \frac{\rho_{k+1}}{\sigma_{k+1}} = \frac{\mathrm{rad}\left(\phi\left(\overline{D}_{\rho_k}(\phi^k(b))\right)\right)}{\mathrm{rad}\left(\phi\left(\overline{D}_{\sigma'_k}(\phi^k(b))\right)\right)} = \frac{\rho_k}{\sigma'_k} > \frac{\rho_k}{\sigma_k} = e_k.$$

Furthermore, $e_k, e_{k+1} \in |K^*|$, so if $e_{k+1} > e_k$, then $e_{k+1} \geqslant |\pi^{-1}|e_k$. The proof of the claim is complete, and the lemma follows. □

LEMMA 3.3. *Let K be a finite extension of $\mathbb{Q}_p$, and let $\phi \in K(z)$ have no recurrent wild critical points in its Julia set. Then there exists some $m \geqslant 0$ such that $T_m = \emptyset$.*

Proof. Note that $T_0 = C_{\mathcal{J}}$, and

$$T_{i+1} = \left\{C_{\mathcal{J}} \text{ points accumulating at wild } T_i \text{ points}\right\}.$$

Therefore, we can write

$$T_i = \left\{a_0 \in C_{\mathcal{J}} \,\middle|\, \left\{\begin{array}{l}\exists a_1, \ldots, a_i \in C_{\mathcal{J}} \text{ wild, and} \\ \forall j = 0, \ldots, i, \;\; a_j \text{ accumulates at } a_{j+1}\end{array}\right\}\right\}.$$

Let $m - 1$ be the number of wild Julia critical points. If T_m were nonempty, then there would be wild Julia critical points $a_1, \ldots, a_m$ with a_j accumulating at a_{j+1}. Thus, there must be j and k with $j < k$ and $a_j = a_k$. Thus, a_j accumulates at $a_k = a_j$; it follows that a_j is a recurrent wild critical point in the Julia set, contradicting the hypotheses of the lemma. So $T_m = \emptyset$. □

4. Proofs of Main Lemma and Theorems

We are now prepared to prove our Main Lemma.

Proof of Main Lemma. Let $C_{\mathcal{J}}$ denote the set of Julia critical points, and let $\alpha = |p|^{(p-1)^{-1}}$; extend K to contain $C_{\mathcal{J}}$ and so that $\alpha \in |K|$. Let π be a uniformizer of K. Define the radius R as in the proof of Lemma 3.2. Let $\mathcal{U}_K$ denote the set of all D-components of $\mathcal{F}$ which contain points of K.

By Lemmas 3.2 and 3.3, we know that all Julia critical points have property $P(|\pi|\alpha, K)$. Let M be the minimum of the lower bounds required in Definition 3.2 for each of the (finitely many) Julia critical points, and let R' be the minimum of the corresponding radii. Decrease R if necessary so that $R \leqslant R'$ and $R \in |K^*|$. Let W be the union of all closed disks of radius R centered at points of $\mathcal{J} \cap K$.

We claim that there are only finitely many D-components $U \in \mathcal{U}_K$ not contained in W. For suppose there were infinitely many. Choose a sequence $\{U_i\} \subset \mathcal{U}_K$ and $a_i \in U_i \cap K$. The sequence $\{a_i\}$ must have an accumulation point in $K \cap \overline{D}_1(0)$, since ϕ is normalized and K is locally compact. This accumulation point cannot be Fatou, or else infinitely many of the $\{a_i\}$ would be in its D-component. But it cannot be Julia either, since $U_i \not\subset W$. Our claim follows from the contradiction. We can therefore

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define $M_0 > 0$ to be the minimum radius of the $U \in \mathcal{U}_K$ outside of W. Decrease M if necessary so that $M \leqslant \max\{R, M_0\}$.

CLAIM 4.1. *For any* $U \in \mathcal{U}_K$, *there exists* $k \geqslant 0$ *such that*

(1) $\Phi^k(U) \not\subset W$, *or*
(2) $\mathrm{rad}(\Phi^k(U)) \geqslant R$, *or*
(3) *there is* $y \in C_{\mathcal{J}}$ *with*

$$\frac{\mathrm{rad}\big(\Phi^k(U)\big)}{\mathrm{dist}\big(\Phi^k(U), y\big)} \geqslant |\pi|\alpha.$$

The key observation used in the proof of the claim is that for any disk $V \subset W$ with $\mathrm{rad}(V) < R$ and

$$\frac{\mathrm{rad}(V)}{\mathrm{dist}(V, C_{\mathcal{J}})} < \alpha,$$

ϕ must be one-to-one on V. To see this, pick $a \in V$, and consider the disk $\overline{D}_R(a) = \overline{D}_R(x)$ for some $x \in \mathcal{J} \cap K$. If $\overline{D}_R(x)$ contains no critical points, then by Proposition 2.1 and our choice of R, ϕ is one-to-one on $\overline{D}_R(x)$ and hence on V. On the other hand, if $\overline{D}_R(x)$ does contain critical points, then it contains exactly one, which lies in $\mathcal{J} \cap K$; we can assume that x is this critical point. By Corollary 2.4, ϕ is one-to-one on V, because the radius of V is less than a factor of α times the distance of V to x.

We prove the claim by contradiction. Pick $U \in \mathcal{U}_K$ for which the claim fails. Pick $b \in U \cap K$. Let $r = \mathrm{rad}(U)$, and let $s \in |K^*|$ be the smallest value in $|K^*|$ which is strictly larger than r. By definition of D-components, $\overline{D}_s(b)$ contains Julia points.

Since the claim fails for $k = 0$, we see that $r < R$ and

$$\frac{r}{\mathrm{dist}(U, C_{\mathcal{J}})} < |\pi|\alpha.$$

Because $|\pi|\alpha, R, \mathrm{dist}(U, C_{\mathcal{J}}) \in |K^*|$, it follows that $s \leqslant R$ and

$$\frac{s}{\mathrm{dist}(U, C_{\mathcal{J}})} \leqslant |\pi|\alpha < \alpha.$$

As we saw above, ϕ must be one-to-one on $\overline{D}_s(b)$, and so, by Proposition 2.1,

$$\frac{\mathrm{rad}(\phi(U))}{\mathrm{rad}\big(\phi\big(\overline{D}_s(b)\big)\big)} = \frac{r}{s}.$$

Similarly, by choosing $k = 1$, it follows that

$$\frac{\mathrm{rad}\big(\phi^2(U)\big)}{\mathrm{rad}\big(\phi^2\big(\overline{D}_s(b)\big)\big)} = \frac{r}{s},$$

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and, continuing the process, for any $k \geqslant 0$,

$$\frac{\mathrm{rad}\big(\phi^k(U)\big)}{\mathrm{rad}\big(\phi^k\big(\overline{D}_s(b)\big)\big)} = \frac{r}{s}.$$

In particular, every $\phi^k\big(\overline{D}_s(b)\big)$ has radius at most $R|\pi|^{-1}$ and is therefore contained in $\overline{D}_{|\pi^{-1}|}(0)$. Hsia's Theorem (see [11]) states that if a family of analytic functions from a disk to $\mathbb{P}^1(\mathbb{C}_p)$ omits at least two points of $\mathbb{P}^1(\mathbb{C}_p)$, then the family is equicontinuous. Since the family $\{\phi^n\}$ on $\overline{D}_s(b)$ omits infinitely many points, it is equicontinuous, and so $\overline{D}_s(b)$ is contained in the Fatou set. But we saw before that it contains Julia points. We have a contradiction, and so the claim follows.

The claim tells us that given any U as in the statement of the Main Lemma, some iterate $\Phi^k(U)$ either has radius at least $\max\{R, M_0\}$, or there is $y \in C_{\mathcal{J}}$ with

$$\frac{\mathrm{rad}\big(\Phi^k(U)\big)}{\mathrm{dist}\big(\Phi^k(U), y\big)} \geqslant |\pi|\alpha. \tag{2}$$

In the former case, we have an iterate of radius at least M, as desired. In the latter case, because y has property $P(|\pi|\alpha, K)$ with lower bound M, we know that some later iterate of U has radius M. Either way, the proof is complete.

Our theorems now follow relatively easily from the Main Lemma.

Proof of Theorem 1.2. Given $\phi \in K(z)$ with no recurrent wild critical points in its Julia set, we can assume that ϕ is normalized. We do so by conjugating the original ϕ by some element of $\mathrm{PGL}(2, \overline{\mathbb{Q}}_p)$; the resulting normalized function is defined over a finite extension of K, so we replace K by this finite extension.

Suppose U is a wandering D-component. Then U must contain some point $b \in \overline{\mathbb{Q}}_p$. Extend K to contain b; thus, U and all its iterates contain points of K.

Select $M > 0$ according to the Main Lemma. Then there must be some iterate $\Phi^{k_0}(U)$ of radius at least M. Applying the Main Lemma to $\Phi^{k_0+1}(U)$, there is some further iterate $\Phi^{k_0+k_1}(U)$ of radius at least M. We can continue this process to produce an infinite sequence of iterates of U, all of radius at least M, all containing points of K, and, because U is wandering, all distinct. Since they are all full D-components, they cannot even intersect.

However, none of the iterates of U can be the D-component at ∞ (which is fixed), and therefore they are all contained in $\overline{D}_1(0)$. Thus, we have infinitely many non-intersecting disks of radius $M > 0$ centered at points of $K \cap \overline{D}_1(0)$. Because K is locally compact, this is impossible; we have the desired contradiction. □

The proof of Theorem 1.3 is similar, and we omit it.

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