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COHOMOLOGICAL CHARACTERIZATIONS OF CHARACTER PSEUDO-AMENABLE BANACH ALGEBRAS

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Abstract

For a Banach algebra $\mathcal{A}$ and a character ϕ on $\mathcal{A}$, we have recently introduced and studied the notion of ϕ-pseudo-amenability of $\mathcal{A}$. Here, we give some characterizations of this notion in terms of derivations from $\mathcal{A}$ into various Banach $\mathcal{A}$-bimodules.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$, the set of all nonzero homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$. $\mathcal{A}$ is called ϕ-amenable if there exists a bounded linear functional $F \in \mathcal{A}^{**}$ satisfying $F(\phi) = 1$ and $a \odot F = \phi(a)F$ for all $a \in \mathcal{A}$, where $\odot$ is the first Arens multiplication on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ defined by the equations

$$(F \odot H)(f) = F(Hf), \quad (Hf)(a) = H(fa), \quad (fa)(b) = f(ab)$$

for all $F, H \in \mathcal{A}^{**}$, $f \in \mathcal{A}^*$, and $a, b \in \mathcal{A}$. This notion is introduced and studied by Kaniuth *et al.* [7]; see also [8]. The notion of character amenability is a generalization of left amenability of the class of F-algebras $\mathcal{L}$ studied in Lau [9] in 1983, known as Lau algebras; see Pier [12]. The class of Lau algebras includes the group algebra $L^1(G)$, and the measure algebra $M(G)$ of an amenable locally compact group G, as well as the Fourier and Fourier–Stieltjes algebras $A(G)$ and $B(G)$ of any G. It also includes the quantum group algebra $L^1(Q)$ when Q is amenable. In this case, the character is taken to be the identity of the von Neumann algebra $\mathcal{L}^*$.

For a Banach $\mathcal{A}$-bimodule X, a derivation $D : \mathcal{A} \to X$ is a linear map such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

For $x \in X$, define $\mathrm{ad}_x : \mathcal{A} \to X$ by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. Then ad_x is a derivation; these are the inner derivations. Note that the dual space X^* of a Banach

$\mathcal{A}$-bimodule X becomes a Banach $\mathcal{A}$-bimodule with

$$\langle a \cdot \xi, x \rangle = \langle \xi, x \cdot a \rangle, \quad \langle \xi \cdot a, x \rangle = \langle \xi, a \cdot x \rangle,$$

for all $x \in X$, $\xi \in X^*$, and $a \in \mathcal{A}$. Following Hu *et al.* [6], for $\phi \in \sigma(\mathcal{A}) \cup \{0\}$, we denote by ${}_\phi\mathcal{M}^{\mathcal{A}}$ the class of Banach $\mathcal{A}$-bimodules X for which the left module action of $\mathcal{A}$ on X is defined by

$$a \cdot x = \phi(a)x \quad (a \in \mathcal{A}, x \in X).$$

The notion of ϕ-amenability is characterized in several different ways; for example, it is equivalent to the notion that for each $X \in {}_\phi\mathcal{M}^{\mathcal{A}}$, every continuous derivation $D : \mathcal{A} \to X^*$ is inner [7]; moreover, it was shown in [6] that ϕ-amenability is equivalent to the existence of a bounded (right) ϕ-approximate diagonal; that is, a bounded net $(\mathbf{m}_\alpha)$ in the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

$$\phi(\pi(\mathbf{m}_\alpha)) \to 1 \quad \text{and} \quad \|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| \to 0$$

for all $a \in \mathcal{A}$, where π denotes the product morphism from $\mathcal{A} \widehat{\otimes} \mathcal{A}$ into $\mathcal{A}$ given by $\pi(a \otimes b) = ab$ for all $a, b \in \mathcal{A}$; see also [1].

Moreover, Ghahramani and Zhang [5] introduced and studied the notion of pseudo-amenability of $\mathcal{A}$, the existence of an approximate diagonal for $\mathcal{A}$; that is, a net $(\mathbf{m}_\alpha)$ in $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

$$\|\pi(\mathbf{m}_\alpha)a - a\| \to 0 \quad \text{and} \quad \|a \cdot \mathbf{m}_\alpha - \mathbf{m}_\alpha \cdot a\| \to 0$$

for all $a \in \mathcal{A}$.

Recently, in [11] we introduced and studied a notion of amenability called ϕ-pseudo-amenability. In this paper we characterize ϕ-pseudo-amenability of $\mathcal{A}$ in terms of derivations from $\mathcal{A}$ into certain Banach $\mathcal{A}$-bimodules.

2. The results

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \to X$ is approximately inner if there is a net $(x_\alpha) \subseteq X$ such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A});$$

see, for example, [2, 4]. The notion of ϕ-approximate diagonal was introduced and studied by Hu *et al.* [6]. We commence this section with the main definition of the paper.

DEFINITION 2.1. Let $\mathcal{A}$ be a Banach algebra and let $\phi \in \sigma(\mathcal{A})$. We say that $\mathcal{A}$ is *ϕ-pseudo-amenable* if it has a (right) ϕ-approximate diagonal (not necessarily bounded).

In our first result, we characterize ϕ-pseudo-amenability of $\mathcal{A}$ in terms of derivations from $\mathcal{A}$ into duals of elements in ${}_\phi\mathcal{M}^{\mathcal{A}}$.

THEOREM 2.2. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. Then the following statements are equivalent.*

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(a) *$\mathcal{A}$ is ϕ-pseudo-amenable.*

(b) *For each $X \in {}_\phi\mathcal{M}^{\mathcal{A}}$, any continuous derivation $D : \mathcal{A} \to X^*$ is approximately inner.*

PROOF. (a) $\Rightarrow$ (b). Suppose that $X \in {}_\phi\mathcal{M}^{\mathcal{A}}$ and $D : \mathcal{A} \to X^*$ is a continuous derivation. Let $(\mathbf{m}_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ be a ϕ-approximate diagonal for $\mathcal{A}$ and let $\Phi : \mathcal{A} \widehat{\otimes} \mathcal{A} \to X^*$ be the bounded linear mapping specified by

$$\Phi(a \otimes b) = D(a)\phi(b)$$

for all $a, b \in \mathcal{A}$. Then $\|\Phi\| \leq \|D\|$ and

$$\Phi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha) = a \cdot \Phi(\mathbf{m}_\alpha) + \phi(\pi(\mathbf{m}_\alpha))D(a) - \phi(a)\Phi(\mathbf{m}_\alpha)$$

for all $a \in \mathcal{A}$. Let $\xi_\alpha := -\Phi(\mathbf{m}_\alpha)$. Then for each $a \in \mathcal{A}$ we have

$$\phi(\pi(\mathbf{m}_\alpha))D(a) = a \cdot \xi_\alpha - \phi(a)\xi_\alpha + \Phi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha).$$

On the other hand,

$$\|\Phi(a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha) \leq \|D\| \|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. It follows that

$$D(a) = \lim_\alpha (a \cdot \xi_\alpha - \xi_\alpha \cdot a) \quad (a \in \mathcal{A}).$$

(b) $\Rightarrow$ (a). Suppose that for each $X \in {}_\phi\mathcal{M}^{\mathcal{A}}$, every continuous derivation $D : \mathcal{A} \to X^*$ is approximately inner. Consider the quotient Banach $\mathcal{A}$-bimodule $X = \mathcal{A}^*/\mathbb{C}\phi$. Let $F_0 \in \mathcal{A}^{**}$ be such that $F_0(\phi) = 1$ and let $\mathrm{ad}_{F_0} : \mathcal{A} \to \mathcal{A}^{**}$ be the inner derivation by F_0. Then the image of ad_{F_0} is a subset of X^*, and hence by our assumption, there exists a net

$$(F'_\alpha) \subseteq X^* = \{F \in \mathcal{A}^{**} : F(\phi) = 0\}$$

such that

$$\mathrm{ad}_{F_0}(a) = \lim_\alpha \mathrm{ad}_{F'_\alpha}(a)$$

for all $a \in \mathcal{A}$. Thus if we define $F_\alpha = F_0 - F'_\alpha$, then

$$F_\alpha(\phi) = (F_0)(\phi) = 1 \quad \text{and} \quad \|a \odot F_\alpha - \phi(a)F_\alpha\| \to 0$$

for all $a \in \mathcal{A}$. We can use Goldstine's theorem to assume that (F_α) is in $\mathcal{A}$ when the convergence in the above equation is the weak convergence. Applying Mazur's theorem, we can get a new net $(a_\alpha) \subseteq \mathcal{A}$ such that

$$\phi(a_\alpha) = 1 \quad \text{and} \quad \|aa_\alpha - \phi(a)a_\alpha\| \to 0.$$

Now, choose $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$ and put

$$\mathbf{m}_\alpha := a_\alpha \otimes a_0.$$

It is easy to check that $(\mathbf{m}_\alpha)$ is a ϕ-approximate diagonal for $\mathcal{A}$. Thus $\mathcal{A}$ is ϕ-pseudo-amenable. □

DEFINITION 2.3. Let $\mathcal{A}$ be a Banach algebra. We say that $\mathcal{A}$ is 0-*pseudo-amenable* if it has a right approximate identity.

THEOREM 2.4. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is 0-pseudo-amenable if and only if every continuous derivation $D : \mathcal{A} \to X^*$ is approximately inner for all $X \in {}_0\mathcal{M}^{\mathcal{A}}$.*

PROOF. The 'if' part follows from [4, Proof of Lemma 2.2]. To prove the converse, suppose that $(a_\alpha) \subseteq \mathcal{A}$ is a right approximate identity for $\mathcal{A}$ and suppose that $D : \mathcal{A} \to X^*$ is a continuous derivation, where $X \in {}_0\mathcal{M}^{\mathcal{A}}$. We set

$$\xi_\alpha := D(a_\alpha)$$

for all α. Therefore for each $a \in \mathcal{A}$ we have

$$\begin{aligned} D(a) &= \lim_\alpha D(aa_\alpha) = \lim_\alpha D(a) \cdot a_\alpha + a \cdot D(a_\alpha) \\ &= \lim_\alpha a \cdot D(a_\alpha) = \lim_\alpha a \cdot \xi_\alpha \\ &= \lim_\alpha a \cdot \xi_\alpha - \xi_\alpha \cdot a. \end{aligned}$$

Thus D is approximately inner. □

A Banach algebra $\mathcal{A}$ is approximately amenable if every continuous derivation from $\mathcal{A}$ into the dual $\mathcal{A}$-bimodule X^* is approximately inner for all Banach $\mathcal{A}$-bimodules X; this notion was introduced and studied by Ghahramani and Loy [4]. Moreover, the notion of character amenability was introduced and studied by Monfared [10]; he called a Banach algebra $\mathcal{A}$ character amenable if it has a bounded right approximate identity and is ϕ-amenable for all nonzero characters ϕ on $\mathcal{A}$. We are thus led to the following definition.

DEFINITION 2.5. A Banach algebra $\mathcal{A}$ is called *character pseudo-amenable* if it is ϕ-pseudo-amenable for all $\phi \in \sigma(\mathcal{A}) \cup \{0\}$.

As an application of Theorems 2.2 and 2.4 we have the following result.

COROLLARY 2.6. *Let $\mathcal{A}$ be an approximately amenable Banach algebra. Then $\mathcal{A}$ is character pseudo-amenable.*

The following example shows that the converse of the above corollary is not true.

EXAMPLE 2.7. (a) Let $1 \leq p < \infty$ and let S be an infinite set. Consider the Banach algebra $\ell^p(S)$ of all complex-valued functions $f := \sum_{s \in S} f(s)\delta_s$ with

$$\|f\|^p := \sum_{s \in S} |f(s)|^p < \infty$$

endowed with the pointwise multiplication, where δ_s is the characteristic function of $\{s\}$. First note that

$$\sigma(\ell^p(S)) = \{\phi_s : s \in S\},$$

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where $\phi_s(f) = f(s)$ for all $f \in \ell^p(S)$. Now, suppose that $X \in {}_{\phi_s}\mathcal{M}^{\ell^p(S)}$ and $D : \ell^p(S) \to X^*$ is a continuous derivation. Then

$$D(f\delta_s) = f \cdot D(\delta_s) + \phi_s(\delta_s)D(f)$$

for all $f \in \ell^p(S)$ and $s \in S$. Since $f\delta_s = f(s)\delta_s = \phi_s(f)\delta_s$ and $\phi_s(\delta_s) = 1$, it follows that

$$D(f) = \phi_s(f)D(\delta_s) - f \cdot D(\delta_s).$$

Thus $D = \mathrm{ad}_{-D(\delta_s)}$ and hence $\ell^p(S)$ is ϕ_s-pseudo-amenable for all $s \in S$ by Theorem 2.2. Moreover, $\ell^p(S)$ has an approximate identity, and consequently it is character pseudo-amenable. On the other hand, it was shown in [3] that the Banach algebra $\ell^p(S)$ is not approximately amenable.

(b) Let G be an infinite compact abelian group. Then since G is compact and abelian, the Feichtinger algebra is

$$S_0(G) = \left\{ f = \sum_{\rho \in \widehat{G}} c_\rho \delta_\rho : \|f\| = \sum |c_\rho| < \infty \right\},$$

where $\widehat{G}$ is the dual group of G. Hence,

$$S_0(G) \cong \ell^1(\widehat{G}),$$

where $\ell^1(\widehat{G})$ is equipped with the pointwise product. Thus $S_0(G)$ is character pseudo-amenable, but not approximately amenable.

Let $\mathcal{M}_\phi^{\mathcal{A}}$ denote the class of Banach $\mathcal{A}$-bimodules X for which the right module action of $\mathcal{A}$ on X is defined by $x \cdot a = \phi(a)x$ $(a \in \mathcal{A}, x \in X)$. We have the following analogue of a result in [6] on ϕ-amenable Banach algebras.

PROPOSITION 2.8. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then the following statements are equivalent.*

(a) *$\mathcal{A}$ is ϕ-pseudo-amenable.*

(b) *For every $X \in \mathcal{M}_\phi^{\mathcal{A}}$, any continuous derivation $D : \mathcal{A} \to X^{**}$ is approximately inner.*

(c) *For every $X \in \mathcal{M}_\phi^{\mathcal{A}}$, any continuous derivation $D : \mathcal{A} \to X$ is approximately inner.*

PROOF. Clearly, (a) implies (b) and (c) implies (a) by Theorem 2.2. Thus it suffices to show that (b) implies (c). Suppose that $X \in \mathcal{M}_\phi^{\mathcal{A}}$ and $D : \mathcal{A} \to X$ is a continuous derivation. Then the map $\Delta : D^{**}|_{\mathcal{A}} \to X^{**}$ is a derivation. By assumption, there exists a net $(\Xi_\alpha) \subseteq X^{**}$ such that

$$\Delta(a) = D(a) = \lim_\alpha (a \cdot \Xi_\alpha - \Xi_\alpha \cdot a).$$

Then by Goldstine's theorem, we can assume that $\Xi_\alpha \in X$ when the above limit converges in the weak topology of X. Applying Mazur's theorem, we can get a new

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net $(x_\alpha) \subseteq X$ such that

$$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a)$$

for all $a \in \mathcal{A}$, and this completes the proof. $\square$

COROLLARY 2.9. *Let $\mathcal{A}$ be a finite-dimensional Banach algebra and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then $\mathcal{A}$ is ϕ-pseudo-amenable if and only if $\mathcal{A}$ is ϕ-amenable.*

PROOF. Suppose that $D : \mathcal{A} \to X^*$ is a continuous derivation for some $X \in {}_\phi\mathcal{M}^{\mathcal{A}}$. Let E be the subspace of X^* generated by $D(\mathcal{A}) + \mathcal{A} \cdot D(\mathcal{A})$. Thus E is a finite-dimensional submodule of X^* and $E \in \mathcal{M}_\phi^{\mathcal{A}}$. Using the fact that

$$\mathcal{B}^1(\mathcal{A}, E) = \{\mathrm{ad}_\xi : \xi \in E\}$$

is a finite-dimensional space, we conclude that it is a closed subspace of $\mathcal{L}(\mathcal{A}, E)$, the set of bounded linear mappings from $\mathcal{A}$ into E, in the strong operator topology. Define $D_E : \mathcal{A} \to E$ by $D_E(a) = D(a)$ for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is ϕ-pseudo-amenable and $E \in \mathcal{M}_\phi^{\mathcal{A}}$, there exists a net (ξ_α) in E such that for each $a \in \mathcal{A}$,

$$D_E(a) = \lim_\alpha \mathrm{ad}_{\xi_\alpha}(a)$$

by Proposition 2.8. It follows that $D_E \in \mathcal{B}^1(\mathcal{A}, E)$, and hence there exists $\xi \in E \subseteq X^*$ such that $D_E = \mathrm{ad}_\xi$. Therefore D is inner. $\square$

Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then we can consider $\ker(\phi) \in \mathcal{M}_\phi^{\mathcal{A}}$ with the right action

$$b \cdot a = \phi(a)b \quad (a \in \mathcal{A}, b \in \ker(\phi)),$$

and the left action to be the natural one. In the following result we consider $\ker(\phi)$ as an $\mathcal{A}$-bimodule with these actions.

COROLLARY 2.10. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then the following statements are equivalent.*

(a) *$\mathcal{A}$ is ϕ-pseudo-amenable.*
(b) *Any continuous derivation $D : \mathcal{A} \to (\ker(\phi))^{**}$ is approximately inner.*
(c) *Any continuous derivation $D : \mathcal{A} \to \ker(\phi)$ is approximately inner.*

PROOF. Clearly, (a) implies (b) and (b) implies (c) by Proposition 2.8. It suffices to show that (c) implies (a). To that end first suppose that $\phi \in \sigma(\mathcal{A})$ and choose any $b \in \mathcal{A}$ with $\phi(b) = 1$. Then

$$D(a) = ab - \phi(a)b \quad (a \in \mathcal{A})$$

defines a continuous derivation from $\mathcal{A}$ into $\ker(\phi)$. Thus D is approximately inner by (c), and so there is a net $(b_\alpha) \subseteq \ker(\phi)$ such that

$$D(a) = \lim_\alpha (ab_\alpha - \phi(a)b_\alpha)$$

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for all $a \in \mathcal{A}$. If we set $a_\alpha := b - b_\alpha$ for all α, then we have $\phi(a_\alpha) = 1$ and for each $a \in \mathcal{A}$,

$$aa_\alpha - \phi(a)a_\alpha \to 0.$$

Trivially,

$$\mathbf{m}_\alpha := a_\alpha \otimes a_0$$

is a ϕ-approximate diagonal for $\mathcal{A}$, where $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$. Therefore $\mathcal{A}$ is ϕ-pseudo-amenable. For the case $\phi = 0$, the proof is similar to the proof of Theorem 2.4. □

Let $\Theta : \mathcal{A} \to \mathcal{B}$ be a Banach algebra homomorphism and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule by the module actions

$$a \cdot b = \Theta(a)b, \quad b \cdot a = \phi(a)b \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

We denote by $\mathcal{B}_\phi^\Theta$ the above $\mathcal{A}$-bimodule in $\mathcal{M}_\phi^{\mathcal{A}}$.

THEOREM 2.11. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A}) \cup \{0\}$. Then the following statements are equivalent.*

(a) *$\mathcal{A}$ is ϕ-pseudo-amenable.*
(b) *For every Banach algebra $\mathcal{B}$ and every homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, any continuous derivation $D : \mathcal{A} \to \mathcal{B}_\phi^\Theta$ is approximately inner.*
(c) *For every Banach algebra $\mathcal{B}$ and every injective homomorphism $\Theta : \mathcal{A} \to \mathcal{B}$, any continuous derivation $D : \mathcal{A} \to \mathcal{B}_\phi^\Theta$ is approximately inner.*

PROOF. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. We show that (c) $\Rightarrow$ (a) holds. Suppose that (c) holds and let $X \in \mathcal{M}_\phi^{\mathcal{A}}$ and $D : \mathcal{A} \to X$ be a continuous derivation. Consider the module extension Banach algebra $X \oplus_1 \mathcal{A}$; that is, the space $X \oplus \mathcal{A}$ endowed with the norm

$$\|(x, a)\| = \|x\| + \|a\| \quad (a \in \mathcal{A}, x \in X)$$

and the product

$$(x_1, a_1)(x_2, a_2) = (x_1 \cdot a_2 + a_1 \cdot x_2, a_1a_2)$$

for all $a_1, a_2 \in \mathcal{A}$ and $x_1, x_2 \in X$. Obviously the map $\Theta : \mathcal{A} \to X \oplus_1 \mathcal{A}$ defined by

$$\Theta(a) = (0, a) \quad (a \in \mathcal{A})$$

is an injective Banach algebra homomorphism. Now, if we define $D_1 : \mathcal{A} \to X \oplus_1 \mathcal{A}$ by

$$D_1(a) = (D(a), 0) \quad (a \in \mathcal{A}),$$

then for each $a, b \in \mathcal{A}$ we have

$$\begin{aligned} D_1(ab) = (D(ab), 0) &= (\phi(b)D(a) + a \cdot D(b), 0) \\ &= \phi(b)(D(a), 0) + (0, a)(D(b), 0) \\ &= \phi(b)D_1(a) + \Theta(a)D_1(b). \end{aligned}$$

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Thus, D_1 is a derivation from $\mathcal{A}$ into $(X \oplus_1 \mathcal{A})_\phi^\Theta$, and so D_1 is approximately inner by assumption. That is, there exist nets $(a_\alpha) \subseteq \mathcal{A}$ and $(x_\alpha) \subseteq X$ such that

$$D_1(a) = \lim_\alpha \mathrm{ad}_{(x_\alpha, a_\alpha)}(a)$$

for all $a \in \mathcal{A}$. Thus, for each $a \in \mathcal{A}$ we have

$$\begin{aligned}(D(a), 0) = D_1(a) &= \lim_\alpha \mathrm{ad}_{(x_\alpha, a_\alpha)}(a)\\ &= \lim_\alpha(\Theta(a)(x_\alpha, a_\alpha) - \phi(a)(x_\alpha, a_\alpha))\\ &= \lim_\alpha((0, a)(x_\alpha, a_\alpha) - \phi(a)(x_\alpha, a_\alpha))\\ &= \lim_\alpha(a x_\alpha - \phi(a)x_\alpha, a a_\alpha - \phi(a)a_\alpha).\end{aligned}$$

Therefore, $D(a) = \lim_\alpha \mathrm{ad}_{x_\alpha}(a)$. So, for each $X \in \mathcal{M}_\phi^{\mathcal{A}}$, any continuous derivation $D : \mathcal{A} \to X$ is approximately inner; this is equivalent to ϕ-pseudo-amenability by Proposition 2.8. □

We end this work with the following description of ϕ-amenability which is of interest in its own right.

THEOREM 2.12. *Let $\mathcal{A}$ be a Banach algebra and $\phi \in \sigma(\mathcal{A})$. Then $\mathcal{A}$ is ϕ-pseudo-amenable if and only if there exists a net $(\mathbf{n}_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $\phi(\pi(\mathbf{n}_\alpha)) \to 1$ and $\|a \cdot \mathbf{n}_\alpha\| \to 0$ for all $a \in \ker(\phi)$.*

PROOF. The 'only if' part follows from the definition of ϕ-pseudo-amenability and ϕ-approximate diagonal. Now, assume that there exists a net $(\mathbf{n}_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ such that

$$\phi(\pi(\mathbf{n}_\alpha)) \to 1 \quad \text{and} \quad \|a \cdot \mathbf{n}_\alpha\| \to 0$$

for all $a \in \ker(\phi)$. Choose $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$. Then $aa_0 - \phi(a)a_0 \in \ker(\phi)$ for all $a \in \mathcal{A}$. Set

$$\mathbf{m}_\alpha := a_0 \cdot \mathbf{n}_\alpha$$

for all α. Thus $\phi(\pi(\mathbf{m}_\alpha)) = \phi(a_0\pi(\mathbf{n}_\alpha)) \to 1$ and, for each $a \in \mathcal{A}$,

$$\|a \cdot \mathbf{m}_\alpha - \phi(a)\mathbf{m}_\alpha\| = \|(aa_0 - \phi(a)a_0) \cdot \mathbf{n}_\alpha\| \to 0.$$

This shows that $(\mathbf{m}_\alpha)$ is a ϕ-approximate diagonal for $\mathcal{A}$. □

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