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ON THE STABLE EQUIVALENCE OF PLAT REPRESENTATIONS OF KNOTS AND LINKS

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1. We are interested in the question of the decideability of the classical knot problem. A *knot* is the embedded image of a circle S^1 in Euclidean 3-space E^3. If L_1, L_2 are knots, then $L_1 \approx L_2$ if there is an orientation-preserving homeomorphism h: $E^3 \rightarrow E^3$ with $h(L_1) = L_2$. By the "knot problem" we mean: given two arbitrary tame knots L_1, L_2 (a knot is tame if it is equivalent to a polygonal knot), decide in a finite number of steps whether $L_1 \approx L_2$. The object of this paper is to show that the knot problem is "stably equivalent" to a problem of deciding membership in the double cosets of a distinguished subgroup K_{2n} of the classical braid group B_{2n} [**1**]. By stably equivalent we mean: In each group B_{2j} we will define a particular elementary braid σ_{2j-2}. We will show that each knot L_i may be associated with a (non-unique) element $\Phi_i \in B_{2n_i}$, for some integer n_i. Our theorem asserts that $L_1 \approx L_2$ if and only if there exists an integer $t \geqq \max(n_1, n_2)$ such that for each $n \geqq t$ the braids $\Phi_i\sigma_{2n_i}\sigma_{2n_i+2} \ldots \sigma_{2n-2} \in B_{2n}$, $i = 1, 2$, are in the same double coset modulo K_{2n}. The question of whether or not there is an effective algorithm to decide if $L_1 \approx L_2$ is thus reduced to two other questions:

(1) Can a bound be placed on the integer t?

(2) Is the double coset problem in B_{2n} modulo K_{2n} decideable?

The smallest possible value of t is the bridge index of L_1. An interesting question is whether it is adequate to choose t to be the bridge index of L_1. If this were true, then the answer to (1) would be affirmative, and the knot problem would be reduced to question (2). For composite knots, we show that choosing t to be the bridge index of L_1 is not adequate (see Section 5). For prime knots the question was open at the time that this paper was written, but new examples of Montesinos now show that even for prime knots some stabilizing is necessary.

With regard to question (2), very little is known, however it is worth noting that the group B_{2n} has been studied extensively in the literature, and that various other related problems in B_{2n} have been solved. For example, the conjugacy problem in B_{2n} has been solved [**10**]; the word problem in B_{2n} has been solved in at least 4 distinct ways [**1**; **2**; **10**; **4**]; the group B_{2n} is residually finite [**3**]; has a (possibly faithful) matrix representation over the rationals (defined by setting the indeterminate in the "Burau representation" [**8**] equal

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to 1/2); and a faithful representation as a subgroup of the automorphism group of a free group. One can, moreover, decide whether an arbitrary element in B_{2n} belongs to the subgroup K_{2n}.

Our main theorem is strikingly similar to a classical result of Singer [**15**] and Reidemeister [**14**], which implies that the homeomorphism problem for closed, orientable 3-manifolds is stably equivalent to the problem of deciding membership in the double cosets of a distinguished subgroup $\mathfrak{N}_g$ of the mapping class group $\mathfrak{M}_g$ of a closed orientable surface of genus g [**5**]. Our algebraic knot problem seems, however, to be easier than this algebraic homeomorphism problem for 3-manifolds, because the group B_{2n} is a much more tractible group than $\mathfrak{M}_g$. On the other hand, our main result is also similar to a theorem of Markov [**13**; **4**] which shows that the knot problem is equivalent to a question about the relationship between the solutions to the conjugacy problem in the groups B_m ($m = 2, 3, 4, \ldots$). The main advantage of the present result over the Markov theorem is that we need only consider the double coset problem in a single group B_{2n}, whereas the Markov approach seems to necessitate the consideration of a chain of conjugacy problems (See [**4**], Markov's Theorem asserts that if two *closed braids* $\beta, \beta' \in B_m$ represent the same knot type, then there is a finite sequence of closed braids $\beta = \beta_1 \rightarrow \ldots \rightarrow \beta_r = \beta'$ joining β to β' such that β_j is either in the same braid group as β_{j-1} and conjugate to β_{j-1}, or can be obtained from β_{j-1} by altering string index in a canonical manner. It is not known whether the operations of altering string index and conjugation in B_m commute. (c.f. our Lemma 10 below: double coset multiplication and altering string index *do* commute.)) However, the conjugacy problem in B_m has been solved, while the double coset problem is open.

The plan of this paper is as follows. In Section 2 we define the groups B_{2n} and K_{2n}, and develop some properties of links which will be needed later. Our main result is Theorem 1, which will be stated and proved in Section 3. In Section 4 we discuss appropriate generalizations of Theorem 1 to links. In Section 5 we discuss the question of placing a bound on the integer t.

2. The groups B_{2n} and K_{2n}. Let E_+^3 denote the subset of Euclidean 3-space E^3 defined by $\{(x, y, z)/z \geqq 0\}$, and let $A = A_1 \cup \ldots \cup A_n$ denote a set of n unknotted and unlinked arcs properly imbedded in E_+^3. To make this choice explicit (Figure 1) we will take A_i, $1 \leqq i \leqq n$, to be the union of the line segments $[(i - .25, 0, 0), (i - .25, 0, 1)]$, $[(i + .25, 0, 0), (i + .25, 0, 1)]$ and the half circles $\{(x, 0, z) | (x - i)^2 + (z - 1)^2 = (.25)^2, z \geqq 1\}$. The points $\partial A \subset \partial E_+^3$ are then the $2n$ points $(i \pm .25, 0, 0)$, $1 \leqq i \leqq n$. The group of isotopy classes $\Phi = [\varphi]$ of orientation-preserving homeomorphisms $\varphi: (\partial E_+^3, \partial A) \rightarrow (\partial E_+^3, \partial A)$ is the classical braid group B_{2n}.

Let E_-^3 denote the lower half-space $z \leqq 0$, and let $A' = A_1' \cup \ldots \cup A_n'$ denote the half circles $\{(x, 0, z) | (x - i)^2 + z^2 = (.25)^2, z \leqq 0\}$. Let $\rho: E_+^3 \rightarrow E_-^3$, with $\rho(x, y, z) = (x, y, -z)$. Then the identification-space $E_+^3 \cup_{\rho\varphi} E_-^3$,

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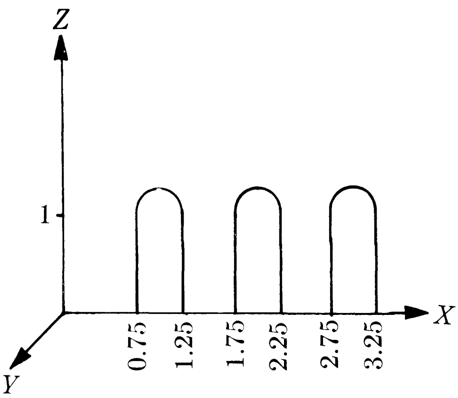
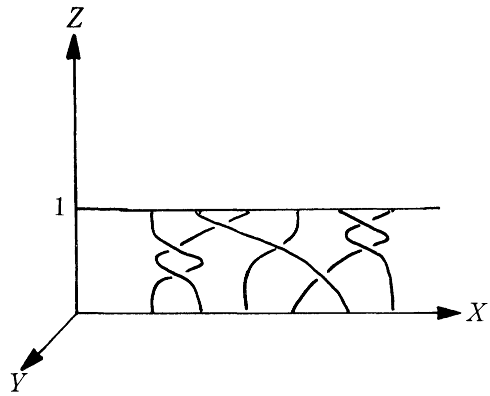
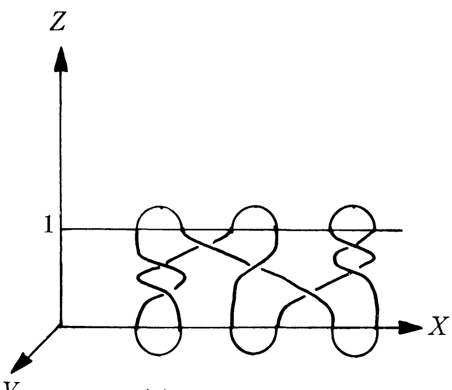


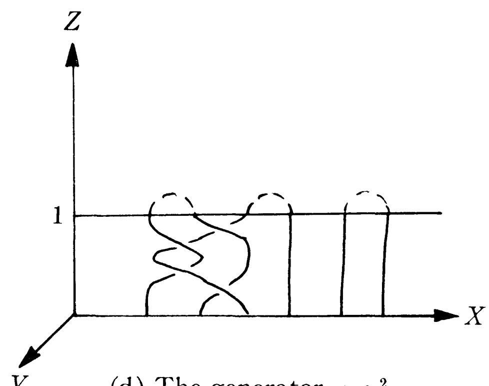
(a) $(E_+{}^3, A)$, $n = 3$.



(b) Geometric braid in the group B_6.



(c) 6-plat defining the geometric braid in Figure (b).



(d) The generator $\sigma_2\sigma_1{}^2\sigma_2$ of K_6 as a geometric braid.

FIGURE 1

defined by the rule $\rho\varphi(x, y, 0) = (x, y, 0)$ for each $(x, y, 0) \in \partial E_+{}^3$, is homeomorphic to E^3 and the subset $L = A \cup_{\rho\varphi} A'$ is a 1-manifold, i.e. a link in E^3. It is immediate that the link type of L depends only on the isotopy class Φ of φ. Thus we have associated a link type with each element $\Phi \in B_{2n}$. The essential features of L are, of course, concealed in the surface mapping φ.

It will sometimes be convenient to visualize the link L in an alternative way. Observe that φ, being an orientation-preserving homeomorphism of $\partial E_+{}^3 \to \partial E_+{}^3$, is isotopic to the identity map via an isotopy φ_t (which will in general move the $2n$ points of ∂A). Adopting the convention $\varphi_0 = \varphi$, $\varphi_1 = \text{id}$, we may

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then define a homeomorphism h: $E_+^3 \cup_{\rho\varphi} E_-^3 \to E_+^3 \cup_\rho E_-^3$ by the rule

$$h(x, y, z) = (\varphi_z(x), \varphi_z(y), z) \text{ if } 0 \leqq z \leqq 1$$
$$= (x, y, z) \text{ if } z > 1 \text{ or } z < 0.$$

Then $h(\partial A\ x\ [0, 1])$ is a *geometric braid* (Figure 1b) and $P^* = h(A \cup_{\rho\varphi} A')$ is a link which is represented as a $2n$-*plat* (Figure 1c).

A link which is represented by a plat has the following properties:

($P1$) It is contained in the subset of E^3 which is represented by the inequalities $-.25 \leqq z \leqq 1.25$.

($P2$) It meets the planes $z = -0.25$ and $z = 1.25$ in precisely n points, and every other plane $z = z_0$, $-0.25 < z_0 < 1.25$ in precisely $2n$ points.

It follows immediately from classical results of Artin [**2**] that:

LEMMA 1. *If a tame link in E^3 satisfies $P1$ and $P2$, then for each sufficiently small $\eta > 0$ the point set $P^* \cap \{(x, y, z)/-0.25 + \eta < z < 1.25 - \eta\}$ determines a well-defined element of the group B_{2n}. Conversely, plats determined by distinct representatives of the same element of B_{2n} define the same link type.*

We will now establish:

LEMMA 2. *Every tame link may be represented as a polygonal link in E^3 which satisfies $P1$ and $P2$ above.*

Proof of Lemma 2. Our link L may without loss of generality be assumed to be polygonal and to satisfy $P1$. We may further assume that:

($P3$) No edge is in a plane parallel to the xy plane.

($P4$) Let $\epsilon > 0$ be an arbitrarily small real number. Then, for any 3 consecutive vertices b_{j-1}, b_j, b_{j+1} of L the line l_j parallel to the z axis through b_j has an ϵ-neighborhood N_j which meets the link only in the edges $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$.

(For, if a vertex sequence b_{j-1}, b_j, b_{j+1} violates either ($P3$) or ($P4$) we may replace it by b_{j-1}, b_j', b_{j+1} for some appropriate b_j' close to b_j, in order to achieve ($P3$) and ($P4$)).

Let $\vec{e}$ denote a unit vector in the direction of the positive z-axis. If b_{j-1} and b_j are any pair of adjacent vertices, the dot product $[b_{j-1}, b_j] \cdot \vec{e} \neq 0$, because L satisfies ($P3$). Let b_0, b_1 be any pair of adjacent vertices which satisfy the condition $[b_0, b_1] \cdot \vec{e} > 0$. (Since each component of L is connected, such a pair exists.) Starting with b_0, b_1, label the vertices of that component of L as $b_0, b_1, \ldots, b_r$ in the order in which they are encountered. Then, there will be some first vertex b_k such that $[b_k, b_{k+1}] \cdot \vec{e} < 0$. (see Figure 2). Let N_k, l_k be as defined in ($P4$) above, and let b_k', b_k'' be the unique points of intersection of ∂N_k with $[b_{k-1}, b_k]$ and $[b_k, b_{k+1}]$ respectively, and b_k^* the unique point of intersection of l_k with the plane $z = 1.25$ (Figure 2). Replace the edge sequence b_{k-1}, b_k, b_{k+1} with b_{k-1}, b_k', b_k^*, b_k'', b_{k+1}. Now repeat the entire procedure, continuing along b_k^*, b_k'', b_{k+1}, b_{k+2} until the first vertex b_m is encountered with $[b_m, b_{m+1}] \cdot \vec{e} > 0$. Just as above, replace b_{m-1}, b_m, b_{m+1} with b_{m-1}, b_m', $b_m^\#$, b_m'', b_{m+1}, where now $b_m^\#$ is the point of intersection of l_m with the plane $z = -.25$.

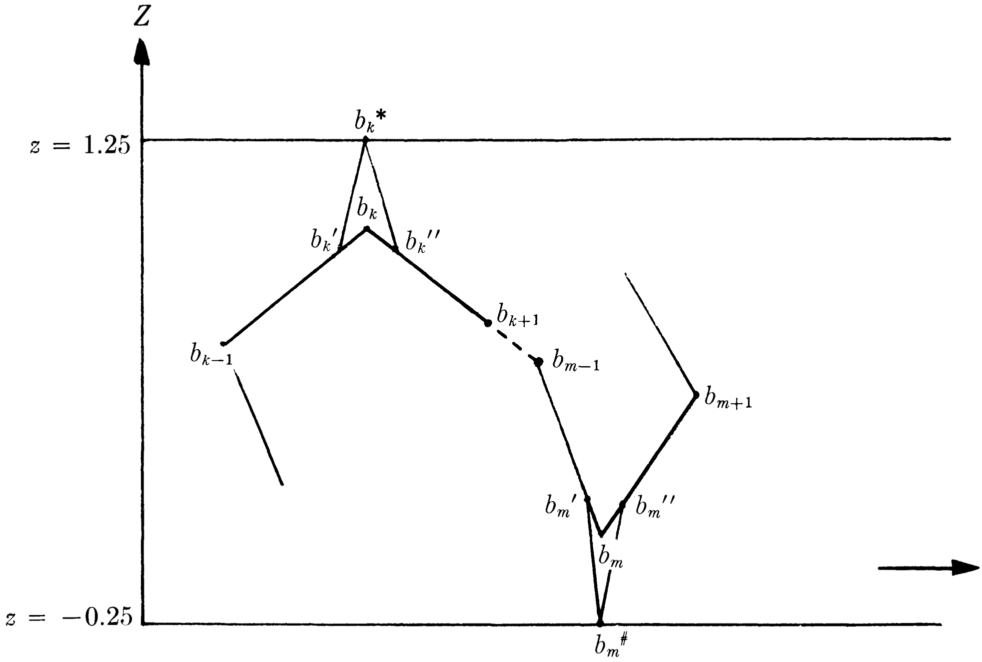

FIGURE 2

After a sufficient number of repetitions of this procedure on each of the components of L, the link will be replaced with a link P^* which defines the same link type as L and satisfies ($P1$) and ($P2$).

Definition. A polygonal link which satisfies properties ($P1$), ($P3$) and ($P4$) above will be said to be in *general position.* If a polygonal link L is in general position, then we will call the plat which is constructed by the method used in the proof of Lemma 2 the *canonical plat* associated with L. Note that the canonical plat associated with a link in general position is unique.

Lemmas (1) and (2) imply immediately:

PROPOSITION 1. *The mapping* $\varphi \rightarrow A \cup_{\rho\varphi} A'$ *induces a surjection from the class of groups* B_{2n}, $n = 1, 2, \ldots$ *onto the class of all tame link types.*

The algebraic structure of the group B_{2n} has been studied extensively in the literature (for a list of references, see [**4**]). A convenient set of generators for B_{2n} will now be defined. Recall that elements of B_{2n} are represented by homeomorphisms of $(\partial E_+^3, \partial A) \rightarrow (\partial E_+^3, \partial A)$. To define one such map h_j, let D_j be the disc of radius 0.5 in the $x - y$ plane with center at $(0.5(j + 1), 0, 0)$, $1 \leqq j \leqq 2n - 1$. Parametrize D_j with polar coordinates, and define h_j by $h_j(r, \theta, 0) = (r, \theta + 4\pi r, 0)$ for points in D_j, extended by the identity map to all of ∂E_+^3. (The map h_j interchanges the jth and $j + 1$st points of ∂A, leaving all others fixed.) The isotopy class of h_j, $1 \leqq j \leqq 2n - 1$, defines an element in B_{2n} which will be denoted σ_j.

LEMMA 3 (Artin [**1**]). *The group B_{2n} is generated by* $\{\sigma_1, \ldots, \sigma_{2n-1}\}$.

It follows immediately, using the definition of a geometric braid given earlier, that by applying an appropriate isotopy φ_t we may view the braid $\sigma_j (j = 1, \ldots, 2n - 1)$ in B_{2n} as the geometric braid which is formed (in a plane projection such as that in Figure 1*b*) by crossing the jth string over the $j + 1$st, leaving all other strings unaltered. In this way we see that the plats belonging to the geometric braids $\Phi \in B_{2n}$ and $\Phi\sigma_{2n} \in B_{2n+2}$ differ in the manner indicated in Figures 3a and 3b. This fact will be needed later.

(a) $A \cup_{\rho\phi} A'$ (b) $A \cup_{\rho\phi\sigma_{2n}} A'$

FIGURE 3

The subgroup K_{2n} of B_{2n} is defined to be the subgroup of those mapping classes $\Phi \in B_{2n}$ which have a representative $\varphi : (\partial E_+^3, \partial A) \rightarrow (\partial E_+^3, \partial A)$ which extends to a map $\hat{\varphi} : (E_+^3, A) \rightarrow (E_+^3, A)$. Generators were recently determined for K_{2n}, in terms of those defined above for B_{2n}.

LEMMA 4 (Hilden [**11**]). *The group K_{2n} is generated by* $\{\sigma_1, \sigma_2\sigma_1^2\sigma_2,$ *and* $\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}, 1 \leqq i \leqq n - 1\}$.

The generator $\sigma_2\sigma_1^2\sigma_2 \in K_6$ is illustrated in Figure 1d as a geometric braid.

One more observation is in order before we proceed to the main part of the paper, and that is to interpret K_{2n} and B_{2n} algebraically as subgroups of the automorphism group of a free group. The group B_{2n} has a faithful representation as a group B_{2n}^* of automorphisms of the free group F_{2n} of rank $2n$, where F_{2n} may be interpreted geometrically as the fundamental group of $\partial E_+^3 - \partial A$. If we choose generators for $\pi_1(\partial E_+^3 - \partial A)$ which are represented by simple loops X_{2i-1}, X_{2i} encircling the points $(i - .25, 0, 0)$, $(i + .25, 0, 0)$ respectively, then the image of σ_j, $1 \leqq j \leqq 2n - 1$ under the isomorphism from B_{2n} to B_{2n}^* will be

$$\sigma_j^* : X_j \rightarrow X_{j+1}, X_{j+1} \rightarrow X_{j+1}^{-1}X_jX_{j+1}, X_k \rightarrow X_k \text{ if } k \neq j, j + 1.$$

With these conventions, we may interpret K_{2n} in another way:

LEMMA 5. *The subgroup $K_{2n}^* \subset B_{2n}^*$ is the group of those braid automorphisms of F_{2n} which leave invariant the normal closure of $X_1X_2, X_3X_4, \ldots, X_{2n-1}X_{2n}$ in F_{2n}.*

Proof of Lemma 5. Let $\Psi \in B_{2n}$ be represented by ψ: $(\partial E_+^3, \partial A) \to (\partial E_+^3, \partial A)$. Clearly a necessary condition for ψ to extend to $\hat{\psi}$: $(E_+^3, A) \to (E_+^3, A)$ is that the induced automorphism ψ_* leave invariant the kernel of the homeomorphism $i_* : \pi_1(\partial E_+^3, \partial A) \to \pi_1(E_+^3, A)$ induced by inclusion. With our choice of generators above for $\pi_1(\partial E_+^3, \partial A)$ ker i_* is precisely the normal closure of $X_1X_2, \ldots, X_{2n-1}X_{2n}$ in $F_{2n} = \pi_1(\partial E_+^3, \partial A)$. The sufficiency of this condition follows from a theorem of MacMillan [**12**].

3. Stable equivalence of plat representations of knots. Using the conventions introduced in Section 2, we may now establish our main result. The proof will occupy the remainder of this section.

THEOREM 1. *Let L_i, $i = 1, 2$, be tame knots. Choose elements $\Phi_i \in B_{2n_i}$ in such a way that the plat $A \cup_{\rho\varphi_i} A'$ defines the same knot type as L_i. (By Proposition 1, this is always possible.) Then $L_1 \approx L_2$ if and only if there exists an integer $t \geqq \max(n_1, n_2)$ such that, for each $n \geqq t$ the elements*

$$\Phi_i' = \Phi_i\sigma_{2n_i}\sigma_{2n_i+2} \ldots \sigma_{2n-2} \in B_{2n}, \quad i = 1, 2$$

are in the same double coset of B_{2n} modulo the subgroup K_{2n}.

Remark. A large part of the proof of Theorem 1 is valid both for knots and links. We will consider the more general case of links as long as we are able to, specializing to knots in Lemmas 9 and 10 below.

Proof of Theorem 1. The first task will be to establish that $\Phi_i' \in B_{2n}$ and $\Phi_i \in B_{2n_i}$ determine the same link type. This will follow by iteration, if we can establish that $\Phi_i\sigma_{2n_i} \in B_{2n_i+2}$ and $\Phi_i \in B_{2n_i}$ determine the same link type, and the easiest way to see that is by a picture. Recall that every plat may be visualized as the link obtained from a geometric braid, by identifying the strings in pairs on the top and on the bottom of the braid (c.f. figure 1b). In this representation, the plats $A \cup_{\rho\Phi} A'$ and $A \cup_{\rho\Phi\sigma_{2n_i}} A'$ will be as illustrated in Figure 3. Clearly, these represent the same link type.

To establish the "if" part of Theorem 1, suppose, then, that $\Phi_1' = \alpha_1\Phi_2'\alpha_2$ for some $\alpha_1, \alpha_2 \in K_{2t}$. Let g_1, g_2 be the self-homeomorphisms of (E_+^3, A) which represent α_1, α_2 respectively. Then, we may define a homeomorphism $h : E_+^3 \cup_{\rho\varphi_1'} E_-^3 \to E_+^3 \cup_{\rho\varphi_2'} E_-^3$ by the rule $h|E_+^3 = g_1$, $h|E_-^3 = \rho g_2\rho^{-1}$. Since $h(A \cup_{\rho\varphi_1'} A') = A \cup_{\rho\varphi_2'} A'$, our links are equivalent. Thus, the "if" part of our assertion is established.

To begin the "only if" part of the proof, we will need some notation. Let L be a polygonal link. If $b_1, b_2, \ldots$ are consecutive vertices of L, then $[b_1, b_2]$, $[b_2, b_3], \ldots$ will denote the corresponding edges. The symbol $[b_i, b_j, b_k]$ will denote the closed simplex with vertices b_i, b_j, b_k. Suppose, then, that L is a link with vertices $b_1, b_2, b_3, \ldots$, and that $[b_1, b_2, b_3] \cap L = [b_1, b_2] \cup [b_2, b_3]$. Then we say that the "move" $\mathfrak{E}_{b_2} = \mathfrak{E}(b_1, b_2, b_3)$ is *applicable* to L, and define a new link L^* by:

(2) $L^* = \mathfrak{E}_{b_2}L = L + [b_1, b_3] - [b_1, b_2] - [b_2, b_3]$.

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The moves $\mathfrak{E}_{b_2}$ and its inverse $(\mathfrak{E}_{b_2})^{-1}$ will be called elementary or *type* $\mathfrak{E}$ deformations. Two links L_1, L_2 are *combinatorially equivalent* if there exists a finite sequence of links joining L_1 to L_2 with the property that each link in the sequence can be obtained from its predecessor by a single type $\mathfrak{E}$ deformation. The sequence joining L_1 to L_2 will be called a *deformation chain.* It is a classical result that tame links are equivalent if and only if they are combinatorially equivalent.

The reader is referred to Section 2 for a definition of "general position". We assert:

LEMMA 6. *If L and L^* are combinatorially equivalent and in general position, then there is a deformation chain from L to L^* such that all links in the chain are in general position.*

Proof of Lemma 6. If $(\mathfrak{E}(a, b, c))^{-1}$ creates an edge $[a, b]$ or $[b, c]$ which violates ($P3$) or ($P4$), we may replace it with $(\mathfrak{E}(a, b', c))^{-1}$, where b' is close to b and is chosen so that $\mathfrak{E}(a, b', c)^{-1}L$ is in general position. Afterwards, replace b by b' in all links which follow. If $\mathfrak{E}(a, b, c)$ creates an edge $[a, c]$ which violates ($P3$) or ($P4$) and if $[c, d]$ is the edge which follows $[a, c]$, choose a point c' close to c and on $[b, c]$, replace $\mathfrak{E}(a, b, c)$ with $\mathfrak{E}(c', c, d)\ \mathfrak{E}(a, b, c')$ $(\mathfrak{E}(b, c', c))^{-1}$, and replace c by c' in all links of the deformation chain. This creates a new deformation chain joining L to L'. Induction on the number of edges which violate $P3$ or $P4$ completes the proof.

Suppose, then, that L and L^* are polygonal links which are joined by a deformation chain, with L and L^* and every link in the chain in general position. By method of Lemma 2, we may associate with each link in the chain a unique canonical plat, as defined in Section 2:

$$\begin{array}{ccccccccc} L = & L_1 & \rightarrow & L_2 & \rightarrow & \ldots . & \rightarrow & L_r & = L^* \\ & \downarrow & & \downarrow & & & & \downarrow & \\ P = & P_1 & & P_2 & & \ldots . & & P_r & = P^* \end{array}$$

Each canonical plat P_j will (later) be associated with a unique element of the group B_{2n_j} for an appropriate integer n_j. Each $L_{j+1} = \mathfrak{E}(a, b, c)^{\pm 1}L_j$ for some vertices a, b, c of L_j. Our first task will be to prove that certain moves are adequate in every case to transform P_j to P_{j+1}. These moves will then be interpreted in terms of operations in the groups B_2, B_4, B_6, . . . , and reduced to the simpler set of moves described in Theorem 1.

A point p in the region between the planes $z = -0.25$ and $z = 1.25$ will be denoted a *lower* (respectively upper) *boundary point* if its z-coordinate is -0.25 (respectively 1.25), otherwise p is an *interior point.* The superscript "#" will be used for lower boundary points, and "$*$" for upper boundary points.

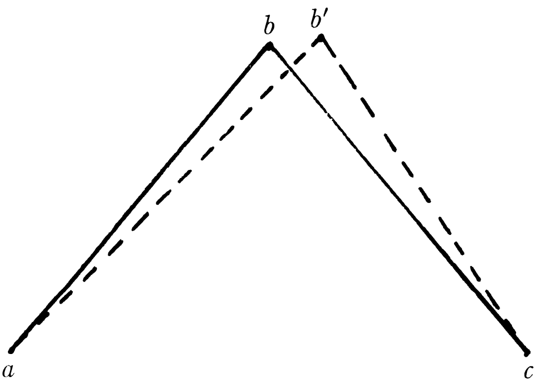
In general, a type $\mathfrak{E}$ move will not take a plat to a plat, however in certain special cases it will, and we denote this subclass of type $\mathfrak{E}$ moves with the symbol *type* $\tilde{\mathfrak{E}}$.

The reader is referred to Figure 4. Let a, b, c be consecutive vertices of P, with b interior. Let b' be a second interior point close to b, satisfying $[a, b, b'] \cap P = [a, b]$ and $[b', b, c] \cap P = [b, c]$. Define:

(3) $\mathfrak{R}_b = \mathfrak{R}(b, b') = \mathfrak{E}(b', b, c)\mathfrak{E}(a, b', b)^{-1}$

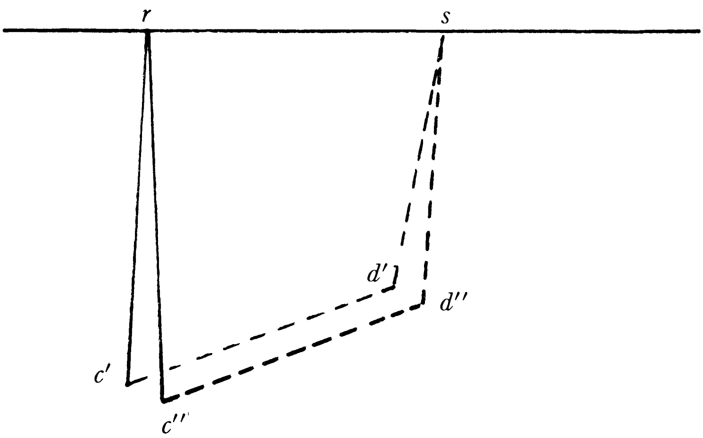
If $\mathfrak{R}_b P$ is a plat, then we say $\mathfrak{R}_b$ is applicable, and call this a move of *type* $\mathfrak{R}$. It is clearly always possible to find an interior point b' close to any interior vertex b of P such that $\mathfrak{R}_b$ is applicable.

The reader is referred next to Figure 5. Let c', r, c'' be consecutive vertices of the plat P, with r a boundary vertex (we illustrate the move with an upper



$\mathfrak{R}(b, b')\ P = P - [a, b] - [b, c] + [a, b'] + [b', c]$

FIGURE 4



$$\mathfrak{B}_r(c', d', s, d'', c'')\ P = P - [c', r] - [r, c''] + [c', d'] + [d', s] + [s, d''] + [d'', c''].$$

FIGURE 5

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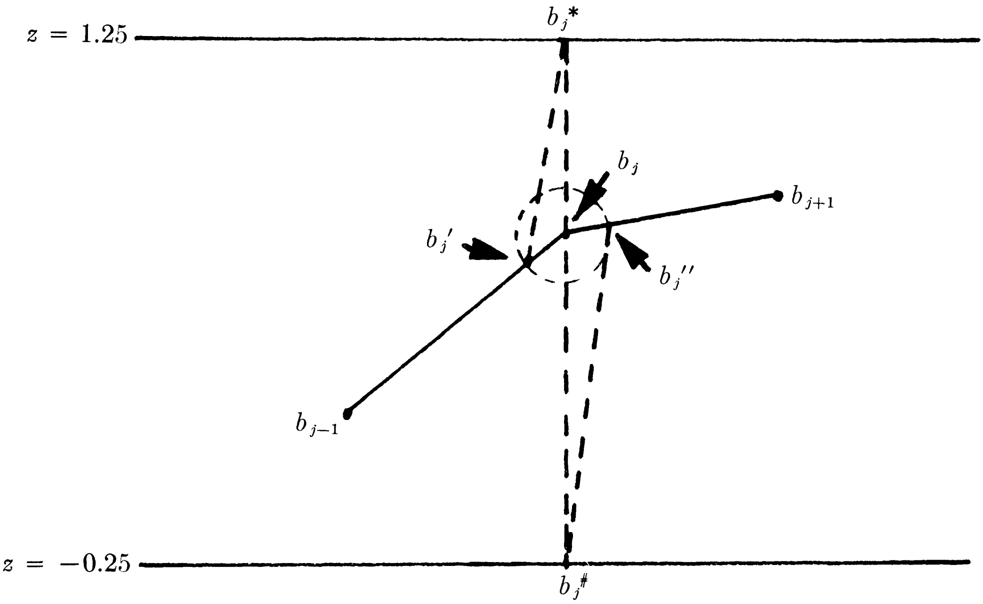
boundary vertex, but the situation is identical if r is a lower boundary vertex) and c' close to c''. It will be assumed that $[c', r, c''] \cap P = [c', r] \cup [r, c'']$. Let $s \neq r$ be a second boundary point, on the same boundary component as r, and let d', d'' be interior points satisfying the conditions $[c', d', d''] \cap P = c'$, $[d', d'', c''] \cap P = c''$, and $[d', s, d''] \cap P = \emptyset$. Then we define:

$$(4) \quad \mathfrak{B}_r = \mathfrak{B}_r(c', d', s, d'', c'') = \mathfrak{E}(d', s, d'')^{-1}\mathfrak{E}(d', d'', c'')^{-1}\mathfrak{E}(c', d', c'')^{-1}\mathfrak{E}(c', r, c'').$$

If $\mathfrak{B}_r P$ is then a plat, we will say that $\mathfrak{B}_r$ is applicable to P. The move $\mathfrak{B}_r$ or its inverse will be denoted *type* $\mathfrak{B}$. We will also sometimes use the degenerate move $\mathfrak{B}_r(c', s, c'')$ which occurs when $d' = c'$, $d'' = c''$. It will also be denoted type $\mathfrak{B}$.

Note that a type $\mathfrak{B}$ move is executed by retracting the "spike" at r to the "base" $c'c''$, and shooting out a new spike to s, which will in general thread in and out of the other edges of the link. In most cases it will not be possible to accomplish this move by sliding the spike at r along the boundary plane to s, because the other edges of P will interfere with such a move.

Moves of type $\tilde{\mathfrak{E}}$, $\mathfrak{R}$ and $\mathfrak{B}$ defined above all leave plat index unaltered. (Plat index is the number of intersections of the plat with the planes $z = 1.25$ and $z = -0.25$.) Our next move changes plat index by ± 1. The reader is referred to Figure 6. Let b_{j-1}, b_j, b_{j+1} be consecutive vertices of the plat P,



$$\mathfrak{A}(b_j)\, P = P - [b_j', b_j] - [b_j, b_j''] + [b_j', b_j^*] + [b_j, b_j^\#] + [b_j^\#, b_j''].$$

FIGURE 6

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with b_j an interior vertex. If P is the unique canonical plat constructed from a link L in general position, then P will also be in general position, hence P satisfies property ($P4$), Lemma 2. Let b_j^*, $b_j^\#$ be the points of intersection of l_j with the upper and lower (or lower and upper) boundary planes, and let b_j', b_j'' be points on $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$ which have distance $\in$ from b_j. Assume further that $[b_{j-1}, b_j] \cdot \vec{e}$ and $[b_j, b_{j+1}] \cdot \vec{e} > 0$ (or < 0). Then we define

$$(5) \quad \mathfrak{A}(b_j) = (\mathfrak{E}(b_j, b_j^\#, b_j''))^{-1}(\mathfrak{E}(b_j', b_j^*, b_j))^{-1}(\mathfrak{E}(b_j, b_j'', b_{j+1}))^{-1} \times (\mathfrak{E}(b_{j-1}, b_j', b_j))^{-1}.$$

Note that $\mathfrak{A}(b_j)$ is always a plat, and that $\mathfrak{A}(b_j)$ is applicable to every interior vertex b_j when P is in general position. The move $\mathfrak{A}(b_j)$ or its inverse will be denoted *type* $\mathfrak{A}$.

LEMMA 7. *Let L, L^* be combinatorially equivalent links which are in general position, and let P, P^* be the canonical plats associated with L, L^*. Then there is a finite sequence of plats*

$$P = P_1^* \rightarrow P_2^* \rightarrow \ldots \rightarrow P_k^* = P^*$$

such that each P_j^* is obtained from P_{j-1}^* by a single move of type $\tilde{\mathfrak{E}}$, $\mathfrak{R}$, $\mathfrak{B}$ or $\mathfrak{A}$.

Proof of Lemma 7. It will be enough to prove the lemma for the case where $L^* = (\mathfrak{E}_b)^{\pm 1}L$, since the general case will then follow by induction on the length of the deformation chain joining L to L^*. It will, moreover, be adequate to consider the case $L^* = \mathfrak{E}_b L$, since the case $L^* = (\mathfrak{E}_b)^{-1}L$ may be handled by the inverse of the sequence used to go from P to P^* if $L^* = \mathfrak{E}_b L$.

Choose an orientation on L. An edge $[b_{j-1}, b_j]$ of L will be colored *blue* if $[b_{j-1}, b_j] \cdot \vec{e} > 0$, and *red* if $[b_{j-1}, b_j] \cdot \vec{e} < 0$. See Figure 7. The vertex b_j will be said to be

type 1 if $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$ have distinct colors, and if b_j is not an interior point;

type 2 if $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$ have distinct colors, and if b_j is an interior point;

type 3 if $[b_{j-1}, b_j]$ and $[b_j, b_{j+1}]$ have the same color. This implies that b_j is interior.

Suppose that $L^* = \mathfrak{E}(b_2, b_3, b_4)L$, where L contains the vertex sequence b_1, b_2, b_3, b_4, b_5. We must treat separately the cases obtained when b_2, b_3, b_4 are each type 1, 2 or 3. There is no loss in generality in assuming that the edge $[b_1, b_2]$ is blue, since the choice of an orientation on L is arbitrary. The type of b_2, b_3, b_4 will then determine uniquely the colors of the edges $[b_2, b_3]$, $[b_3, b_4]$, $[b_4, b_5]$. It may happen that there will be two possibilities for the color of $[b_2, b_4] \in L^*$, however once this color has been fixed, the type of the vertices b_2 and b_4 of L^* will be fixed. Thus there will be at most 54 cases to treat, depending on the type of b_2, b_3, b_4 and the color of $[b_2, b_4]$, and each such case may be associated with a unique mapping $(i, j, k) \rightarrow (l, m)$, where $i, j, k, l, m = 1, 2, 3$ denote the vertex types of the vertices b_2, b_3, b_4 of L and b_2, b_4 of L^*.

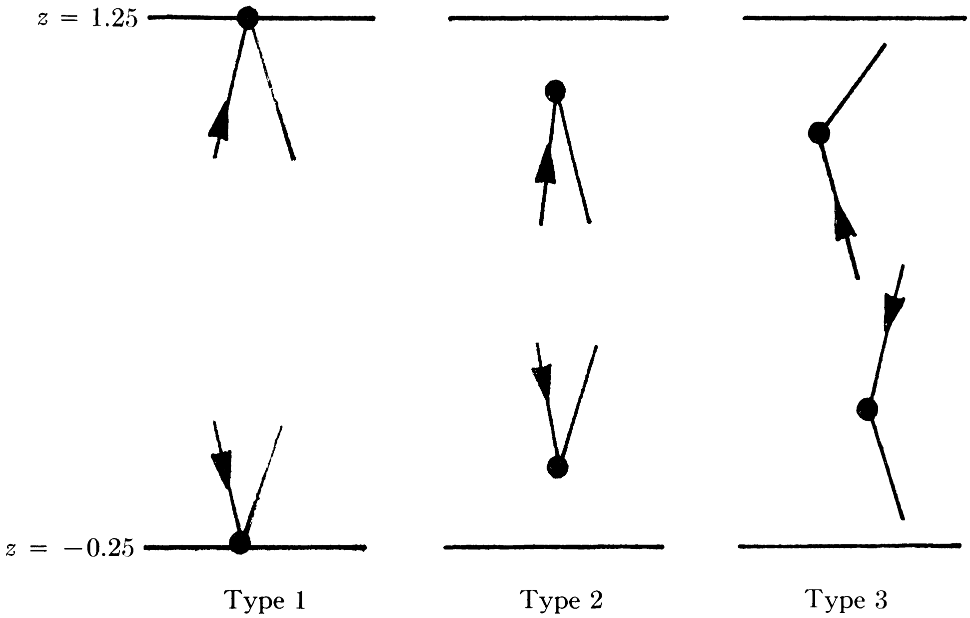

FIGURE 7

In Table 1 we examine these 54 cases. Some turn out to be impossible. For example (cases 1, 2) if b_2, b_3, b_4 had types 1 1 1 then after applying $\mathfrak{S}_{b_3}$ the link L^* would have an edge which violated ($P3$), so that L^* would not be in general position. Or (cases 19, 20) if b_2, b_3, b_4 are types 2 1 1, then $[b_2, b_4]$ will always be blue, never red. The 54 cases are moreover not all independent, because of symmetry consideration, since we note that a replacement sequence for the case $(i, j, k) \rightarrow (l, m)$ may be obtained from one for the case $(k, j, i) \rightarrow (m, l)$ by reflecting each plat in the original sequence about the plane $z = 0.50$.

In the cases not handled as above, replacement sequences will be given in Table 1 whenever it is possible to define them unambiguously, or else discussed below Table 1 in further detail. Our notation will always be the same: The vertices of the link L which are involved in our move will be denoted b_1, b_2, b_3, b_4, b_5. We define $L^* = \mathfrak{S}(b_2, b_3, b_4)L$. If any vertex of L or L^* (say b_k) is type 2, then in constructing the canonical plat associated with L or L^*, we must shoot out a spike at b_k to b_k^* (an upper boundary point) if $[b_{k-1}, b_k]$ is blue, or to $b_k^\#$ (a lower boundary point) if $[b_{k-1}, b_k]$ is red. The unique points of intersection of the boundary ∂N_k of an ϵ-neighborhood of b_k with the edges $[b_{k-1}, b_k]$ and $[b_k, b_{k+1}]$ which are needed to construct the canonical plat will be denoted b_k' and b_k'' (c.f. Figure 2). The first non-trivial case, (1 1 2) $\rightarrow$ (1 3) is worked out with pictures, to illustrate the procedure.

Case 3. (1 1 2) $\rightarrow$ (1 3). Figure 8 shows the portions of the links L and L^* starting at b_1 and ending at b_5. Since $b_4 \in L$ is a type 2 vertex, we must alter it to produce the canonical plat P, which will include the vertex sequence

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TABLE 1

Code: Column (i) Case number
Columns (ii), (iii), (iv) Type of vertices $b_2, b_3, b_4 \in L$
Column (v) Color of $[b_2, b_4] \in L^*$
Columns (vi), (vii) Type of vertices $b_2, b_4 \in L^*$
Column (viii) Replacement sequence, if possible.

i	ii	iii	iv	v	vi	vii	viii	i	ii	iii	iv	v	vi	vii	viii
1	1	1	1	r	*	*	impossible	28	2	2	2	b	3	2	ref. case 27
2	1	1	1	b	*	*	impossible	29	2	2	3	r	2	2	see below
3	1	1	2	r	1	3	see below	30	2	2	3	b	3	3	see below
4	1	1	2	b	*	*	impossible	31	2	3	1	r	2	1	ref. case 15
5	1	1	3	r	1	2	see below	32	2	3	1	b	*	*	impossible
6	1	1	3	b	*	*	impossible	33	2	3	2	r	2	2	$\mathfrak{E}(b_2, b_3, b_4)$
7	1	2	1	r	*	*	impossible	34	2	3	2	b	*	*	impossible
8	1	2	1	b	*	*	impossible	35	2	3	3	r	2	3	$\mathfrak{E}(b_2, b_3, b_4)$
9	1	2	2	r	1	3	see below	36	2	3	3	b	*	*	impossible
10	1	2	2	b	*	*	impossible	37	3	1	1	r	2	1	ref. case 5
11	1	2	3	r	1	2	see below	38	3	1	1	b	*	*	impossible
12	1	2	3	b	*	*	impossible	39	3	1	2	r	2	2	ref. case 23
13	1	3	1	r	1	1	$\mathfrak{E}(b_2, b_3, b_4)$	40	3	1	2	b	3	3	ref. case 24
14	1	3	1	b	*	*	impossible	41	3	1	3	r	2	3	see below
15	1	3	2	r	1	2	$\mathfrak{E}(b_2, b_3, b_4)$	42	3	1	3	b	3	2	ref. case 41
16	1	3	2	b	*	*	impossible	43	3	2	1	r	2	1	ref. case 11
17	1	3	3	r	1	3	$\mathfrak{E}(b_2, b_3, b_4)$	44	3	2	1	b	*	*	impossible
18	1	3	3	b	*	*	impossible	45	3	2	2	r	2	2	ref. case 29
19	2	1	1	r	*	*	impossible	46	3	2	2	b	3	3	ref. case 30
20	2	1	1	b	3	1	ref. case 3	47	3	2	3	r	2	3	see below
21	2	1	2	r	2	3	see below	48	3	2	3	b	3	2	ref. case 47
22	2	1	2	b	3	2	ref. case 21	49	3	3	1	r	*	*	impossible
23	2	1	3	r	2	2	see below	50	3	3	1	b	3	1	ref. case 17
24	2	1	3	b	3	3	see below	51	3	3	2	r	*	*	impossible
25	2	2	1	r	*	*	impossible	52	3	3	2	b	3	2	ref. case 35
26	2	2	1	b	3	1	ref. case 9	53	3	3	3	r	*	*	impossible
27	2	2	2	r	2	3	see below	54	3	3	3	b	3	3	$\mathfrak{E}(b_2, b_3, b_4)$

$b_1, b_2, b_3, b_4', b_4^*, b_4'', b_5$. Since none of the vertices b_1, b_2, b_4, b_5 of L^* is type 2, this same vertex sequence b_1, b_2, b_4, b_5 will be included in the canonical plat P^*. To get from P to P^* by moves of type $\tilde{\mathfrak{E}}$, $\mathfrak{R}$, $\mathfrak{B}$ and $\mathfrak{A}$, first apply $\mathfrak{R}(b_4', b_4)$,

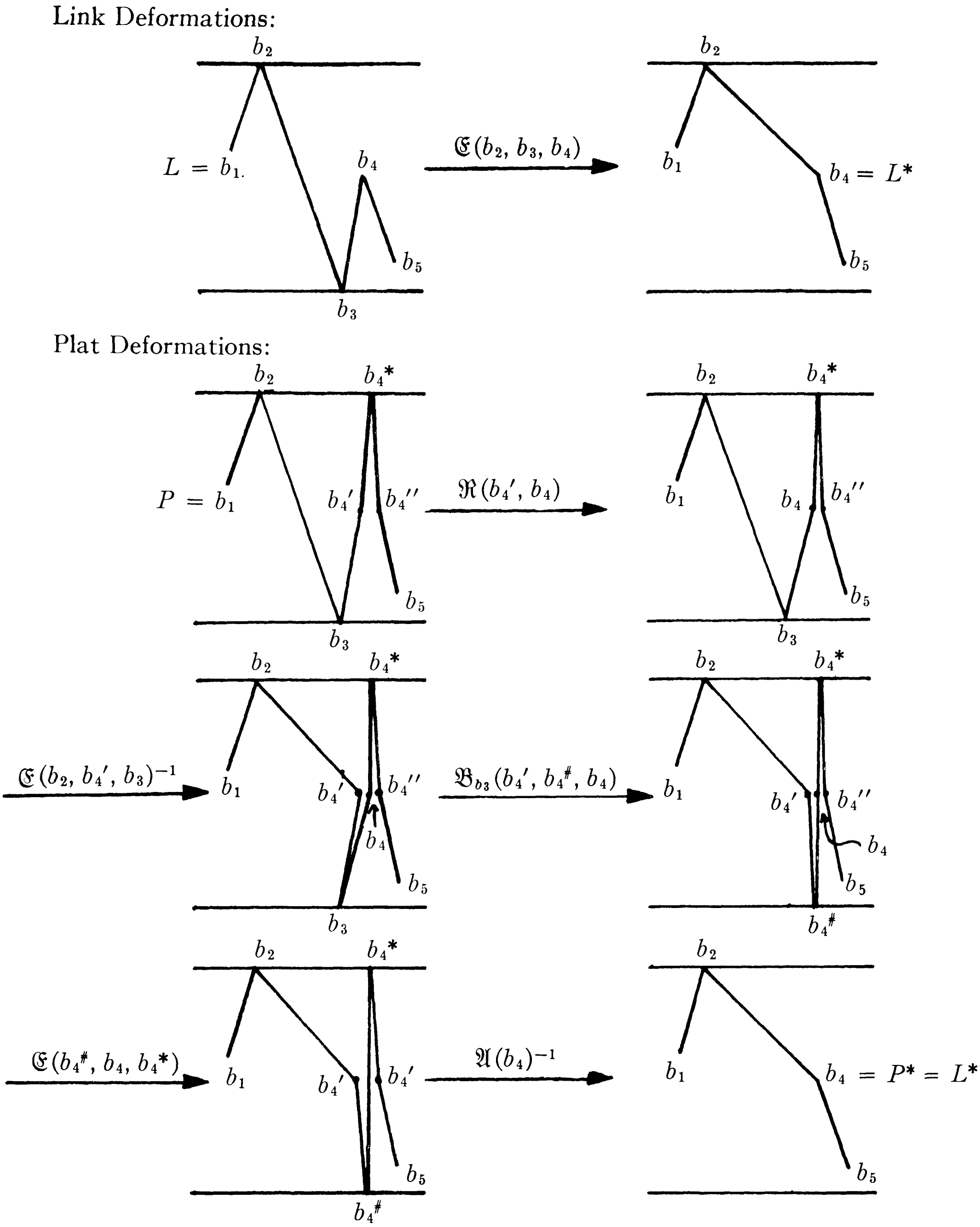

FIGURE 8

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moving the vertex b_4' to b_4. Then apply $(\tilde{\mathfrak{C}}(b_2, b_4', b_3))^{-1}$. It is applicable because $\mathfrak{C}(b_2, b_3, b_4)$ was applicable to L. Next, apply $\mathfrak{B}_{b_3}(b_4', b_4^{\#}, b_4)$ to move the tip of the spike at b_3 to the point $b_4^{\#}$. Finally, apply $\mathfrak{C}(b_4^{\#}, b_4, b_4^*)$ and $(\mathfrak{A}(b_4))^{-1}$, to obtain P^*.

Case 5. (1 1 3) $\rightarrow$ (1 2). The plat $P = L$. Apply $\mathfrak{R}(b_4, b_4'')$ to P to move b_4 to b_4''. Apply $(\mathfrak{C}(b_2, b_4', b_3))^{-1}$. It is applicable because $\mathfrak{C}(b_2, b_3, b_4)$ is applicable to L. Apply $\mathfrak{B}_{b_3}(b_4', b_4^{\#}, b_4'')$ to move the spike at b_3 to $b_4^{\#}$, obtaining P^*.

Case 9. (1 2 2) $\rightarrow$ (1 3). The plat P contains the vertex sequence $b_1, b_2, b_3', b_3^{\#}, b_3'', b_4', b_4^*, b_4'', b_5$. Apply $\mathfrak{R}(b_4', b_4)$ to move b_4' to b_4. Then apply $(\mathfrak{C}(b_2, b_4', b_3'))^{-1}$. Next apply $(\mathfrak{B}_{b_4^{\#}}(b_4', b_3', b_3^{\#}, b_3'', b_4))^{-1}$ to move the spike at $b_3^{\#}$ to $b_4^{\#}$. Apply $\mathfrak{C}(b_4^{\#}, b_4, b_4^*)$ to delete the vertex b_4. Finally, apply $\mathfrak{A}(b_4)^{-1}$ to obtain $P^* = L^* = \mathfrak{C}(b_2, b_3, b_4)L$.

Case 11. (1, 2, 3) $\rightarrow$ (1, 2). Apply $\mathfrak{C}(b_2, b_4', b_3')$ to P. Next apply $\mathfrak{R}(b_4, b_4'')$. Finally, apply $(\mathfrak{B}_{b_4^{\#}}(b_4', b_3', b_3^{\#}, b_3'', b_4''))^{-1}$ to obtain P^*.

Case 21. (2, 1, 2) $\rightarrow$ (2, 3). Apply $\mathfrak{R}(b_4', b_4)$ to P. Then apply $(\mathfrak{C}(b_2'', b_4', b_3))^{-1}$. Next, move the spike at b_3 to $b_4^{\#}$ by applying $\mathfrak{B}_{b_3}(b_4', b_4^{\#}, b_4)$. Delete the vertex b_4 by applying $\mathfrak{C}(b_4^{\#}, b_4, b_4^*)$. Finally, apply $(\mathfrak{A}(b_4))^{-1}$.

Case 23. (2, 1, 3) $\rightarrow$ (2, 2). Apply $\mathfrak{R}(b_4, b_4'')$ to P. Then apply $(\mathfrak{C}(b_2'', b_4', b_3))^{-1}$. Finally, apply $(\mathfrak{B}_{b_4^{\#}}(b_4', b_3, b_4''))^{-1}$ to obtain p^*.

Case 24. (2, 1, 3) $\rightarrow$ (3, 3). Apply $\mathfrak{R}(b_2'', b_2)$, $(\mathfrak{C}(b_3, b_2'', b_4))^{-1}$, $\mathfrak{B}_{b_3}(b_2, b_2^{\#}, b_2'')$, $\mathfrak{C}(b_2^*, b_2, b_2^{\#})$, $(\mathfrak{A}(b_2))^{-1}$.

Case 27. (2, 2, 2) $\rightarrow$ (2, 3). Apply $\mathfrak{R}(b_4', b_4)$ to P. Then apply $\mathfrak{C}(b_2'', b_4', b_3')$. Next apply $(\mathfrak{B}_{b_4^{\#}}(b_4', b_3', b_3^{\#}, b_3'', b_4))^{-1}$. Now apply $\mathfrak{C}(b_4^{\#}, b_4, b_4^*)$ to delete b_4. Finally, apply $(\mathfrak{A}(b_4))^{-1}$ to obtain P^*.

Case 29. (2, 2, 3) $\rightarrow$ (2, 2). Apply $(\mathfrak{C}(b_2'', b_4', b_3'))^{-1}$. $\mathfrak{R}(b_4, b_4'')$, $(\mathfrak{B}_{b_4^{\#}}(b_4', b_3', b_3^{\#}, b_3'', b_4''))^{-1}$.

Case 30. (2, 2, 3) $\rightarrow$ (3, 3). Apply $\mathfrak{R}(b_2'', b_2)$, then $(\mathfrak{C}(b_3'', b_2'', b_4))^{-1}$. Now move the spike at $b_3^{\#}$ to $b_2^{\#}$ by $(\mathfrak{B}_{b_2^{\#}}(b_2, b_3', b_3^{\#}, b_3'', b_2''))^{-1}$. Delete b_2 by applying $\mathfrak{C}(b_2^*, b_2, b_2^{\#})$. Apply $(\mathfrak{A}(b_2))^{-1}$ to obtain P^*.

Case 41. (3, 1, 3) $\rightarrow$ (2, 3). Apply $\mathfrak{R}(b_2, b_2')$, $\mathfrak{C}(b_3, b_2'', b_4)$, $\mathfrak{B}_{b_3}(b_2', b_2^*, b_2'')$.

Case 47. (3, 2, 3) $\rightarrow$ (2, 3). Apply $\mathfrak{R}(b_2, b_2')$, then $\mathfrak{C}(b_3'', b_2'', b_4)$, then $\mathfrak{B}_{b_2^*}(b_2', b_3', b_3^*, b_3'', b_2'')^{-1}$.

This completes the proof of Lemma 7.

Having established Lemma 7, it will no longer be necessary to restrict our attention to polygonal links and combinatorial deformations. Accordingly, we may replace any plat P_j^* in the deformation chain joining P to P^* by any other plat which is the image of P_j^* under a homeomorphism $h: E^3 \rightarrow E^3$, where h is required to be isotopic to the identity map via an isotopy h_t such that $h_0 = h$, $h_1 =$ identity, and, for each $0 \leqq t \leqq 1$, $h_t(x, y, z) = (x, y, z)$ whenever $z \leqq -0.25 + \eta$ or $z \geqq 1.25 - \eta$, for some small $\eta > 0$. At the same time, we replace the p. l. deformations of types $\mathfrak{B}$ and $\mathfrak{A}$ by moves $\tilde{\mathfrak{B}}$ and $\tilde{\mathfrak{A}}$ which are isotopic to $\mathfrak{B}$ and $\mathfrak{A}$ via an isotopy which is the identity for all points $(x, y, z) \in E^3$ such that $z \leqq -0.25 + \eta$ or $z \geqq 1.25 - \eta$. It is immediate that the composition of a move of type $\mathfrak{B}$ (or $\mathfrak{A}$) and any number of moves of type

$\mathfrak{C}$ and $\mathfrak{R}$ will be a move of type $\tilde{\mathfrak{B}}$ (or $\tilde{\mathfrak{A}}$). Thus we may sharpen Lemma 7 to:

LEMMA 8. *Let P, P^* be two plats which define the same tame link type. Then, there is a finite sequence of plats*

$$P = P_1^* \to P_2^* \to \ldots . \to P_k^* = P^*$$

such that each P_{j+1}^* is obtained from P_j^* by a single move of type $\tilde{\mathfrak{B}}$ or $\tilde{\mathfrak{A}}$.

To proceed further, we wish to specialize the move $\tilde{\mathfrak{A}}$. Intuitively, $\tilde{\mathfrak{A}}$ has the effect of adding a "trivial loop" to the plat P at any point b. We now wish to show that it is adequate to do this in a very special way, choosing b to be arbitrarily close to the lower boundary plane, and to have larger x coordinate than any other point of P. Thus b is in the lower right corner of P, relative to a projection of P onto the $x - z$ plane (cf. Figure 3). We will, further, require that the new double point introduced by a type $\tilde{\mathfrak{A}}$ move applied at b have smaller z-coordinate than any other double point of P. We will also restrict ourselves to the case where the new twist introduced into the projection of P when $\tilde{\mathfrak{A}}$ is applied has the sense illustrated in Figure 3b, and never the opposite sense. We will call this special type $\tilde{\mathfrak{A}}$ move "type $\mathfrak{L}$", with $\mathfrak{L}^+$ denoting the addition of a loop at b, and $\mathfrak{L}^-$ denoting the deletion of a loop at b. Note that in Lemma 9, below, we restrict our attention to *knots*. (The corresponding assertion for links is a little more complicated).

LEMMA 9. *Let q be an arbitrary point of P, which is assumed to have a single component. Let $\tilde{\mathfrak{A}}$ be a type $\tilde{\mathfrak{A}}$ move applied at q. Then there exists a sequence $\mathfrak{B}_1, \ldots, \mathfrak{B}_z$ of type $\mathfrak{B}$ moves such that*

$$\mathfrak{A} = \mathfrak{B}_z \ldots \mathfrak{B}_1\mathfrak{L}^+, \quad \mathfrak{A}^{-1} = \mathfrak{L}^-\mathfrak{B}_1^{-1} \ldots \mathfrak{B}_z^{-1}.$$

Proof. Note that the second assertion follows from the first because $\mathfrak{L}^-$ is the inverse of $\mathfrak{L}^+$. Hence we may restrict our attention to the case where a loop is *added* at q.

We first show that it is possible to reverse the direction of the twist in a type $\mathfrak{L}^+$ move by a sequence of moves of type $\mathfrak{B}$ and $\mathfrak{L}^+$. See Figure 9. Choose points b', b'' close to $s^\#$, and apply a type $\mathfrak{B}$ move, retracting the spike at $s^\#$ to the base $b'b''$ and shooting out a new spike along the indicated path (*through* the loop at $r^\#$) to a new location $t^\#$ (picture 2). After an isotopy (picture 3), use a second type $\mathfrak{B}$ move to return the spike to $s^\#$, this time choosing a path that goes behind the loop at $r^\#$ (picture 4), achieving thereby the desired result. Thus we may, without loss of generality, assume that any type $\mathfrak{L}^+$ move which is applied has the sense illustrated in Figure 3b.

Let b be the point at which our type $\mathfrak{L}$ move is to be applied (Figure 10). Since P has one component, b is connected to q by a sequence of arcs which go between the upper and lower boundary planes, touching each k times. See Figure 10. Let r_0^*, $r_1^\#$, r_1^*, $r_2^\#$, $\ldots$, r_{k-1}^*, $r_k^\#$, r_k^* be the ordered sequence of upper and lower boundary points of P which are encountered in going along P from b to q, where b is located on $r_1^*r_1^\#$ and q is located on $r_k^\#r_k^*$.

Apply a type $\mathfrak{L}^+$ move at b (picture 2, Figure 10). This will create two new boundary points, $s_1^\#$ and s_0^*, with $s_1^\#$ very close to $r_1^\#$. Choose points

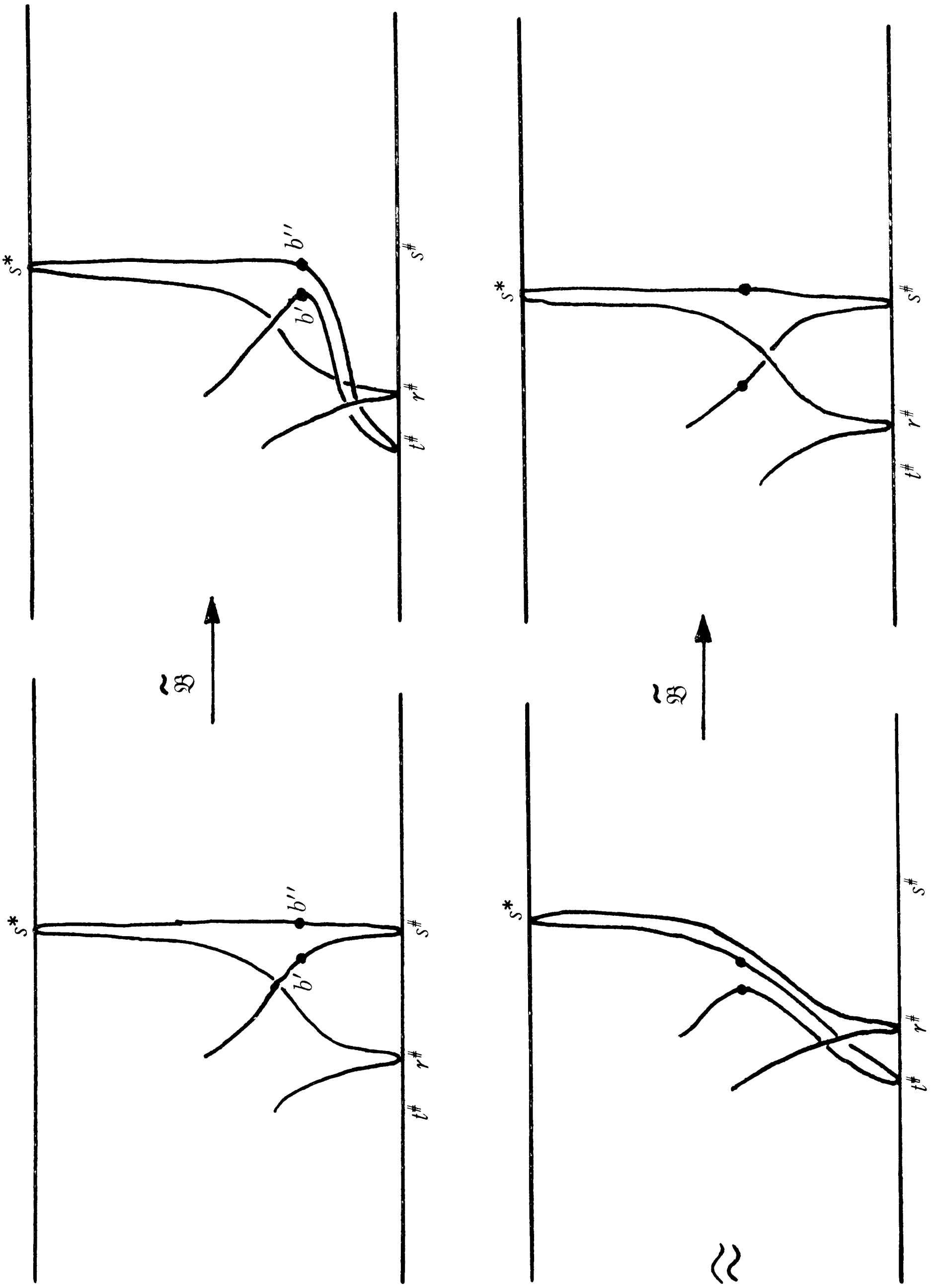

FIGURE 9

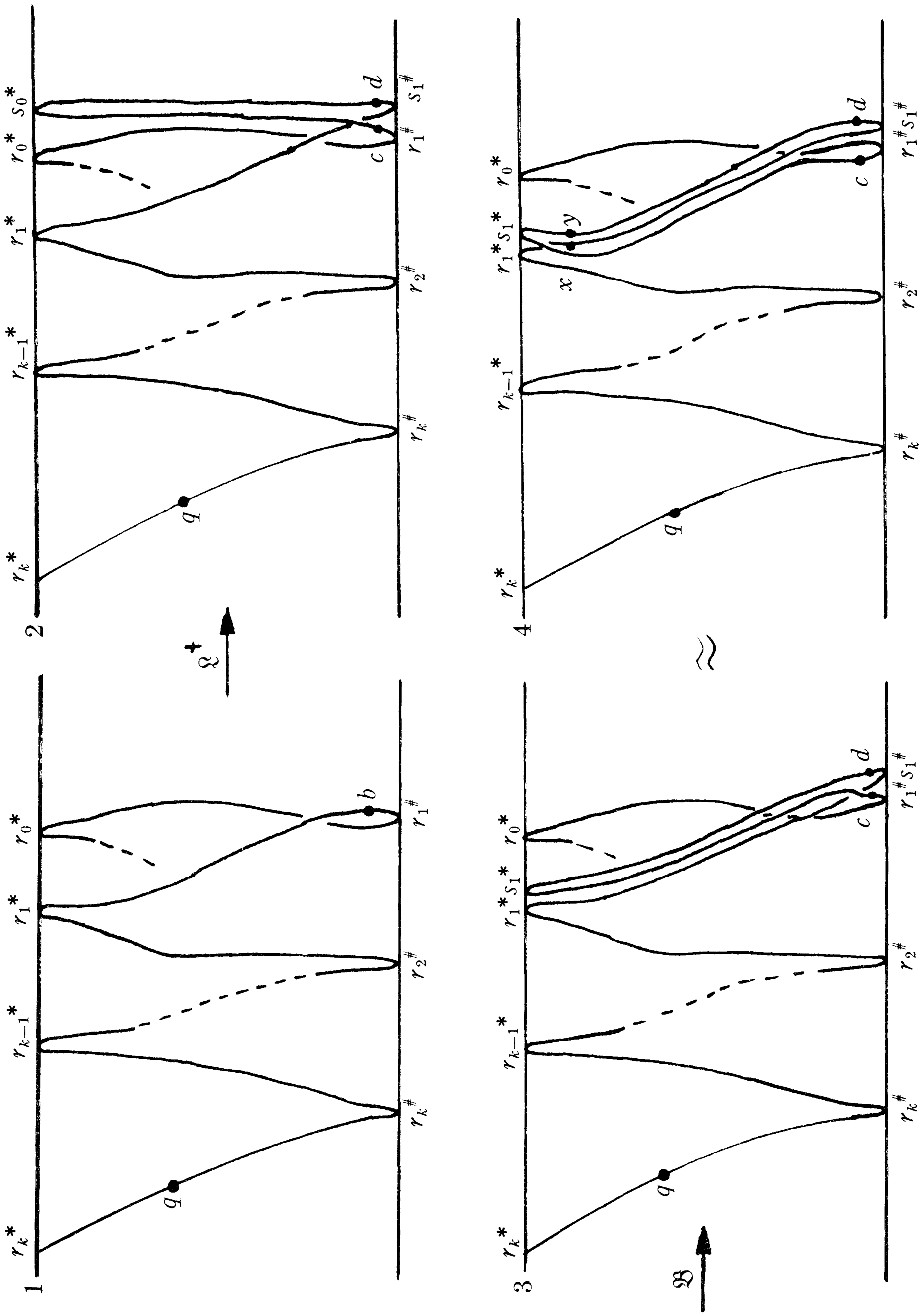
1
2
3
4
q
b
c
d
x
y
r_k^*
r_{k-1}^*
r_1^*
r_0^*
s_0^*
s_1^*
$r_k^\#$
$r_2^\#$
$r_1^\#$
$s_1^\#$
$\mathfrak{L}^+$
$\mathfrak{B}$
$\approx$

FIGURE 10

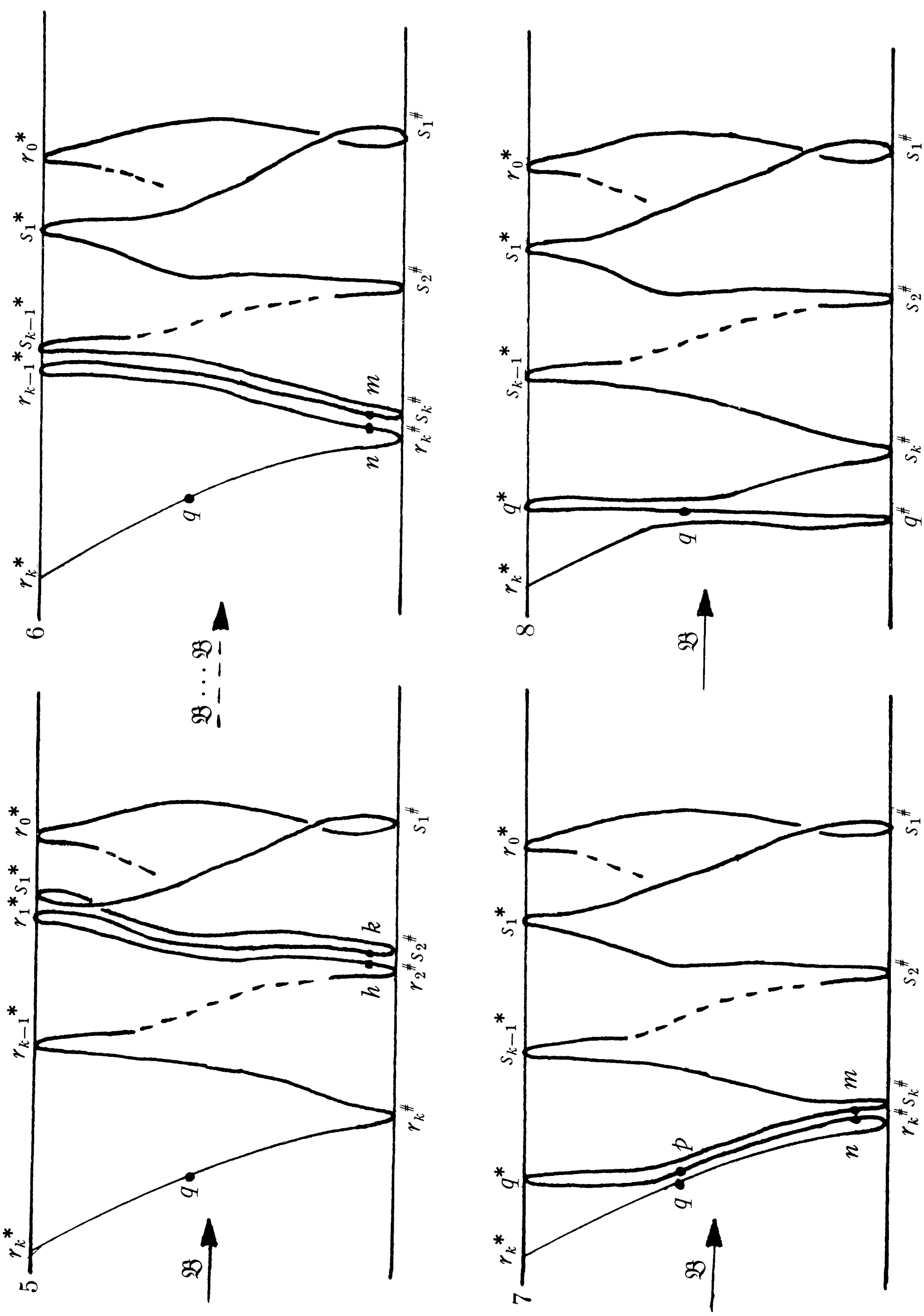

FIGURE 10 (*Concluded*)

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$c \in [r_1^{\#}, s_0^*]$ and $d \in [s_0^*, s_1^{\#}]$, with c and d very close to $r_1^{\#}$ and $s_1^{\#}$ respectively. Now perform a type $\mathfrak{B}$ move, pulling the spike at s_0^* back to the base cd, and shooting out a new spike to s_1^*, which is close to r_1^*, with the entire path $c\, s_1^*\, d$ very close to $s_1^{\#} r_1^*$ (Picture 3). Picture 4 is obtained from picture 3 by an isotopy. Next choose points y on $s_1^* s_1^{\#}$ and x on $s_1^{\#} r_1^*$, both very close to the upper boundary (and hence also to s_1^*). Perform a type $\mathfrak{B}$ move, retracting the spike at $s_1^{\#}$ to the base xy, and shooting out a new spike, along a path very close to $r_1^* r_2^{\#}$, to the point $s_2^{\#}$ on the lower boundary plane. Let h be a point on $r_1^* r_2^{\#}$ which is very close to $r_2^{\#}$. Then we have produced a move of type $\mathfrak{A}$ at the point h as the product $\mathfrak{B}_2 \mathfrak{B}_1 \mathfrak{L}^+$.

Repeating the construction above, it is now easy to see (picture 6) that there exist moves $\mathfrak{B}_3, \mathfrak{B}_4, \ldots, \mathfrak{B}_{2k}$ such that a type $\mathfrak{A}$ move at a point n on $r_{k-1}^* r_k^{\#}$, where n is close to $r_k^{\#}$, may be obtained as a product $\mathfrak{B}_{2k} \mathfrak{B}_{2k-1} \ldots \mathfrak{B}_1 \mathfrak{L}^+$.

Now apply a type $\mathfrak{B}$ move at the base nm (close to $r_k^{\#}$), pulling in the spike at r_{k-1}^* to nm, and shooting out a new spike along a path which stays close to $r_k^{\#}\, q$ up to q, and then goes straight up to q^* on the upper boundary plane. Finally, choose p on $q^* r_k^{\#}$, close to q, and apply $\mathfrak{B}_{2k+2}$ by pulling back the spike at $r_k^{\#}$ to the base qp, and shooting out a new spike straight to $q^{\#}$ on the lower boundary plane. Thus we have produced a type $\mathfrak{A}$ move at q as the product $\mathfrak{B}_{2k+2} \ldots \mathfrak{B}_1 \mathfrak{L}^+$, and the lemma is established.

LEMMA 10. *Let P have one component, and suppose that P^* is obtained from P by applying a type $\tilde{\mathfrak{B}}$ move, followed by a type $\mathfrak{L}^+$ move. Then, we may also obtain P^* from P, by first applying a type $\mathfrak{L}^+$ move, and following it by an appropriate sequence of type $\tilde{\mathfrak{B}}$ moves.*

Proof. If the type $\tilde{\mathfrak{B}}$ move was applied to move a spike on the upper boundary plane there is no problem, since a type $\mathfrak{L}$ move is always applied at a point arbitrarily close to the lower boundary plane, so that the two operations necessarily commute. Suppose that a type $\tilde{\mathfrak{B}}$ move has been applied to move a spike from $r^{\#}$ on the lower boundary to a new location at $s^{\#}$ on the lower boundary, by retracting it to the base $c'c''$ and shooting out a new spike from $c'c''$ (Figure 11). If a type $\mathfrak{L}^+$ move is then applied at the point b, and if b does not belong to the arc $c's^{\#}c''$, then again $\tilde{\mathfrak{B}}$ and $\mathfrak{L}^+$ commute, hence we may assume that b is on $c's^{\#}c''$, say on the segment $s^{\#}c''$, which is assumed to have larger x coordinates than $c's^{\#}$. It will be assumed further that b is very close to $s^{\#}$. After the type $\mathfrak{L}$ move our plat will be in the situation of picture 3 (Figure 11). We must show that it is possible to get from the situation of picture 3 to picture 1 by a series of moves of type $\tilde{\mathfrak{B}}$, followed by (at the very last step) a single move of type $\mathfrak{L}^-$. Note that since P is a plat, the arc $s^{\#}c''$ may be continued to a point t^* on the upper boundary plane. We now apply a type $\tilde{\mathfrak{B}}$ move, at the base $b'b''$ (picture 3), to move the spike at s_1^* to t_1^*, close to t^*. This motion will be assumed to be done by pulling back the spike at s_1^* to $b'b''$, and then shooting out a new spike along a path in a very small neighborhood of the arc $s_1^{\#}c''t^*$. An isotopy brings us to the situation of picture 5.

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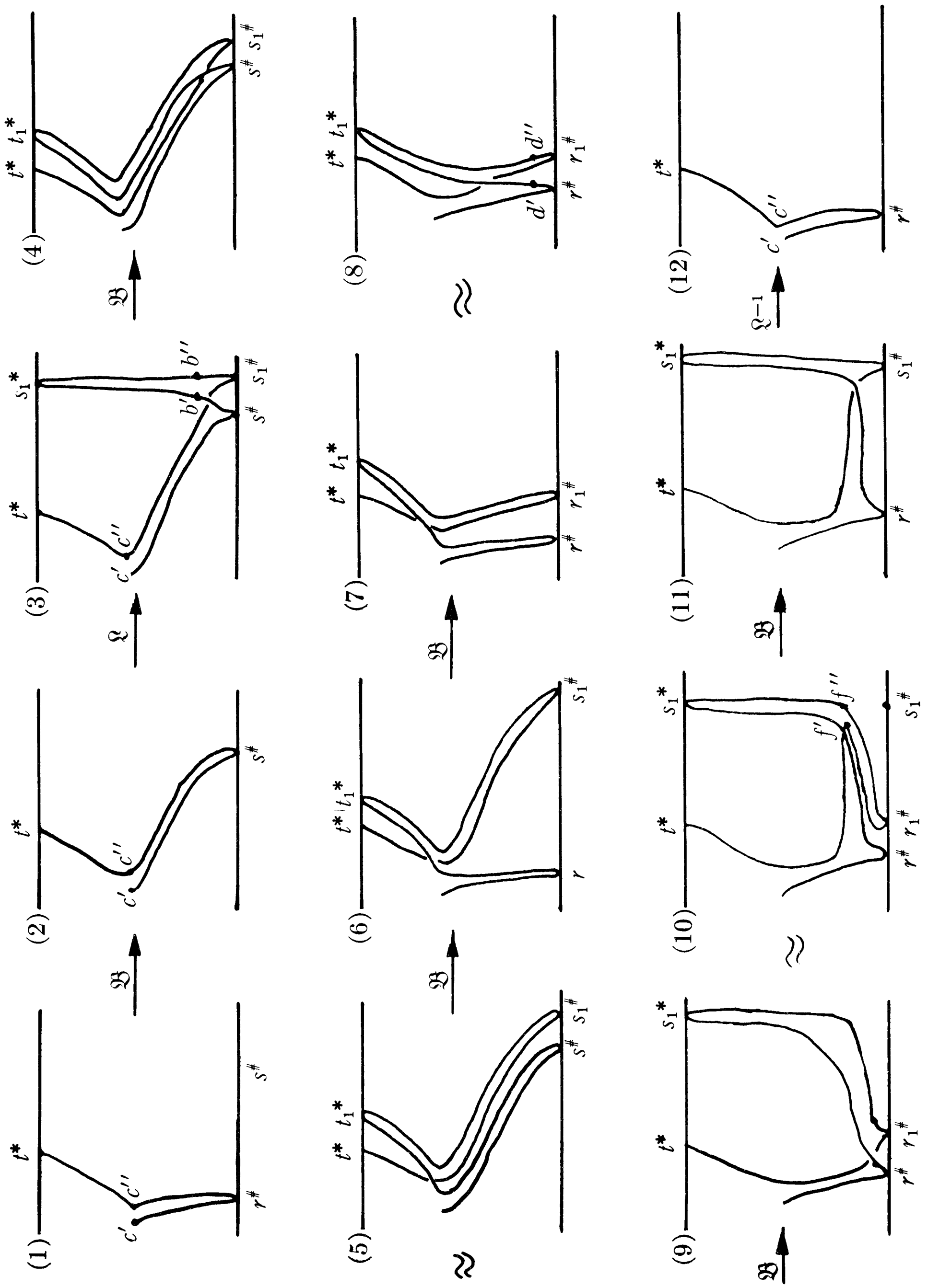

FIGURE 11

(The isotopies which are illustrated in Figure 11 could of course be absorbed in the type $\mathfrak{B}$ moves, however the procedure is easier to visualize if the isotopies are carried out in stages.) We now apply another type $\mathfrak{B}$ move, retracting the spike at $s^\#$ to the base $c'c''$ and shooting out a new spike to $r^\#$ along the path used in picture 1. To go from picture 6 to 7, copy this last move with the spike at $s_1^\#$ to move it to $r_1^\#$ (close to $r^\#$), along a path in a neighborhood of $c''r^\#$. An isotopy takes us to picture 8. Now pick two points d', d'' close to $r_1^\#$, and using them as a base retract the spike at t_1^* to $d'd''$ and shoot it out again to s_1^*, along a path which is at first very close to the lower boundary, and then straight up to s_1^* (recall that s_1^* has a maximal x-coordinate, by our original choice of s_1^* via a type $\mathfrak{K}$ move). See 9. Another isotopy takes us to picture 10. In picture 10, choose points f', f'' on the plat close to $s_1^\#$ on the lower boundary edge. Then (picture 11) apply a type $\tilde{\mathfrak{B}}$ move, pulling back the spike at $r_1^\#$ to f', f'', and shooting it out to $s_1^\#$. We are now ready to delete the trivial loop at $s_1^\#$ and s_1^* by a type $\mathfrak{K}$ move, to get picture 12. A final isotopy takes up to a picture, which is identical with picture 1. This completes the proof of Lemma 10.

LEMMA 11. *Let P, P^* be plats which define the same knot type. Suppose that $P = P_1 \to P_2 \to \ldots \to P_s = P^*$ is a sequence of plats joining P to P^*, with $P_j = \mathcal{M}_j P_{j-1}$ for each $j = 2, \ldots, s$, each $\mathcal{M}_j$ being a single move of type $\mathfrak{B}$, $\mathfrak{K}^+$ or $\mathfrak{K}^-$. Then there is another sequence $P = P_1^* \to P_2^* \to \ldots \to P_q^* = P^*$ joining P to P^*, with $P_j^* = \mathcal{M}_j^* P_{j-1}^*$, and*

$$\mathcal{M}_j^* \text{ is type } \mathfrak{K}^+ \text{ if } j = 2, \ldots, m \leqq q;$$

$$\mathcal{M}_j^* \text{ is type } \mathfrak{B} \text{ if } j = m+1, \ldots, s \leqq q;$$

$$\mathcal{M}_j^* \text{ is type } \mathfrak{K}^- \text{ if } j = s+1, \ldots, q.$$

Proof. If there exists an integer k such that $\mathcal{M}_k$ is type $\mathfrak{K}^+$ but $\mathcal{M}_{k-1}$ is not type $\mathfrak{K}^+$, then let k be the smallest such integer. If $\mathcal{M}_{k-1}$ is type $\mathfrak{B}$, apply Lemma 10 to replace the sequence $\mathcal{M}_k\mathcal{M}_{k-1} = \mathfrak{K}^+\mathfrak{B}$ by $\mathfrak{B}_t \ldots \mathfrak{B}_1\mathfrak{K}^+$. If $\mathcal{M}_{k-1}$ is type $\mathfrak{K}^-$, then $\mathfrak{K}^-$ and $\mathfrak{K}^+$ will cancel each other, and we may replace the sequence $\mathcal{M}_{k+1}\mathcal{M}_k\mathcal{M}_{k-1}\mathcal{M}_{k-2}$ with $\mathcal{M}_{k+1}\mathcal{M}_{k-2}$. This process may be repeated as often as necessary until we finally obtain a new chain $P = P_1^* \to P_2^* \to \ldots \to P_m^* \to \hat{P}_{m+1} \to \ldots \to \hat{P}_v = P^*$, where $P_j^* = \mathfrak{K}^+ P_{j-1}^*$ for each $j = 2, \ldots, m$, and each $\hat{P}_j$ is obtained from its predecessor by a single move of type $\mathfrak{B}$ or $\mathfrak{K}^-$.

If there now exists an integer t such that $\hat{\mathcal{M}}_t$ is a type $\mathfrak{K}^-$, but $\hat{\mathcal{M}}_{t+1}$ is type $\mathfrak{B}$, then let t be the largest such integer. Then, necessarily, $\hat{\mathcal{M}}_{t+1}$ is type $\mathfrak{B}$. Applying Lemma 10, we may now replace the sequence $\hat{\mathcal{M}}_{t+1}\hat{\mathcal{M}}_t = \mathfrak{B}\mathfrak{K}^-$ by $\mathfrak{K}^-\mathfrak{B}_1 \ldots \mathfrak{B}_n$, thereby reducing t. This process may be repeated as often as needed, until all $\mathfrak{K}^-$ moves are collected at the right of the chain. This proves Lemma 11.

We are now ready to prove Theorem 1. From Lemma 11 we know that, if P, P^* are plat representatives of the same knot type, then we may find sequences of plats

$$P = P_1^* \to P_2^* \to \ldots \to P_m^*,$$
$$P^* = P_q^* \to P_{q-1}^* \to \ldots \to P_s^*$$

such that for $2 \leqq j \leqq m$ the plat P_j^* is obtained from P_{j-1}^* by a move of type $\mathfrak{L}^+$, and if $q - 1 \geqq j \geqq s$ the plat P_{j-1}^* is obtained from P_j^* by a move of type $\mathfrak{L}^+$, while P_s^* is obtained from P_m^* by a sequence of moves of type $\mathfrak{B}$.

It only remains to interpret our moves algebraically. To do this we refer the reader back to Section 2, where the group B_{2n} and its subgroup K_{2n} were defined.

By Lemma 1, each plat P_j^* determines a well-defined element Φ_j of the group B_{2n_j}. Thus we have:

$$\begin{array}{ccccccccccccc} P = P_1^* & \overset{\mathfrak{L}^+}{\to} \ldots \overset{\mathfrak{L}^+}{\to} & P_m^* & \overset{\tilde{\mathfrak{B}}}{\to} \ldots \overset{\tilde{\mathfrak{B}}}{\to} & P_s^*, & \quad & P^* = P_q^* & \overset{\mathfrak{L}^+}{\to} \ldots \overset{\mathfrak{L}^+}{\to} & P_s^* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Phi_1 & \to \ldots \to & \Phi_m & \to \ldots \to & \Phi_s & & \Phi_q & \to \ldots \to & \Phi_s \end{array}$$

In Section 2, we defined the particular elements σ_i of B_{2n} as self-homeomorphisms of $(\partial E_+^3, \partial A)$. Also in Section 2, we described a method for constructing a geometric braid from a self-homeomorphism of $(\partial E_+^3, \partial A)$. Applying that construction to the elements $\sigma_i \in B_{2n}$ $(i = 1, 2, \ldots, 2n - 1)$ one obtains immediately the well-known fact that σ_i may be interpreted as an elementary geometric braid in which the ith string crosses over the $i + 1$st, all other strings being left invariant. Thus the plats defined by $\Phi \in B_{2n}$ and $\Phi\sigma_{2n} \in B_{2n+2}$ differ in the manner illustrated in Figure 3, i.e. the plat defined by the latter is obtained from that defined by the former by a single move of type $\mathfrak{L}^+$. Conversely, if $\hat{P}_j^*$ is obtained from $\hat{P}_{j-1}^*$ by a single move of type $\mathfrak{L}^+$, and if P_{j-1}^* is the plat defined by $\Phi_{j-1} \in B_{2n_{j-1}}$, then it must follow that $\hat{P}_j^*$ is the plat defined by $\Phi_j = \Phi_{j-1}\sigma_{2n_{j-1}} \in B_{2n_{j-1}+2}$. Thus, after a sequence of $m - 1$ moves of type $\mathfrak{L}^+$ we must have $\Phi_m = \Phi_1\sigma_{2n_1}\sigma_{2n_1+2} \ldots \sigma_{2n_1+2m-2}$. (Our convention here is the common algebraic convention of composing free automorphisms from left to right. This does not agree with the usual geometric convention of composing mappings from right to left.) Similarly, we obtain $\Phi_s = \Phi_q\sigma_{2n_q}\sigma_{2n_q+2} \ldots \sigma_{2n_q+2(q-s-1)}$.

To relate Φ_m to Φ_s we need the algebraic analogue of $P_j^* = \mathfrak{B}_j P_{j-1}^*$. Suppose first that $\mathfrak{B}$ moves a spike in the upper boundary plane from r^* to s^*, by retracting it to the base $c'c''$, and then sending it out again along a new path in the complement of P_{j-1}^* (cf Figure 5). We may without loss of generality assume that c' and c'' lie on a plane parallel to the xy plane, and that the projection of this plane onto the xz plane contains no double points of P_{j-1}^*. This plane will then divide the braid Φ_{j-1} into an "upper braid" Ψ_1 and a "lower braid" Ψ_2, each of which defines an element in the group $B_{2n_{j-1}}$, with $\Phi_{j-1} = \Psi_1\Psi_2$. Then, $\Phi_j = \Psi_1'\Psi_2$ for some $\Psi_1' \in B_{2n_{j-1}}$, where $\Phi_j\Phi_{j-1}^{-1} =$

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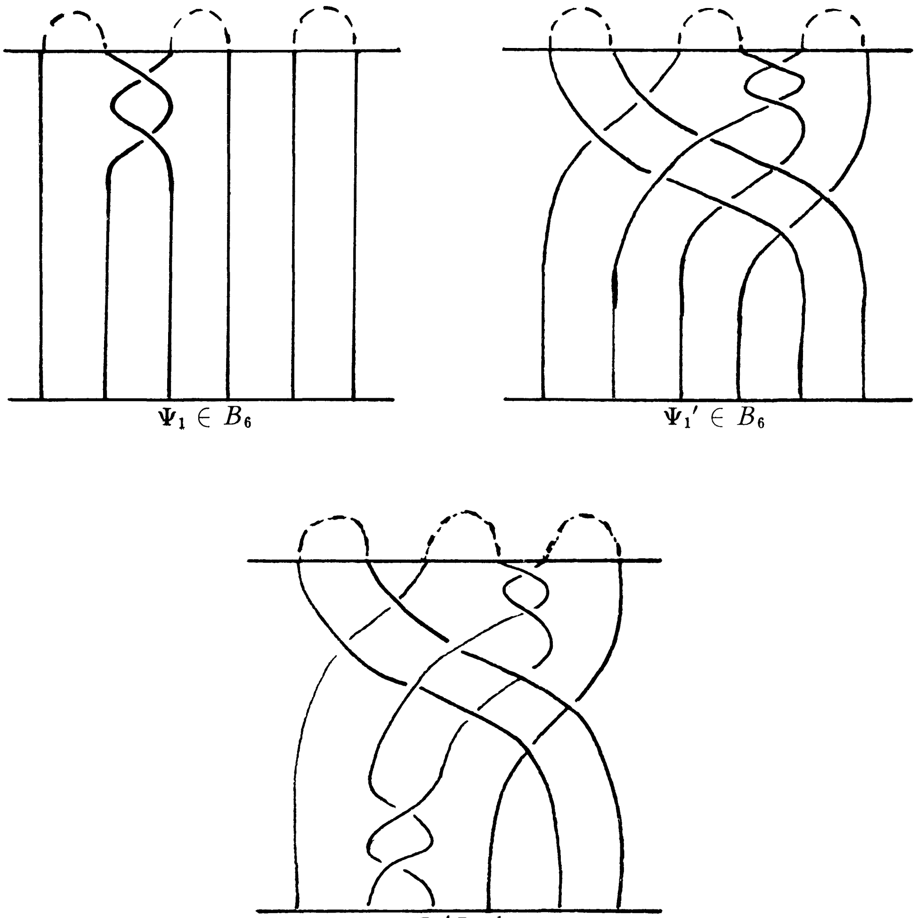
$\Psi_1'\Psi_1^{-1}$ is in the subgroup $K_{2n_{j-1}}$ of $B_{2n_{j-1}}$ defined in Section 2. This is illustrated in Figure 12, for a typical case. Thus $\Psi_1' = \kappa\Psi_1$, for some $\kappa \in K_{2n_{j-1}}$, and $\Phi_j = \kappa\Phi_{j-1}$.



$\Psi_1 \in B_6$ $\Psi_1' \in B_6$

$\Psi_1'\Psi_1^{-1}$

FIGURE 12

If, instead, $\mathfrak{B}$ had moved a spike on the *lower* boundary plane from $r^\#$ to $s^\#$, then $\Phi_j = \Psi_1\Psi_2'$, with $\Phi_{j-1}^{-1}\Phi_j \in K_{2n_{j-1}}$, so that $\Psi_2^{-1}\Psi_2' = \tau \in K_{2n_{j-1}}$, and $\Phi_j = \Phi_{j-1}\tau$, $\tau \in K_{2n_{j-1}}$. Thus, in the general case, Φ_j is in the same double coset (mod $K_{2n_{j-1}}$) as Φ_{j-1}. The "only if" part of Theorem 1 then follows immediately.

4. Stable equivalence of plat representations of links. Lemmas 6, 7, and 8 of Section 3 apply to both knots and links, but Lemmas 9 and 10 apply

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only to knots. We may, however, easily modify Lemmas 9 and 10 to cover the general case of links, and so obtain a generalization of Theorem 1 to links. The appropriate modification is to apply Lemmas 9 and 10 to the individual components of L. The analogue of Theorem 1 may then be seen to be as follows:

THEOREM 1′. *Let L_i $(i = 1, 2)$ be tame links. Let $\Phi_i \in B_{2n_i}$ be a plat representative of L_i, $i = 1, 2$. Then $L_1 \approx L_2$ if and only if, after adding a suitable number of "trivial loops" to each component of L_1 and of L_2 we obtain new plat representatives, $\Phi_1' \in$, $\Phi_2' \in B_{2n}$, for some $n \geqq \max(n_1, n_2)$, such that Φ_1' and Φ_2' are in the same double coset of B_{2n} modulo the subgroup K_{2n}.*

5. How much stabilizing is needed? Theorem 1 tells us that the questions of whether there is an effective algorithm to decide if an arbitrary pair of tame knots L_1, L_2 determine the same knot type is equivalent to two other questions:

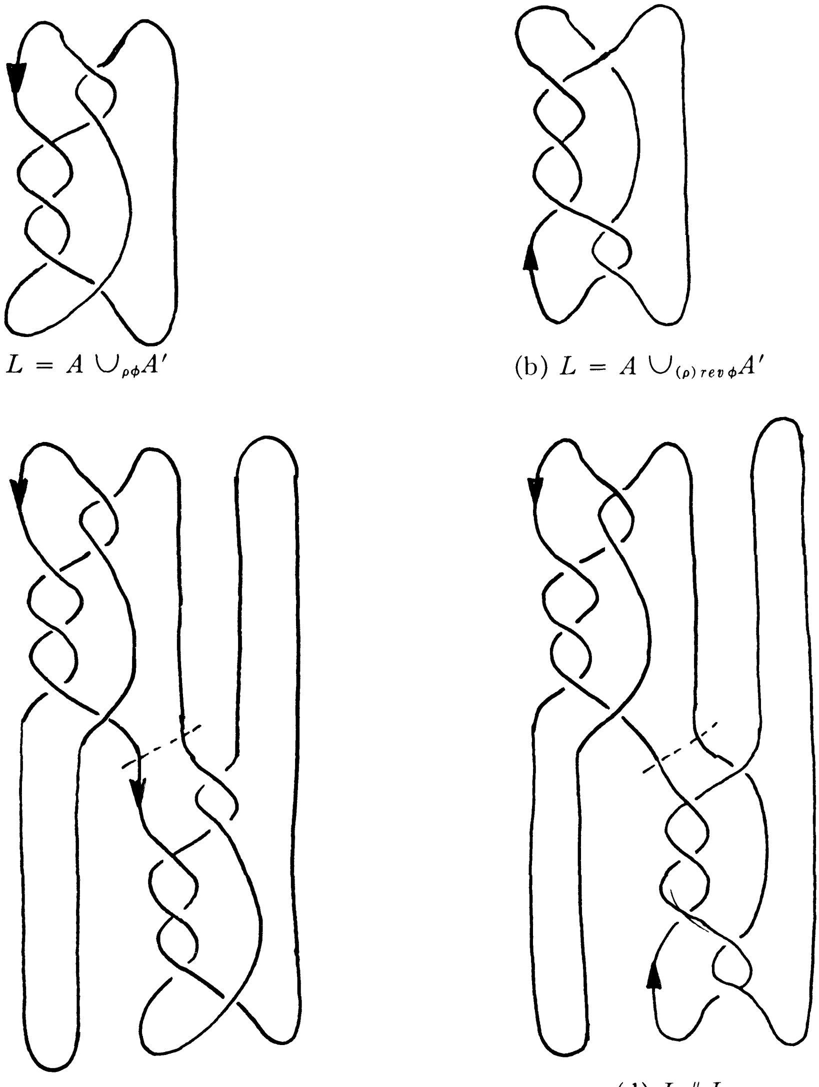
(1) Can a bound be placed on the integer t?

(2) Is the double coset problem in B_{2n} modulo K_{2n} decideable?

We consider here the first question.

A *lower* bound on t is easily obtained, because according to [**4**, Theorem 5.2], the minimum string index $2n$ of a plat representative of a link is precisely the bridge number of the link, a familiar link invariant. If it happened that, for knots of bridge number b, there was a $1 - 1$ correspondence between double cosets mod K_{2b} in B_{2b} and knot type, it would then be unnecessary to consider (1) any further, since by Lemma 10 the operations of altering string index (i.e. applying a type $\mathfrak{L}^+$ move, or (algebraically) of adding a braid generator σ_{2n} and increasing braid index by 2) is commutative with double coset multiplication mod K_{2n}; Theorem 1 would thus imply that, given *any* plat representatives $\Phi_i \in B_{2n_i}$ of L_i, $i = 1, 2$, with say $n_1 \leqq n_2$, than $L_1 \approx L_2$ if and only if $\Phi_1\sigma_{2n_1}\sigma_{2n_1+2} \ldots \sigma_{2n_2-2}$ is in the same double coset mod K_{2n_2} as Φ_2. This naive possibility is now shown to be false, however the particular manner in which it fails suggests interesting possibilities.

If L is a knot, and $P = A \cup_{\rho\varphi} A'$, $\Phi \in B_{2n}$ a plat representative for L, then another plat representative for L is obtained by turning the plat "upside down", i.e. looking at the image of P under a rotation $h: E^3 \to E^3$ of 180° about an axis parallel to the x axis through the point $(0, 0, 0.75)$. If Φ is expressed as a word $\sigma_{\mu_1}^{\epsilon_1}\sigma_{\mu_2}^{\epsilon_2} \ldots \sigma_{\mu_r}^{\epsilon_r}$ in the generators $\sigma_1, \ldots, \sigma_{2n-1}$ of B_{2n}, then $h(P) = A \cup_{\rho\varphi'} A'$ is represented by $\Phi' = \text{Rev } \Phi = \sigma_{\mu_r}^{\epsilon_r} \ldots \sigma_{\mu_2}^{\epsilon_2}\sigma_{\mu_1}^{\epsilon_1}$. In particular, let $n = 2$, and let L be the 2-bridge knot defined by the 4-plat $\Phi = \sigma_2^2\sigma_1^3\sigma_2^{-1} \in B_4$, so that $\text{Rev } \Phi = \sigma_2^{-1}\sigma_1^3\sigma_2^2$ (see Figure 13(a) and (b)). It follows from [**6**] that the 2-fold covering space of S^3 branched over the knot L is the lens space $L(7, 2)$, and from [**5**] that Φ and Rev Φ are in distinct double cosets modulo K_4. (In [**5**] a correspondence is set up between equivalence classes of Heegard splittings of 3-manifolds and double cosets in the group $\mathfrak{M}_g$ mod $\mathfrak{N}_g$, defined in Section 1 of this paper. The lens space $L(7, 2)$ is shown in [**5**] to admit 4 genus one Heegaard splittings which are equivalent, but



(a) $L = A \cup_{\rho\phi} A'$ (b) $L = A \cup_{(\rho)\, rev\, \phi} A'$

(c) $L \# L$ (d) $L \# L$

FIGURE 13

no 2 of which are strongly equivalent. This implies that Φ and Rev Φ are in distinct double cosets mod K_4). Thus it appears that if we wish to restrict our attention to small values of b, it will be necessary to admit at least *two* double cosets mod K_{2t}, that of Φ and of Rev Φ. (Of course, the double cosets of Φ and Rev Φ are not necessarily distinct; this will depend on knot type.)

A more serious difficulty arises when we consider the knot $L \# L$ formed by taking the connected sum of L with itself (see [**9**]). since there are several "natural" 6-plats which represent $L \# L$, one obtained by pasting the projec-

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tion of Figure (13a) to itself, and another by pasting the projection of Figure (13a) to that of Figure (13b) (see Figures 13c, 13d respectively). Again, the methods of [**5**] (see in particular Theorem 2) imply that the 6-braids determined by (13a) and (13b) are in distinct double cosets modulo K_6.

More subtle examples have been obtained recently by Montesinos of *prime* knots which admit two plat representations of minimum string index which are in distinct double cosets modulo K_{2n}. (Montesinos' examples were obtained after this paper was completed [see Montesinos, "Plat representations of prime links are not unique"], but before it went to press.) Thus the problem of stabilizing appears to be an unavoidable part of the knot problem.

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