ON THE STRUCTURE OF SEMI-PRIME RINGS AND THEIR RINGS OF QUOTIENTS

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We are mainly interested in the study of prime and semi-prime rings and their rings of quotients. However, our argument proceeds largely in the category of modules (§ 1 to 4) and bimodules (§ 5 to 7).

After a brief description of the generalized rings of quotients introduced recently by Johnson, Utumi, and Findlay and the present author, we study a closure operation on the lattice of submodules of a module. For the lattice of left ideals of a ring, the concept of closed submodules reduces to the M-ideals of Utumi. The lattice of closed submodules of a module is always a complete modular lattice. We are specially interested in the case when it is a complemented lattice. This happens, in particular, when the singular submodule of Johnson and Wong vanishes. We consider the lattice of closed right ideals of a prime ring S and determine the maximal ring of right quotients of S in the case when this lattice has atoms. Our results for such prime rings are closely related to recent results by Goldie, Lesieur and Croisot, and Johnson.

All proofs in § 2 and § 3, concerning the closure operation on the lattice of all submodules of a module, have been carefully designed to carry over to an essentially different situation in § 5. There we study a closure operation, called *b-closure*, on the lattice of all submodules of a bimodule. This does not reduce to the original closure operation, even when the bimodule is converted into a right module. The connection between the two closure operations is rather exemplified by the following: Call a submodule *dense* (*b-dense*) if its closure (b-closure) is the whole module. Then an ideal in a ring is b-dense if and only if it is dense both as a right ideal and as a left ideal.

Each bimodule M possesses a *b-completion*, that is a largest bimodule in which M is b-dense. The b-completion of a ring S is also a ring and coincides with the so-called maximal ring of right and left quotients, first introduced in a special case by Utumi and defined in general by Johnson and Wong. The b-completion of a prime ring with non-zero socle is described symmetrically in terms of dual vector spaces.

The b-closed ideals of a semi-prime ring S are precisely its annihilator ideals. They form a complete Boolean algebra, which is isomorphic with the algebra of regular open sets in the prime ideal space of S. If S is also b-complete, the b-closed ideals are precisely the direct summands of S. This fact is exploited to obtain a structure theorem: Every such ring S is the direct sum of two

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rings C and C^*, where C is the complete direct product of b-complete prime rings and the lattice of annihilator ideals of C^* has no atoms.

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1. Survey of generalized rings of quotients.

1.1. If S is any associative ring, a right S-module M_S consists of an additive abelian group M and a mapping $(m, s) \to ms$ of $M \times S$ into M satisfying the obvious distributive and associative laws. Left modules are defined dually. The ring S gives rise, in an obvious way, to the right module S_S and the left module ${}_SS$.

A right module M_S is called *unitary* if S has a unity element 1 and $m1 = m$ for all $m \in M$. Every right module M_S can be converted into a unitary module $M_{S^\#}$ as follows: $S^\#$ is the ring consisting of the additive group $S \oplus Z$, Z the ring of integers, with multiplication defined by

$$(s + z)(s' + z') = (ss' + sz' + zs') + zz',$$

for $s, s' \in S$ and $z, z' \in Z$. One then puts

$$m(s + z) = ms + mz,$$

for $m \in M$, $s \in S$, and $z \in Z$.

1.2. Findlay and the present author **(5)** investigated a relation among three modules A_S, B_S, and C_S. They wrote $A \leqslant B(C_S)$ as an abbreviation for any of the following three equivalent statements:

(1) A_S is a submodule of B_S and, for any submodule E_S of $A_S - B_S$, $\mathrm{Hom}_S(E, C) = 0$. Here $A - B$ is the difference (or quotient) module of A modulo B.

(2) A_S is a submodule of B_S and, if $\phi \in \mathrm{Hom}_S(D, C)$, where D_S is any submodule of B_S and $A \subset \ker \phi$, the kernel of ϕ, then the image $\mathrm{im}\, \phi = 0$.

(3) A_S is a submodule of B_S and, for any $b \in B$ and any $0 \neq c \in C$, there exists an $s \in S$ and an integer z such that $bs + bz \in A$ and $cs + cz \neq 0$. If the modules in question are unitary, z can be taken to be 0.

1.3. If $A \leqslant B(B_S)$, B_S was called a *rational* extension of A_S. It was shown that any module M_S possesses a largest rational extension (*rational completion*) $\bar{M}_S$, unique up to isomorphism over M_S. $\bar{M}_S$ is *rationally complete* in the following sense: If $A \leqslant B(\bar{M}_S)$, then every $\phi \in \mathrm{Hom}_S(A, \bar{M})$ can be extended to a (unique) $\bar{\phi} \in \mathrm{Hom}_S(B, \bar{M})$. Two constructions of $\bar{M}_S$ were given:

(1) Let $M_S{}^I$ be the minimal injective extension **(4)** of M_S, then $\bar{M}_S$ consists of all those elements of $M_S{}^I$ which are annihilated by every endomorphism of $M_S{}^I$ which annihilates M_S.

(2) The right ideals D of $S^{\#}$ such that $D \leqslant S^{\#}(M_S)$ form a directed set under inclusion, and the additive groups $\mathrm{Hom}_S(D, M)$ form a direct system. Their direct limit is turned into an S-module $\bar{M}_S$ in a natural way. If M_S is unitary, one may replace the $S^{\#}$ of this construction by S.

1.4. Johnson and Wong **(19)** called a submodule L_S of M_S *large* if it has non-zero intersection with every non-zero submodule of M_S. They introduced the *singular* submodule $J(M_S)$ of a module M_S. It consists of all elements of M which annihilate a large right ideal of S. They showed that if $J(M_S) = 0$, then also $J(\bar{M}_S) = 0$ and $\bar{M}_S$ is injective. Moreover, the ring of endomorphisms of $\bar{M}_S$ is regular (in the sense of von Neumann) and injective as a right module.

1.5. If S is any associative ring, Q_S the rational completion of S_S, then Q is actually a ring extending S. Q coincides with the *maximal ring of right quotients* of S, previously defined by Johnson **(10)** and Utumi **(17)** in the following important cases.

Johnson's case. The singular submodule of S_S is actually an ideal, call it the *right singular ideal.* Johnson assumed that this ideal vanishes. He showed that the right singular ideal of Q then also vanishes and that Q is regular and injective as a right Q-module.

Utumi's case. Utumi assumed that, for any non-zero element s of S, $sS \neq 0$. It is, in fact, easily seen that this is a necessary and sufficient condition for Q to contain a unity element **(5**, 6.2**)**.

Among many other interesting applications, Utumi computed the maximal ring of left quotients of any primitive ring S with non-zero socle **(17**, 5.1**)**. Thus, let $V = eS$ be a minimal right ideal of such a ring, e an idempotent element of S **(9**, p. 57, Proposition 1**)**. Then $D = eSe$ is known to be a skew-field, and V is a vector space ${}_D V$. Utumi showed that $\mathrm{Hom}_D(V, V)$ is the maximal ring of left quotients of S.

1.6. For an integral domain S, the maximal ring Q of right quotients coincides with the classical field of quotients. If S is not an integral domain, there may also exist a "classical" ring of quotients. For example, if S is commutative, then this classical ring of quotients Q_{cl} consists of all ratios s/s', where $s \in S$ and s' is any *regular* element of S, in the sense that $s''s' \neq 0$ for any non-zero element s''. However, Q_{cl} may be smaller than Q. For instance **(2)**, if S is any Boolean ring, then $Q_{\mathrm{cl}} \cong S$, but Q is the Dedekind–MacNeille completion of S.

1.7. The ternary relation $A \leqslant B(C_S)$ has a number of properties, which are easily derived from the definition. We state them here for later reference.

P0. If $B - A \cong B' - A'$ and $C \cong C'$, then $A \leqslant B(C_S)$ implies $A' \leqslant B'(C_S')$.

P1. If $0 \leqslant C(C_S)$, then $C = 0$.

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P2. If $A \leqslant B(C_S)$ and D_S is a submodule of C_S, then $A \leqslant B(D_S)$.

P3. If D_S is a submodule of B_S containing the submodule A_S (that is, $A \subset D \subset B$), then

$$A \leqslant B(C_S) \Leftrightarrow \text{both } A \leqslant D(C_S) \text{ and } D \leqslant B(C_S).$$

P4. $A \leqslant A(C_S)$.

P5. If $A \leqslant B(C_S)$ and $C \leqslant D(D_S)$, then $A \leqslant B(D_S)$.

Actually, it was shown in **(5)** that the second condition of P5 can be replaced by the weaker assumption that C_S is a large submodule of D_S. This stronger result will not be used here.

We mention also the following property, which has to do with change of rings **(5**, 5.5**)**.

(†) For any modules A_T, B_T, and C_T, if S is a subring of T such that $S \leqslant T(C_S)$, then

$$A \leqslant B(C_T) \Leftrightarrow A \leqslant B(C_S).$$

2. The lattice associated with a module. All modules are understood to be right S-modules.

2.1. Let A be a submodule of M. There is a largest submodule A^c of M containing A such that $A \leqslant A^c(M)$. This may be constructed as follows:

$$\begin{aligned} A^c &= \{m \in M | \quad A \leqslant A + mS^{\#}(M_S)\} \\ &= \{m \in M | \quad m^{-1}A \leqslant S^{\#}(M_S)\}. \end{aligned}$$

The second formula is due to Findlay. Here

$$m^{-1}A = \{x \in S^{\#} | \quad mx \in A\}.$$

2.2. The assignment $c: A \rightarrow A^c$ is a closure operation on the lattice of all submodules of M. It has the following properties:

C1. $0^c = 0$.

C2. $(A \cap B)^c = A^c \cap B^c$.

C3. If $\phi \in \mathrm{Hom}_S(M, M)$, then $\phi(A^c) \subset (\phi A)^c$.

These correspond to A1, A2, and half of A3 of Johnson's "structures" on rings **(11)**.

Proof.

(C1) Since $0 \leqslant 0^c(M)$, therefore $0 \leqslant 0^c(0^c)$, by P2, hence $0^c = 0$, by P1.

(C2) Since c is a closure operation, $(A \cap B)^c \subset A^c \cap B^c$. To show the converse, observe that $A \leqslant A^c(M)$. From this we deduce that $A \leqslant A^c \cap (A + B)(M)$, by P3, that is $A \leqslant A + (A^c \cap B)(M)$, by the modular law. Now

$$(A + (A^c \cap B)) - A \cong (A^c \cap B) - (A \cap B),$$

by one of the isomorphism theorems of group theory. Therefore $A \cap B \leqslant A^c \cap B(M)$, by P0. Similarly we deduce from $B \leqslant B^c(M)$ that $A^c \cap B \leqslant$

$A^c \cap B^c(M)$. In view of P3, both these results together imply that $A \cap B \leqslant A^c \cap B^c(M)$.

(C3) Let $\phi \in \mathrm{Hom}_S(M, M)$, $K = A^c \cap \ker \phi$. Now $A \leqslant A^c(M)$, $\phi A^c \cong A^c - K$, $\phi A \cong (A + K) - K$, and $\phi A^c - \phi A \cong A^c - (A + K)$. By P3 and P0, $\phi A \leqslant \phi A^c(M)$, hence $\phi A^c \subset (\phi A)^c$, as required.

2.3. PROPOSITION. *The lattice $L(M)$ of closed submodules of M is a complete modular lattice, with set-intersection as meet.*

Proof. That we have a complete lattice follows from the fact that we have a closure operation. The join of two or more submodules of M is defined by

$$A \vee B = (A + B)^c, \quad \bigvee_{i \in I} A_i = \Big(\sum_{i \in I} A_i\Big)^c.$$

Finally, let A, B, and C be submodules of M and assume that $B \subset A$. Then

$$\begin{aligned} A \cap (B \vee C) &= A^c \cap (B + C)^c \\ &= (A \cap (B + C))^c \\ &= (B + (A \cap C))^c \\ &= B \vee (A \cap C), \end{aligned}$$

using C2 and the modular law for the lattice of all submodules of M.

2.4. A submodule K of M will be called *dense* if $K^c = M$. One easily verifies that every dense submodule is large.

LEMMA. *If K is dense in M, A any submodule of K, then $A^c \cap K$ is the closure of A in K.*

Proof. Since $A \leqslant A^c$, we have $A \leqslant A^c \cap K(K)$, by P2 and P3. Therefore $A^c \cap K$ is contained in the closure A^d of A in K.

Now $A \leqslant A^d(K)$, hence $A \leqslant A^d(M)$, by P5. Therefore $A^d \subset A^c$, and so $A^d \subset A^c \cap K$.

Note. We should really write $A^{c(M)}$ for A^c and $A^{c(K)}$ for A^d, but we have endeavoured not to make the notation too heavy.

2.5. The following partly generalizes a result by Utumi (**17**, Theorem 2).

PROPOSITION. *If K is dense in M, then $L(K)$ and $L(M)$ are isomorphic lattices under the inverse correspondences*

$$A \to A^c, \qquad B \to B \cap K,$$

where $A \in L(K)$ and $B \in L(M)$.

Proof. Again let d denote the closure operation in K. We observe that clearly $A^c \in L(M)$ and that $B \cap K \in L(K)$, since

$$(B \cap K)^d = B^d \cap K^d = B^c \cap K \cap K = B \cap K,$$

by C2 and the above lemma.

Next, we note that the two mappings are inverses. For $A^c \cap K = A^d = A$, by the lemma, and $(B \cap K)^c = B^c \cap K^c = B \cap M = B$, by C2 and the fact that K is dense in M.

Finally, we observe that the two mappings are meet-isomorphisms, hence lattice isomorphisms. For $B \cap B' \cap K = B \cap K \cap B' \cap K$ and $(A \cap A')^c = A^c \cap A'^c$.

2.6. Proposition. *If K is a closed submodule of M, then any closed submodule of K is closed in M.*

Proof. Let A be a closed submodule of K, then $A \leqslant A^c(M)$, hence $A \leqslant A^c \cap K(K)$, by P2 and P3. Thus, $A^c \cap K \subset A$, which is closed in K. But $A \subset A^c$ and $A \subset K$, hence $A = A^c \cap K = A^c \cap K^c = (A \cap K)^c = A^c$, in view of C2.

2.7. Examples of closed submodules are the following submodules K of M:

(1) K is maximal such that $K \cap L = 0$, for some submodule L of M.

(2) $K = \{m \in M | \ Fm = 0\}$, for some subset F of $\mathrm{Hom}_S(M, M)$.

(3) K is a direct summand of M.

(4) K is rationally complete.

Indeed, (1) follows easily from the known fact that every dense submodule is large, (2) follows immediately from C3, and (3) is a special case of (2). Finally, assume that K is rationally complete, K^c its closure in M. Then K^c is a rational extension of K and therefore coincides with K, and so K is closed in M.

2.8. By the *socle* of a complete lattice we shall understand the join of all its *atoms*, that is its minimal non-zero elements.

Proposition. *The socle of $L(M)$ is contained in every large closed submodule of M. It is mapped into itself by every endomorphism of M.*

Proof. Let A be an atom of $L(M)$, L a large closed submodule of M. Since $A \neq 0$, we have $A \cap L \neq 0$. Since A and L are closed, so is $A \cap L$. Since A is an atom, $A \cap L = A$, that is $A \subset L$. Thus L contains all atoms, hence their join.

Let $\phi \in \mathrm{Hom}_S(M, M)$ and let $\{A_i\}_{i \in I}$ be the set of all atoms of $L(M)$. By C3,

$$\phi\Big(\sum_{i \in I} A_i\Big)^c \subset \Big(\phi \sum_{i \in I} A_i\Big)^c \subset \Big(\sum_{i \in I} (\phi A_i)^c\Big)^c.$$

The result will follow if we show that the $(\phi A_i)^c$ are all 0 or atoms.

Let $A = A_i$ be any atom of $L(M)$. Any submodule of ϕA has the form ϕB, where $K \subset B \subset A$, K being the kernel of ϕ. Assume $B \neq 0$, then $\phi A - \phi B \cong A - B$. Now $B \leqslant A(M)$, hence $\phi B \leqslant \phi A(M)$, by P0. Therefore $\phi A \subset (\phi B)^c$, and so $(\phi A)^c \subset (\phi B)^c$.

Now let C be any closed submodule of M such that $0 \neq C \subset (\phi A)^c$. Since

ϕA is a large submodule of $(\phi A)^c$, $C \cap \phi A$ is a non-zero submodule of ϕA, hence has the form ϕB, where $B \neq 0$. By the above and C2,

$$(\phi A)^c \subset (C \cap \phi A)^c = C^c \cap (\phi A)^c = C.$$

Thus $(\phi A)^c$ is an atom, as remained to be shown.

2.9. PROPOSITION. *If M is any module, the socle of $L(M)$ is the closure of the discrete direct sum of some of its atoms. If $L(M)$ is a distributive lattice, then its socle is even the closure of the discrete direct sum of all the atoms.*

Proof. The argument for the first result is standard, for example, (**9**, p. 61). Indeed, let $\{A_i\}_{i \in I}$ be the set of all atoms of $L(M)$. By Zorn's lemma, one finds a maximal subset J of I such that, for all $i \in I$,

$$A_i \cap \bigvee_{j \in J - \{i\}} A_j = 0.$$

Now, for any $i \in I$, $A_i \cap \bigvee_{j \in J} A_j = 0$ or $= A_i$. By maximality of J, it is easily shown to be not 0, hence $A_i \subset \bigvee_{j \in J} A_j$. The first result now follows.

Next, assume that $L(M)$ is a distributive lattice. We will show that

$$A_i \cap \sum_{j \in I - \{i\}} A_j = 0,$$

for any $i \in I$. Thus, suppose that m belongs to the set denoted by the left side of this equation. Then there is a finite subset F of $I - \{i\}$ such that

$$m \in A_i \cap \sum_{j \in F} A_j \subset \bigvee_{j \in F} (A_i \cap A_j),$$

by the distributive law. Since $i \notin F$, A_i and A_j are distinct atoms, hence $A_i \cap A_j = 0$ for all $j \in F$. Therefore $m = 0$, as required.

3. Complemented lattices. Unless otherwise stated, all modules are still assumed to be right S-modules.

3.1. Of special interest is the case where the lattice of closed submodules of a module is complemented.

LEMMA *$L(M)$ is complemented if and only if every large submodule of M is dense.*

We recall that a large submodule is one that has non-zero intersection with every non-zero submodule.

Proof. Assume $L(M)$ is complemented. This means that for every closed submodule A there is a closed submodule B such that $A \cap B = 0$ and $A \vee B = M$. Let L be any large submodule of M, then L^c will have a complement $K = K^c$. But then $K \cap L = 0$ and so $K = 0$. Hence $L^c = L^c \vee K = M$.

Conversely, assume the condition and let A be any closed submodule of M. Using Zorn's lemma, we find a maximal B such that $A \cap B = 0$. By 2.7(1),

B is closed. A well-known argument **(10)** now shows that $A+B$ is a large submodule of M. By assumption, $A + B$ is dense, hence its closure $A \vee B = M$. Thus B is a complement of A.

Our proof is now complete. Incidentally, we have shown:

If $L(M)$ is complemented and $A \in L(M)$, then any maximal submodule B of M such that $A \cap B = 0$ is a complement of A.

3.2. PROPOSITION. *If the lattice $L(M)$ associated with a module M is complemented then so is the corresponding lattice of any submodule and of any rational extension of M.*

Proof. Let $L(M)$ be complemented. If N is a rational extension of M, then $L(M) \cong L(N)$, by 2.5, hence $L(N)$ is also complemented.

Now let A be any submodule of M. Since A is dense in A^c, $L(A)$ and $L(A^c)$ are isomorphic, by 2.5. Thus it suffices to show that $L(K)$ is complemented, for any closed submodule K.

Let $B \in L(K) \subset L(M)$, by 2.6. Hence there exists $C \in L(M)$ such that $B \cap C = 0$ and $B \vee C = M$. We claim that $(C \cap K)^d$ is a complement of B in $L(K)$, where d is the closure operation for submodules of K.

Indeed, $C \cap K$ is a large submodule of $(C \cap K)^d$, hence $B \cap (C \cap K)^d = 0$. Moreover, by the modular law and C2,

$$(B + (C \cap K))^c = ((B + C) \cap K)^c = (B + C)^c \cap K^c = M \cap K = K.$$

Therefore, in view of 2.6,

$$\begin{aligned} K = (B + (C \cap K))^c &\subset ((B + (C \cap K))^d)^c \\ &= (B + (C \cap K))^d \subset (B + (C \cap K)^d)^d, \end{aligned}$$

hence the right side $= K$, as required.

3.3. THEOREM. *If M is rationally complete and $L(M)$ is complemented, then the following conclusions hold:*

(a) *Every closed submodule of M is a direct summand.*

(b) *For any submodule D of M, any $\phi \in \mathrm{Hom}_S(D, M)$ may be extended to an endomorphism of M.*

(c) *$F = \mathrm{Hom}_S(M, M)$ is a regular ring.*

(d) *The lattice $L(M)$ is isomorphic with the lattice of principal right ideals of F.*

(e) *F is injective as a right F-module.*

Proof.

(a) Let A be a closed submodule of M. By assumption, it has a complement B so that $A \cap B = 0$ and $A \vee B = M$. Consider the map $\phi \in \mathrm{Hom}_S(A + B, M)$ defined by $\phi(a + b) = a$. By rational completeness, this may be extended to $\psi \in \mathrm{Hom}_S(M, M)$. We have

$$\psi M = \psi(A + B)^c \subset (\psi(A + B))^c = A^c = A,$$

by C3. Thus, for any $m \in M$,

$$\psi^2 m = \psi(\psi m) = \phi(\psi m) = \psi m,$$

and so ψ is a decomposition operator.

(b) Let $D \subset M$, $\phi \in \mathrm{Hom}_S(D, M)$. By rational completeness of M, ϕ may be extended to $\phi' \in \mathrm{Hom}_S(D^c, M)$. By (a), D^c is a direct summand of M, hence ϕ' may be extended further to an element of $\mathrm{Hom}_S(M, M)$.

(c) Let $f \in F = \mathrm{Hom}_S(M, M)$. We observe that $K = \ker f$ is closed by 2.7(2). By (a), K is a direct summand, hence $M = K + H$ and $K \cap H = 0$. Thus f induces an isomorphism $g: H \to fH$. By (b), $g^{-1}: fH \to H$ may be extended to $f' \in F$. For any $k \in K$, $h \in H$, we thus have

$$ff'f(k + h) = ff'0 + fg^{-1}gh = fk + fh = f(k + h).$$

Therefore $ff'f = f$.

(d) This is proved like Johnson's theorem **(12**, II, 7.5**)**, by showing that, for any idempotent $e \in F$, the principal right ideal eF of F determines the direct summand eM of M and vice versa. Thus $eM = (eF)M$ and $eF = \{f \in F | fM \subset eM\}$.

(e) This is proved like **(19**, Theorem 5**)**.

3.4. Looking at the above proof, we find that the conditions of the theorem can be somewhat relaxed. Instead of rational completeness, it suffices to assume this:

For any submodule D of M, every $\phi \in \mathrm{Hom}_S(D, M)$ can be extended to some (necessarily unique) $\phi' \in \mathrm{Hom}_S(D^c, M)$.

It is easily seen that this condition is equivalent to the following:

M is mapped into itself by every endomorphism of the rational completion $\bar{M}$ of M.

3.5. *Examples.* The lattice $L(M)$ will be complemented if the singular submodule $J(M) = 0$. Johnson and Wong proved (c) and (e) for this important case. However this is not the only example.

The ring $S = Z_p$ of integers modulo the prime p may be regarded as a right Z-module. As such, its singular submodule $J(S_Z) = Z_p \neq 0$. Now, $L(S_Z)$ has only two elements, hence is trivially complemented. It can also be shown that S_Z is rationally complete.

Johnson and Wong **(19**, Theorem 5**)** have also shown that $\bar{M}$ is injective when $J(M) = 0$. This result cannot be generalized to the case when $L(M)$ is complemented. For Z_p, regarded as a Z-module, is not divisible.

3.6. We may ask when the lattice associated with a module consists of only two elements, that is, every non-zero submodule is dense. Goldie **(5)** has called a non-zero module *uniform* if every non-zero submodule is large. Thus, by 3.1, $L(M)$ has exactly two elements if and only if M is uniform and $L(M)$ is complemented.

Of special interest is the case when $S = Z$, the ring of integers.

PROPOSITION. *If M is an additive abelian group (Z-module), then $L(M)$ has exactly two elements if and only if M is cyclic of prime order or a subgroup of the additive group of rationals.*

We shall omit the proof, which depends on standard theorems in the theory of abelian groups.

3.7. A lattice is called *atomic* if every non-zero element contains ($\geqslant$) an atom, or minimal non-zero element.

PROPOSITION. *If the lattice $L(M)$ is complemented and atomic, then its socle is M. If the socle of $L(M)$ is M, then $L(M)$ is complemented.*

Proof. Assume that $L(M)$ is atomic and complemented. Let C be its socle, D a complement of C. Since $C \cap D = 0$, D contains no atoms, hence $D = 0$. Therefore $M = (C + D)^c = C^c = C$.

Conversely, suppose that $C = M$. By 2.8, every large, closed submodule of M coincides with M. By 3.1, $L(M)$ is complemented.

4. On prime rings.

4.1. An associative ring S is called *prime* if it has any one of the following equivalent properties:

(1) For any non-zero ideals A and B of S, $AB \neq 0$.
(2) For any non-zero elements s, s' of S, $sSs' \neq 0$.
(3) For any non-zero ideal A of S, $A^r = 0$.
(4) For any non-zero ideal B of S, $B^l = 0$. Here

$$A^r = \{s \in S | \, As = 0\}, \qquad B^l = \{s \in S | \, sB = 0\}$$

are the right and left annihilators of A and B respectively.

If S is a ring for which $S^l = 0$, it is well known (**5**, 6.4) and easily shown that an ideal A of S is dense as a submodule of S_S if and only if $A^l = 0$. It follows that every two-sided ideal in a prime ring is dense.

LEMMA. *If S is a prime ring, the socle of $L(S_S)$ is either 0 or S.*

Proof. Suppose the socle of $L(S_S)$ is not 0. By 2.8, it is an ideal, hence dense. But, by definition, the socle is closed, hence it coincides with S.

4.2. If S is a prime ring, Q any ring of right quotients of S, then Q is also a prime ring. (It suffices to assume that S_S be a large submodule of Q_S.)

Indeed, let A and B be non-zero ideals of Q. Then $A \cap S$ and $B \cap S$ are non-zero ideals of S, hence $(A \cap S)(B \cap S) \neq 0$, and so $AB \neq 0$.

4.3. The following theorem owes its present form to a discussion with R. E. Johnson. (An independent proof of it was also found by Utumi.)

THEOREM. *If S is a prime ring such that the lattice $L(S_S)$ has non-zero socle, then its maximal ring of right quotients is a complete ring of linear transformations of a right vector space.*

Proof. We are given that S is a prime ring such that $L(S_S)$ has non-zero socle. Let Q be its maximal ring of right quotients, this is also prime, by 4.2. Moreover $L(Q_S) \cong L(S_S)$, by 2.5. We shall verify below that the closed submodules of Q_S are actually closed right ideals of Q, hence $L(Q_Q) = L(Q_S)$. Therefore, $L(Q_Q)$ also has non-zero socle, which must coincide with Q, by 4.1. Now, by 3.7, $L(Q)$ is complemented. Since $S^l = 0$, Q contains a unity element (see, for example (**5**, 6.2)). Therefore $Q \cong \mathrm{Hom}_Q(Q, Q)$, and this is a regular ring, by 3.3. Thus every principal right ideal of Q is a direct summand, hence a closed right ideal. Therefore, every atom of $L(Q_Q)$ is a minimal right ideal. Thus Q has non-zero socle. (The usual socle of Q is the socle of the lattice of all right ideals of Q.) Moreover Q_Q is rationally complete. By Utumi's theorem, mentioned in 1.5, $Q \cong \mathrm{Hom}_D(V', V')$, where D is a skew-field and V'_D is a right vector space.

4.4. The proof given above depended on the following lemma, which is implicit in the work of Utumi.

LEMMA. *If Q is the maximal ring of right quotients of S then any closed submodule of Q_S is a closed right ideal of Q.*

Proof. Let A be a closed submodule of Q_S, and let $a \in A$. Take any $q' \in Q$ and $0 \neq q \in Q$. Since $S \leqslant Q(Q_S)$, we can find $x \in S^\#$ such that $q'x \in S$ and $qx \neq 0$. Now take any $a' \in A$, then $(a' + aq')x \in A$ and $qx \neq 0$. Thus $A \leqslant A + aQ(Q_S)$, and so $A + aQ \subset A^c = A$, hence $aQ \subset A$. Therefore A is a right ideal. To see that it is closed, assume $A \leqslant B(Q_Q)$. Then also $A \leqslant B(Q_S)$, by 1.7 (†), hence $B \subset A^c = A$, as required.

4.5. As has also been observed by Johnson, Theorem 4.3 partly generalizes a recent result of Goldie (**8**). Goldie obtained the conclusion of Theorem 4.3 (even using the classical ring of quotients) for prime rings satisfying the following ascending chain conditions as well as their symmetric duals:

($1r$) Every direct sum of non-zero right ideals of S has a finite number of terms.

($2l$) The ascending chain condition holds for the annihilator left ideals of S.

It is not difficult to show that the assumption of Theorem 4.3 for a prime ring S is implied by ($1r$) and ($2l$), or even by ($1r$) and ($2r$), the symmetric dual of ($2l$) (**15**, Propriété 12). In this connection we shall only establish one lemma.

4.6. A ring without non-zero, nilpotent ideals is called *semi-prime*. Clearly, every prime ring is semi-prime.

LEMMA. *If S is any semi-prime ring satisfying* ($2l$), *then* $J(S_S) = 0$.

Proof. Let $\{L_i | \, i \in I\}$ be the set of all closed large right ideals of S and consider

$$J = \sum_{i \in I} L_i{}^l.$$

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We have

$$J^{rl} = \Big(\sum_{i \in I} L_i{}^l\Big)^{rl} = \Big(\sum_{i \in F} L_i{}^l\Big)^{rl},$$

for a finite subset F of I, by (2l). Therefore

$$J^r = J^{rlr} = \Big(\bigcap_{i \in F} L_i{}^{lr}\Big)^{lr} = \bigcap_{i \in F} L_i{}^{lr},$$

since $A \to A^{lr}$ is also a closure operation on the lattice of right ideals of S, and the intersection of "closed" right ideals is "closed." Now a finite intersection of large right ideals is large, hence $L = J^r$ is a large right ideal. Thus $(J \cap L)^2 \subset JL = 0$. Since S is semi-prime, $J \cap L = 0$, hence $J = 0$.

Thus $L_i{}^l = 0$, for all closed, large right ideals L_i of S. This easily implies that $L'^l = 0$, for any large right ideal L' of S, as was to be shown.

5. Rational completions of bimodules.

5.1. If R and S are associative rings, a *bimodule* ${}_RM_S$ consists of a right module M_S and a left module ${}_RM$ with the same additive group such that

$$(rm)s = r(ms) \qquad (r \in R,\ m \in M,\ s \in S).$$

By a standard trick, ${}_RM_S$ may be regarded as a right module, even a unitary right module M_T. Thus let R' be anti-isomorphic with R, then we put $T = S^\# \otimes_Z R'^\#$ and write

$$m(x \otimes y') = ymx \qquad (m \in M,\ x \in S^\#,\ y \in R^\#).$$

In view of this identification it is clear that, for R- S-bimodules A, B, and C, $A \leqslant B({}_RC_S)$ must mean that ${}_RA_S$ is a submodule of ${}_RB_S$ and $\mathrm{Hom}_{R,S}(E, C) = 0$, for every submodule ${}_RE_S$ of ${}_RB_S - {}_RA_S$. We can also speak of the rational completion ${}_R\bar{M}_S$ of ${}_RM_S$, meaning that $\bar{M}_T$ is the rational completion of M_T.

5.2. Theorem. *Let ${}_RM_S$ be any bimodule, ${}_R\bar{M}_S$ its rational completion. Then the rational completions of M_S and ${}_RM$ are also bimodules ${}_R\vec{M}_S$ and ${}_R\overleftarrow{M}_S$ respectively. They are isomorphic over ${}_RM_S$ to unique submodules of ${}_R\bar{M}_S$ and will be identified with these. Their intersection ${}_R\hat{M}_S$ in ${}_R\bar{M}_S$ is the largest extension of ${}_RM_S$ satisfying $M \leqslant \hat{M}(\hat{M}_S)$ and $M \leqslant \hat{M}({}_R\hat{M})$. ${}_R\hat{M}_S$ is "b-complete" in the following sense:*

If ${}_RA_S$ and ${}_RB_S$ are any bimodules such that $A \leqslant B(\hat{M}_S)$ and $A \leqslant B({}_R\hat{M})$, then any element of $\mathrm{Hom}_{R,S}(A, \hat{M})$ can be extended to a unique element of $\mathrm{Hom}_{R,S}(B, \hat{M})$.

Proof. Every $r \in R$ determines an element of $\mathrm{Hom}_S(M, M)$, namely the map $m \to rm$, $m \in M$. Since M_S is dense in $\vec{M}_S$, this map may be extended to a unique element of $\mathrm{Hom}_S(\vec{M}, \vec{M})$, by 1.3. We may as well write this map $n \to rn$, $n \in \vec{M}$. Thus $\vec{M}$ is also an R- S-bimodule.

Now $M \leqslant \vec{M}(\vec{M}_S)$, *a fortiori* $M \leqslant \vec{M}({}_R\vec{M}_S)$. By 1.3, the injection of M into $\vec{M}$ can be extended to a unique element of $\mathrm{Hom}_{R,S}(\vec{M}, \bar{M})$, and this is easily seen to be a monomorphism.

We may identify $\vec{M}$ with its isomorphic image in $\bar{M}$. Similarly $\overleftarrow{M}$ may be regarded as a submodule of $\bar{M}$. Put $\hat{M} = \vec{M} \cap \overleftarrow{M}$. From $M \leqslant \vec{M}(\vec{M}_S)$ we immediately deduce that $M \leqslant \hat{M}(\hat{M}_S)$. By symmetry, we have also $M \leqslant \hat{M}({}_R\hat{M})$. We defer the proof that $\hat{M}$ is the largest extension of M with these two properties.

Now let $A \leqslant B(\hat{M}_S)$, $A \leqslant B({}_R\hat{M})$, and $\phi \in \mathrm{Hom}_{R,S}(A, \hat{M})$. *A fortiori*, $\phi \in \mathrm{Hom}_S(A, \hat{M})$, hence it may be extended to a unique $\vec{\phi} \in \mathrm{Hom}_S(B, \vec{M})$. Take any $r \in R$ and compare $r\vec{\phi}$ with $\vec{\phi}r \in \mathrm{Hom}_S(B, \vec{M})$. These two maps coincide on A, hence on B, since $A \leqslant B(\vec{M}_S)$. (This last statement follows from $A \leqslant B(\hat{M}_S)$ and $\hat{M} \leqslant \vec{M}(\vec{M}_S)$ by P5, where the second statement follows from $M \leqslant \vec{M}(\vec{M}_S)$ by P3.)

Thus $\vec{\phi} \in \mathrm{Hom}_{R,S}(B, \vec{M})$. In the same way, we extend ϕ to a unique $\overleftarrow{\phi} \in \mathrm{Hom}_{R,S}(B, \overleftarrow{M})$. Now both $\vec{\phi}$ and $\overleftarrow{\phi}$ may be regarded as elements of $\mathrm{Hom}_{R,S}(B, \bar{M})$. They agree on A, hence on B, since $A \leqslant B({}_R\bar{M}_S)$. (This last statement follows by P5 from $A \leqslant B({}_R\hat{M}_S)$, which is a trivial consequence of $A \leqslant B(\hat{M}_S)$, and $\hat{M} \leqslant \bar{M}({}_R\bar{M}_S)$, an immediate consequence of $M \leqslant \bar{M}({}_R\bar{M}_S)$.)

Now the image of $\vec{\phi}$ lies in $\vec{M}$, the image of $\overleftarrow{\phi}$ in $\overleftarrow{M}$, hence their common image lies in $\vec{M} \cap \overleftarrow{M} = \hat{M}$, and so we obtain an element of $\mathrm{Hom}_{R,S}(B, \hat{M})$.

Finally, assume that $M \leqslant N(N_S)$ and $M \leqslant N({}_RN)$. It follows by a standard argument that N may be regarded as a unique submodule of $\hat{M}$. (Indeed, since $N \leqslant \hat{M}(\hat{M}_S)$, P5 yields $M \leqslant N(\hat{M}_S)$, and similarly $M \leqslant N({}_R\hat{M})$. In view of the completeness property just proved, the injection of M into $\hat{M}$ may be extended to a unique element of $\mathrm{Hom}_{R,S}(N, \hat{M})$, and this is easily seen to be a monomorphism.) Thus $\hat{M}$ is, up to isomorphism, the largest bimodule N with the prescribed properties.

5.3. THEOREM. *Let S be a ring, ${}_S\bar{S}_S$ its rational completion as a bimodule. The maximal rings $\vec{S}$ and $\overleftarrow{S}$ of right and left quotients of S, regarded as S-S-bimodules, are isomorphic to unique submodules of ${}_S\bar{S}_S$, and will be identified with these. Their intersection $\dot{S}$ is a subring of both. It is the largest ring extension of S which is both a right and a left ring of quotients of S.*

$\dot{S}$ is the maximal ring of right and left quotients of S of Johnson and Wong

(19, 8**)**. A special case had previously been studied by Utumi **(17**, 5.3**)**. The present construction is more symmetrical than these earlier ones.

Proof. All of this follows immediately from 5.2, with the exception of the fact that the operations of multiplication in the rings $\vec{S}$ and $\overleftarrow{S}$ coincide on their intersection $\dot{S}$.

As was shown in **(5)**, $\vec{S}$ is a ring with multiplication $*$ (say) such that $q * s = qs$, for all $q \in \vec{S}$ and $s \in S$. By symmetry, $\overleftarrow{S}$ is a ring with multiplication $\circ$ (say), such that $s \circ p = sp$, for all $s \in S$ and $p \in \overleftarrow{S}$. We wish to show that $p \circ q = p * q \in \dot{S}$, for all p and q in $\dot{S}$.

Let us write

$$Y = \{y \in S^{\#} | \; qy \in S\}, \qquad X = \{x \in S^{\#} | \; xp \in S\}.$$

A simple calculation show sthat

$$x(p \circ q)y = (xp)(qy) = x(p * q)y \qquad (x \in X,\ y \in Y),$$

and so $X(p \circ q - p * q)Y = 0$. Since

$$S^{\#} - Y \cong (qS^{\#} + S) - S,$$

we deduce from P3 that $Y \leqslant S^{\#}(S_S)$, and similarly that $X \leqslant S^{\#}({}_SS)$. The result now follows if, for any $m \in \bar{S}$, $XmY = 0$ implies $m = 0$. In view of the representation of bimodules as unitary right modules (see 5.1), this may be inferred from the following lemma.

5.4. Lemma. *If M_R, N_S, A_R, B_S and $C_{R\otimes S}$ are right modules such that $A \leqslant M(C_R)$ and $B \leqslant N(C_S)$ then $[A \otimes B] \leqslant M \otimes N(C_{R\otimes S})$.*

Here $[A \otimes B]$ is the set of all

$$\sum_{i=1}^{k} a_i \otimes b_i \in M \otimes N$$

with $a_i \in A$ and $b_i \in B$.

Proof. Let D be any $R \otimes S$-submodule of $M \otimes N$ and consider $\phi \in \mathrm{Hom}_{R\otimes S}(D, C)$ such that $[A \otimes B] \subset \ker \phi$. We wish to show that $\mathrm{im}\, \phi = 0$.

Take a k-tuple $(a_1, \ldots, a_k)$ of elements of A. Let D' be the set of all k-tuples $(n_1, \ldots, n_k)$ of elements of N such that

$$\sum_{i=1}^{k} a_i \otimes n_i \in D.$$

Clearly, D' is an S-submodule of $kN = N \oplus \ldots \oplus N$. Let

$$\phi'(n_1, \ldots, n_k) = \phi\Big(\sum_{i=1}^{k} a_i \otimes n_i\Big),$$

then $\phi' \in \mathrm{Hom}_S(D', C)$ and $kB \subset \ker \phi'$. Now, $B \leqslant N(C_S)$, hence $0 \leqslant N - B(C_S)$, and therefore $0 \leqslant k(N - B)(C_S)$. But $k(N - B) \cong kN - kB$, hence $kB \leqslant kN(C_S)$. Therefore $\mathrm{im}\, \phi' = 0$, and so $[A \otimes N] \subset \ker \phi$. Repeating the whole argument on the other side, we finally obtain $M \otimes N \subset \ker \phi$, as required.

5.5. Let D be a skew-field, ${}_DV$ and V'_D left and right D-modules (vector spaces) respectively. Put

$$B = \mathrm{Hom}_{D,D}(V \otimes_Z V', D);$$

this is clearly an additive group. We may regard B as the module of *bilinear forms* from $V \times V'$ into D. There is a canonical isomorphism

$$B \cong \mathrm{Hom}_D(V, \mathrm{Hom}_D(V', D)).$$

Thus, for any $b \in B$ and $v \in V$, we may regard vb as an element of $\mathrm{Hom}_D(V', D)$ such that

$$(vb)v' = vbv' \qquad (v' \in V').$$

(We write vbv' in place of $b(v \otimes v')$.)

An element b_0 of B is called *non-degenerate* if

$$vb_0 = 0 \Rightarrow v = 0 \quad \text{and} \quad b_0v' = 0 \Rightarrow v' = 0 \quad (v \in V, v' \in V').$$

If B contains a non-degenerate element b_0, V and V' are called *dual vector spaces* (**9**, p. 69).

Put $S = V' \otimes_D V$. This is turned into a ring with an obvious multiplication, as illustrated by

$$(v'_1 \otimes v_1)(v'_2 \otimes v_2) = v'_1 \otimes (v_1b_0v'_2)v_2.$$

Moreover, one obtains in a natural way the bimodules ${}_DV_S$, ${}_SV'_D$, and ${}_SB_S$.

If V'_D and ${}_DV$ are dual vector spaces, the mapping $v \to vb_0$ is a monomorphism of ${}_DV$ into $\mathrm{Hom}_D(V', D)$, and this induces a monomorphism of $\mathrm{Hom}_D(V, V)$ into B, its image being $\{b \in B | \ Vb \subset Vb_0\}$. We also have an isomorphic embedding of ${}_SS_S$ into ${}_SB_S$. Indeed, the element

$$s = \sum_{i=1}^{n} v'_i \otimes v_i$$

of S gives rise to the bilinear form (s) where

$$v(s)v' = \sum_{i=1}^{n} (vb_0v'_i)(v_ib_0v').$$

THEOREM. *Let V'_D and ${}_DV$ be dual vector spaces with a non-degenerate bilinear form b_0, and let S be the ring $V' \otimes_D V$. Then the bimodule ${}_SB_S$ of bilinear forms from $V \times V'$ into D is a rational extension of ${}_SS_S$, and the maximal rings of right quotients, left quotients, and right and left quotients of S may be realized thus:*

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$$\overrightarrow{S} = \{b \in B |\ bV' \subset b_0 V'\},$$
$$\overleftarrow{S} = \{b \in B |\ Vb \subset Vb_0\},$$
$$\dot{S} = \{b \in B |\ bV' \subset b_0 V' \text{ and } Vb \subset Vb_0\}.$$

We omit the proof of this theorem, which is another formulation of Utumi's results (**17**, 5.1, 5.3), in the hope of improving it at a later time in two directions: to identify the rational completion of ${}_S S_S$ and to extend the result to projective modules over prime rings.

5.6. The preceding theorem may be applied to obtain $\dot{S}$ for any prime ring S with non-zero socle. As is well known (**16**), such a ring is a primitive ring, hence we may apply Utumi's result (see 1.5). Thus $\overleftarrow{S} \cong \mathrm{Hom}_D(V, V)$, where $D = eSe$ is a skew-field and $V = eS$ is a left D-module. Dually, also $\overrightarrow{S} \cong \mathrm{Hom}_D(V', V')$, where $V' = Se$ is a right D-module. Utumi also computed $\dot{S}$ (**17**, 5.3), but a more symmetric form of $\dot{S}$ may be obtained by 5.5.

Indeed, it is well known (**9**, page 77) that ${}_D V$ and V'_D are dual vector spaces. One easily verifies that S is a ring of right and left quotients of SeS, the latter being isomorphic to $V' \otimes_D V = S_0$, say. Thus $\dot{S} = \dot{S}_0$, and this is determined by 5.5.

6. On semi-prime rings.

6.1. With any bimodule ${}_R M_S$ we may associate the lattices $L({}_R M_S)$, $L({}_R M)$, and $L(M_S)$. In addition, we shall be interested in the lattice $L^b({}_R M_S)$, which consists of all *b-closed* submodules of ${}_R M_S$, where b is a closure operation defined on the lattice of all submodules of ${}_R M_S$ as follows:

Let ${}_R A_S$ be any submodule of ${}_R M_S$ then ${}_R A^b{}_S$ is the largest submodule ${}_R B_S$ of ${}_R M_S$ such that

$$(\ddagger) \qquad A \leqslant B(M_S) \quad \text{and} \quad A \leqslant B({}_R M).$$

We will show that ${}_R A^b{}_S$ is in fact the intersection of the closure of A_S in M_S with the closure of ${}_R A$ in ${}_R M$.

Indeed, let the closure operation for submodules of M_S be denoted by c. Take any element r of R, then

$$r(A^c) \subset (rA)^c \subset A^c,$$

by C3 and the fact that A is a left R-module. Thus we have a bimodule ${}_R A^c{}_S$. In the same way, if the closure operation for submodules of ${}_R M$ is denoted by d, we obtain a bimodule ${}_R A^d{}_S$. Put $B = A^c \cap A^d$, then B is a bimodule and ($\ddagger$) holds.

On the other hand, assume that ${}_R B_S$ is any submodule of ${}_R M_S$ satisfying ($\ddagger$). Then $B \subset A^c$ and $B \subset A^d$, hence $B \subset A^c \cap A^d$, as was to be shown.

Henceforth we write $A^b = A^c \cap A^d$.

6.2. We now make the blanket assertion:

All results obtained for $L(M_S)$ in § 2 and § 3 remain valid for $L^b({}_RM_S)$, *mutatis mutandis.*

Indeed, the results in § 2 and § 3 were based only on these facts: the existence of a closure operation, the existence of a rational completion, and properties P0 to P5 for the ternary relation $A \leqslant B(C_S)$ among right modules.

Since we have already established the existence of a b-closure and a b-completion, it remains to verify properties P0 to P5 for the ternary relation

$$A \leqslant B(C_S) \quad \text{and} \quad A \leqslant B({}_RC)$$

among bimodules. This is a routine verification. For example, P5 asserts for bimodules that

$$[A \leqslant B(C_S) \text{ and } A \leqslant B({}_RC) \text{ and } C \leqslant D(D_S) \text{ and } C \leqslant D({}_RD)] \Rightarrow [A \leqslant B(D_S) \text{ and } A \leqslant B({}_RD)].$$

This implication clearly follows from the separate implications for left modules and right modules.

From the above blanket assertion we must except the special construction in 2.1.

In translating results from one situation to the other, we must make the following replacements:

Replace	c	by	b,
	$L(M)$	by	$L^b(M)$,
	closed	by	*b-closed*,
	dense	by	*b-dense*,
	rationally complete	by	*b-complete*,
	rational extension	by	*right and left rational extension.*

Here a submodule ${}_RA_S$ of ${}_RM_S$ is called *b-dense* if $A^b = M$.

In future, the analogue of (let us say) 2.5 for b-closure will be denoted by 2.5^b.

6.3. If S is a ring, we are particularly interested in $L^b({}_SS_S)$, which we shall denote more briefly by $L^b(S)$.

PROPOSITION. *For an associative ring S, $L^b(S)$ has at most two elements if and only if either S is a prime ring or $S^2 = 0$ and the additive group of S is cyclic of prime order or a subgroup of the additive group of rationals.*

Proof. We proceed in three steps.

(1) If S is a non-zero prime ring, then $L^b(S)$ has exactly two elements.

Indeed, let A be any non-zero ideal, then $A^l = 0$. From this one easily deduces that A_S is dense in S_S. See, for example (**5**, 6.4). Similarly ${}_SA$ is dense in ${}_SS$, hence A is b-dense in S.

(2) If $L^b(S)$ has at most two elements and $S^2 \neq 0$, then S is a prime ring.

Indeed, assume that every non-zero ideal is b-dense and $S^2 \neq 0$. Suppose $S^l \neq 0$, then $sS = 0$, for some $s \neq 0$. Then $S^{\#}s$ is a non-zero ideal, hence it is b-dense in S. But $S^{\#}sS = 0$, hence $SS = 0$, contrary to assumption. Thus $S^l = 0$, and similarly $S^r = 0$.

Now suppose $sSs' = 0$, $s' \neq 0$. Since $S^r = 0$, $Ss' \neq 0$. Since, $S^l = 0$, $Ss'S \neq 0$. Thus $Ss'S$ is b-dense in S. But $sSs'S = 0$, hence $sS = 0$. Since $S^l = 0$, we have $s = 0$. Therefore S is a prime ring.

(3) If $S^2 = 0$, then $L^b(S) = L^b({}_ZS_Z) = L(S)$, where Z is the ring of integers. The result now follows from 3.6.

6.4. We may also ask when $L^b(S)$ is a complemented lattice. Essentially, this implies that $\dot{S}$ is a semi-prime ring and that $L^b(S)$ is a Boolean algebra, as we shall see.

Porposition. *If S is a ring for which $L^b(S)$ is complemented, then $L^b(S) \cong L^b(\dot{S})$, where $\dot{S}$ is the maximal ring of right and left quotients of S. If furthermore $S^l = 0$, then $\dot{S}$ is semi-prime.*

Proof. Clearly, ${}_SS_S$ is b-dense in ${}_S\dot{S}_S$. Hence, by 2.5^b, $L^b(S) \cong L^b({}_S\dot{S}_S)$. We claim that the latter is actually $L^b({}_{\dot{S}}\dot{S}_{\dot{S}}) = L^b(\dot{S})$. Indeed, this will follow from the lemma below, which asserts that all b-closed submodules of ${}_S\dot{S}_S$ are b-closed ideals in $\dot{S}$.

If $S^l = 0$ then $\dot{S}^l \cap S = 0$, hence, also $\dot{S}^l = 0$, since $S \leqslant \dot{S}(\dot{S}_S)$. For the remainder of the proof we may as well assume that $\dot{S} = S$ and $S^l = 0$. Suppose that A is a non-zero, nilpotent ideal of S, say $A^k = 0$ and $A^{k-1} \neq 0$, for $k \geqslant 2$. Let $B = A^{k-1}$ and consider its b-closure B^b. Now $B \leqslant B^b(S_S)$, $BB^b \subset S$, and $B^2 = 0$, hence $BB^b = 0$. Applying 3.3^b, we obtain $S = B^b \oplus C$, where C is another ideal of S. Therefore $BC \subset B^bC = 0$, hence $BS = BB^b + BC = 0$. Since $S^l = 0$, we deduce $B = 0$, a contradiction. Thus S contains no non-zero, nilpotent ideal, and so is semi-prime.

6.5. Lemma. *If S is a ring such that $L^b(S)$ is complemented, $\dot{S}$ its maximal ring of right and left quotients, then any b-closed submodule of ${}_S\dot{S}_S$ is a b-closed ideal of $\dot{S}$.*

Proof. Let $A \in L^b({}_S\dot{S}_S)$. By 3.3^b, $\dot{S} = A + B$, $A \cap B = 0$. Now $A \cap S$ and $B \cap S$ are ideals of S, and

$$(A \cap S)(B \cap S) \subset A \cap B = 0.$$

By 2.5^b, the b-closure of $A \cap S$ in ${}_S\dot{S}_S$ is A and that of $B \cap S$ is B.

Take any element a of $A \cap S$, then $aB \subset S$, $a(B \cap S) = 0$ and $B \cap S \leqslant B(\dot{S}_S)$, hence $aB = 0$. Thus $(A \cap S)B = 0$. Arguing similarly on the other side, we obtain $AB = 0$. By symmetry also $BA = 0$, and so A and B are ideals of $\dot{S}$.

Let A' be the b-closure of A in ${}_{\dot{S}}\dot{S}_{\dot{S}}$, then $A \leqslant A'(\dot{S}_{\dot{S}})$. But $S \leqslant \dot{S}(\dot{S}_S)$, hence $A \leqslant A'(\dot{S}_S)$, by 1.7(†). Similarly $A \leqslant A'({}_S\dot{S})$, and so A' is contained in the b-closure of A in ${}_S\dot{S}_S$, which is just A. Thus $A' = A$, as required.

6.6. Johnson has shown in **(12**, II**)**—among many other interesting results—that the annihilator ideals of a semi-prime ring form a complete Boolean algebra. This is also contained in the following:

THEOREM. *If S is a semi-prime ring, then $L^b(S)$ is a complete Boolean algebra, whose elements are the annihilator ideals of S. If $\dot{S}$ is the maximal ring of right and left quotients of S, then $\dot{S}$ is also semi-prime, $L^b(\dot{S}) \cong L^b(S)$, and the elements of $L^b(\dot{S})$ are the direct summands of $\dot{S}$.*

Proof. Let S be semi-prime. We first verify the following condition:

(*) For each ideal A of S there exists an ideal A^* such that, for any ideal B of M, $A \cap B = 0$ if and only if $B \subset A^*$.

Indeed, let A^r be the right annihilator of A in S, then $(A \cap A^r)^2 \subset AA^r = 0$, and so $A \cap A^r = 0$, since S is semi-prime. If B is any ideal such that $A \cap B = 0$, then $AB = 0$, hence $B \subset A^r$. Thus the condition holds with $A^* = A^r$.

The following consequences of (*) are immediate:

(1) A^* is uniquely determined.
(2) $A \subset A^{**}$, $A^{***} \subset A^*$, $A \subset B \Rightarrow B^* \subset A^*$.
(3) $A \to A^{**}$ is a closure operation.
(3) A^{**} is the largest ideal of S in which A is a large S-S-submodule.
(5) For any collection $\{A_i\}_{i\in I}$ of ideals of S,

$$\Big(\sum_{i\in I} A_i\Big)^* = \bigcap_{i\in I} A_i^*.$$

(6) The ideals A of S such that $A^{**} = A$ form a complete Boolean algebra with set intersection as meet and * as complementation. The join of a family $\{A_i\}_{i\in I}$ of elements of this Boolean algebra is given by

$$\bigvee_{i\in I} A_i = \Big(\sum_{i\in I} A_i\Big)^{**} = \Big(\bigcap_{i\in I} A_i^*\Big)^*.$$

We omit the straightforward derivations of (1) to (6) from (*). Since we could also have taken $A^* = A^l$, the left annihilator of A in S, it follows from (1) that $A^* = A^r = A^l$. Thus the right annihilator ideals of S are the same as the left annihilator ideals.

Next, we shall show that

$$(**) \qquad A \leqslant A^{**}(S_S).$$

Take $x \in A^{**}$, $0 \neq s \in S$, we seek $y \in S^\#$ such that $sy \neq 0$ and $xy \in A$. In fact, we shall find y in S. We have apparently three cases:

Case 1. $sA \neq 0$. Take $y \in A$ such that $sy \neq 0$. Then $xy \in SA \subset A$.

Case 2. $sA^* \neq 0$. Take $y \in A^*$ such that $sy \neq 0$. Then $xy \in A^{**}A^* = 0 \subset A$.

Case 3. $sA = 0$ and $sA^* = 0$. Then $s(A + A^*) = 0$, hence $s \in (A + A^*)^* = A^* \cap A^{**} = 0$. Since $s \neq 0$, this case does not really arise.

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Now A^{**} is a closed submodule of S_S, by 2.7 (1). Hence, by (**), it is the closure of A in S_S. By symmetry, it is also the closure of A in ${}_SS$, hence it is the b-closure of A in ${}_SS_S$. Thus the annihilator ideals coincide with the b-closed ideals, and so $L^b(S)$ is a complete Boolean algebra, by (6) above.

It follows from 6.4 that $L^b(S) \cong L^b(\dot{S})$ and that $\dot{S}$ is semi-prime. Hence the annihilator ideals of S are also the b-closed ideals of S, and these are the direct summands of $\dot{S}$, in view of 3.3^b. The proof is now complete.

6.7. The *Dedekind–MacNeille* completion of a partially ordered set S is a complete lattice, whose elements are the subsets of S of the form (see **(1**, p. 58, Theorem 12**)**): the set of all lower bounds of the set of all upper bounds of a non-empty subset K of S. The following corollary to Theorem 6.6 contains a new proof of the main result of **(2)** for Boolean rings with 1.

COROLLARY. *The Dedekind–MacNeille completion of a Boolean ring with* 1 *is given by*

$$L^b(S) \cong L^b(\dot{S}) \cong \dot{S} = \vec{S}.$$

Proof. Let K be any non-empty subset of S, then the set of its upper bounds is

$$K' = \{s \in S |\ \forall_{k \in K}\ sk = k\} = \{s \in S |\ 1 - s \in K^*\},$$

and the set of all lower bounds of K' is

$$\{t \in S |\ \forall_{s \in K'}\ st = t\} = \{t \in S |\ K^*t = 0\} = K^{**}.$$

Thus the Dedekind–MacNeille completion of S consists precisely of the annihilator ideals of S, hence coincides with $L^b(S) \cong L^b(\dot{S})$, by Theorem 6.6. Now it is easily verified that $\dot{S} = \vec{S}$ is a Boolean ring (see **(2**, Corollary 2**)**). Therefore $L^b(\dot{S}) \cong \dot{S}$, by the last part of Theorem 6.6.

6.8. We have called a ring *semi-prime* if it has no non-zero, nilpotent ideals. It is known (see, for example **(9**, p. 196**)**) that a semi-prime ring may also be characterized as a ring in which the intersection of all prime ideals is 0. A *prime ideal* of S is any ideal P such that $S - P$ is a prime ring. The following two assertions are equivalent characterizations of prime ideals:

(a) For all ideals A and B, if $AB \subset P$ then $A \subset P$ or $B \subset P$.

(b) For all elements s and s', if $sSs' \subset P$, then $s \in P$ or $s' \in P$.

In what follows, $\mathscr{P}(S)$ will denote the set of proper prime ideals of S.

It is easily verified that, for any ideal A of a semi-prime ring S,

$$A^* = \cap\{P \in \mathscr{P}(S) |\ A \not\subset P\}.$$

This was used in a different approach to Theorem 6.6 by the author in (Amer. Math. Soc. Notices, 7 (1960), p. 92). It turns out that every proper prime ideal P is either b-closed ($P^{**} = P$) or b-dense ($P^{**} = S$) in S. The former are also the maximal proper b-closed ideals of S.

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Condition (*), which is responsible for associating a Boolean algebra with the ring S, might also hold for rings which are not semi-prime. It can be shown that (*) is in fact equivalent to the vanishing of the intersection of all ideals P' of S such that

$$A \cap B = 0 \Rightarrow \text{ either } A \subset P' \text{ or } B \subset P',$$

for any ideals A and B. A similar result holds for modules, but it would take us too far afield to go into further details here.

6.9. As was pointed out by McCoy **(16)**, the set $\mathscr{P}(S)$ of proper prime ideals of a semi-prime ring becomes a topological space under the usual Stone topology, the open sets being precisely the sets

$$\Gamma A = \{P \in \mathscr{P}(S) | A \not\subset P\},$$

where A is any ideal of S.

If V is any open subset of $\mathscr{P}(S)$, we introduce the ideal

$$\Delta V = \cap_{P \in V} P.$$

Then $\Delta\Gamma A = A^*$, the annihilator of A. On the other hand, $\Gamma\Delta V = V^\perp$ is easily seen to be the interior of the complement of V, also called the *exterior* of V. A set of the form $V^\perp$ is called a *regular open* set. The open set U is regular open if and only if $(U^\perp)^\perp = U$.

THEOREM. *If S is a semi-prime ring, the mapping $A \to \Gamma A$ is an isomorphism of the complete Boolean algebra of annihilator ideals of S onto the algebra of regular open sets in the prime ideal space $\mathscr{P}(S)$.*

Proof. One easily verifies that $\Gamma(A^*) = (\Gamma A)^\perp$ and $\Gamma(A \cap B) = \Gamma(A) \cap \Gamma(B)$, for annihilator ideals A and B. Thus Γ is a lattice homomorphism. Now $\Delta\Gamma\Delta$ is the inverse mapping of Γ; for let A be any annihilator ideal, V any regular open set, then

$$(\Delta\Gamma\Delta)\Gamma A = (A^*)^* = A, \qquad \Gamma(\Delta\Gamma\Delta)\, V = (V^\perp)^\perp = V.$$

The analogous result for maximal ideal spaces of commutative semi-simple rings with 1 was recently obtained by Fine, Gillman, and the present author. The proofs of these two results are practically identical.

7. On the structure of semi-prime rings.

We wish to present some results on the structure of semi-prime rings, which resemble those of Dieudonné **(3)**. We require three lemmas which we have stated together for convenience.

7.1. LEMMA. *Let $S = C \oplus D$ as a direct sum of rings.*

(1) *If S is a b-complete ring, then so is C.*

(2) $L^b({}_SC_S) = L^b(C)$.

(3) ${}_SC_S$ *is a b-complete bimodule, if C is a b-complete ring, and $S^l = 0 = S^r$.*

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Proof.

(1) Let $A \leqslant B(C_C)$, $A \leqslant B({}_CC)$, and $\phi \in \mathrm{Hom}_{C,C}(A, C)$. We may turn A and B into S-S-bimodules by demanding that $DB = 0$ and $BD = 0$. Then also $A \leqslant B(C_S)$ and $A \leqslant B({}_SC)$. Now ϕ may be regarded as an element of $\mathrm{Hom}_{S,S}(A, S)$, hence it may be extended to an element ψ of $\mathrm{Hom}_{S,S}(B, C)$. Then $\pi\psi \in \mathrm{Hom}_{C,C}(B, S)$ extends ϕ, where π is the projection $C \oplus D \to C$.

(2) If $A \in L^b({}_SC_S)$, a striaghtforward argument shows that $A \in L^b(C)$. The converse is a bit more difficult: Let A be a b-closed ideal in C, B its b-closure in ${}_SC_S$. Then $A \leqslant B(C_S)$, and so, for any $b \in B$ and $0 \neq c \in C$, we can find $x \in S^{\#}$ such that $bx \in A$ and $cx \neq 0$. Now $x = c + d + z$, where $c \in C$, $d \in D$, and z is an integer. Since $bd = 0$ and $cd = 0$, we may as well take $d = 0$, so that $x \in C^{\#}$. Thus $A \leqslant B(C_C)$ and, by symmetry, $A \leqslant B({}_CC)$. Since A was a b-closed ideal of C, we have $A = B$, and so A is also b-closed in ${}_SC_S$, as required.

(3) Let $A \leqslant B(C_S)$, $A \leqslant B({}_SC)$, and $\phi \in \mathrm{Hom}_{S,S}(A, C)$. Let $0 \neq c \in C$ and $b \in B$, we can find $x \in S^{\#}$ such that $cx \neq 0$ and $bx \in A$. Now $S^l = 0$, hence $cxC = cxS \neq 0$, and so there exists $c' \in C$ such that $cxc' \neq 0$. But $bxc' \in A$ and $xc' \in C$, hence $A \leqslant B(C_C)$. Similarly $A \leqslant B({}_CC)$. Since C is b-complete, ϕ can be extended to $\psi \in \mathrm{Hom}_{C,C}(B, C)$. We will show that $\psi \in \mathrm{Hom}_{S,S}(B, C)$.

Indeed, it suffices to show that $\psi(db) = d(\psi b)$, for any $b \in B$ and $d \in D$. Given d, the mapping $b \to \psi(db)$ belongs to $\mathrm{Hom}_C(B, C)$, and $\psi(dA) = \phi(dA) = d(\phi A) = 0$. Now we recall that $A \leqslant B(C_C)$, hence $\psi(dB) = 0$. Since also $d(\psi B) \subset dC = 0$, the result follows.

7.2. Proposition. *If S is the weak direct sum of the set of rings $\{S_i\}_{i \in I}$ and $S^l = 0 = S^r$, then its maximal ring of right and left quotients $\dot{S}$ is the complete direct sum of the $\dot{S}_i$.*

This is the two-sided analogue of **(17**, 2.1**)**.

Proof. We shall prove this in three steps.

(1) A complete direct sum of b-complete bimodules is b-complete.

Indeed, let

$$M = \prod_{i \in I} M_i$$

where the M_i are b-complete R-S-modules. Suppose $A \leqslant B(M_S)$ and $A \leqslant B({}_RM)$. Let $\phi \in \mathrm{Hom}_{R,S}(A, M)$. Since there is a well-known monomorphism of M_i into M, we have $A \leqslant B(M_{iS})$ and $A \leqslant B({}_RM_i)$. Now let π_i be the canonical epimorphism of M onto M_i, then $\pi_i\phi \in \mathrm{Hom}_{R,S}(A, M_i)$. By b-completeness of M_i, this may be extended to $\psi_i \in \mathrm{Hom}_{R,S}(B, M_i)$. By definition of direct products, there exists a unique $\psi \in \mathrm{Hom}_{R,S}(B, M)$ such that $\pi_i\psi = \psi_i$, for all $i \in I$. Now, for any $a \in A$, $\pi_i(\psi a) = \psi_i a = \pi_i(\phi a)$, hence $\psi a = \phi a$, and so ψ extends ϕ.

(2) If

$$S = \dot{\sum_{i \in I}} S_i$$

is the weak direct sum of rings S_i, and T_i is a ring of right quotients of S_i, then

$$T = \prod_{i \in I} T_i$$

is a ring of right quotients of S.

Actually we only require the known case $S^l = 0$. In the general case one might proceed thus:

Let $t \in T$, $s \in S$. Denoting by t_i the ith component of t, then $(ts)_i = t_i s_i \in T_i S_i \subset T_i$, hence T is a right S-module. We claim that $S \leqslant T(T_S)$. Indeed, let $t' \neq 0$ and $t \in T$, we seek $x \in S$ such that $t'x \neq 0$ and $tx \in S$.

Since $t' \neq 0$, there exists $k \in I$ such that $t_k' \neq 0$. Now T_k is a ring of right quotients of S_k, hence we can find $x_k \in S_k^{\#}$ such that $t_k' x_k \neq 0$ and $t_k x_k \in S_k$. Putting $x_i = 0$ for $i \neq k$, we obtain an element x of $S^{\#}$, for which it is easily verified that $t'x \neq 0$ and $tx \in S$.

(3) We now prove the proposition. By (2) and symmetry,

$$T = \prod_{i \in I} \dot{S}_i$$

is a ring of right and left quotients of

$$S = \dot{\sum_{i \in I}} S_i.$$

Now each of the $\dot{S}_i$ is b-complete as a ring, hence also as a T-T-bimodule, by Lemma 7.1. Therefore, by (1), T is also b-complete. Now $T \leqslant \dot{S}(T_S)$ and $S \leqslant T(T_S)$, hence $T \leqslant \dot{S}(T_T)$, by 1.7 (†). By symmetry also $T \leqslant \dot{S}({}_T T)$, hence $T = \dot{S}$.

7.3. We recall that a ring is called *b-complete* if it coincides with its maximal ring of right and left quotients.

THEOREM. *If S is a b-complete semi-prime ring, then $S = C \oplus C^*$, where C is the socle of $L^b(S)$. Let $\{A_i\}_{i \in I}$ be the set of all atoms of $L^b(S)$, then*

$$C \cong \prod_{i \in I} A_i, \qquad C^* = \bigcap_{i \in I} A_i^*.$$

The A_i are b-complete prime rings and C^* is a b-complete semi-prime ring such that $L^b(C^*)$ has no atoms.

Proof. Since $L^b(S)$ is a Boolean algebra, it is a complemented distributive lattice. By 2.9^b, the socle C of $L^b(S)$ is the b-closure of the weak direct sum of the A_i, and by 6.6, $S = C \oplus C^*$. By Lemma 7.1, C and C^* are also b-complete rings. The A_i are prime rings by 2.6 and 6.3, they are b-complete by 7.1.

Let B be the sum of the atoms of $L^b(C)$, then C is the b-closure of B in S, hence $B \leqslant C(S_S)$, and so $B \leqslant C(C_S)$. We claim that $B \leqslant C(C_B)$.

Indeed, let $c' \neq 0$ and $c, c' \in C$. Since $S^l = 0$, we have $c'S \neq 0$. But $c'C^* = 0$, hence $c'C \neq 0$. Now $B \leqslant C(C_S)$, hence $c'B \neq 0$. Thus we can pick

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$b \in B$ such that $c'b \neq 0$. Since B is an ideal of C, we also have $cb \in B$, and therefore $B \leqslant C(C_B)$.

By symmetry also $B \leqslant C(_BC)$, and so C is a ring of right and left quotients of B. Since C is b-complete, $C = \dot{B}$, the maximal ring of right and left quotients of B. By 7.2, we have

$$C \cong \prod_{i \in I} A_i.$$

We now turn our attention to C^*. We have

$$C^* = B^{***} = B^* = \Big(\sum_{i \in I} A_i\Big)^* = \bigcap_{i \in I} A_i^*.$$

As pointed out before, C^* is a b-complete ring. Suppose there is an atom A of $L^b(C^*)$. By Lemma 7.1, this lattice is the same as $L^b({}_SC^*{}_S)$. Now all b-closed submodules of ${}_SC^*{}_S$ are b-closed ideals of S, by 2.6^b. Thus A is a b-closed ideal of S.

Suppose A contains the non-zero ideal J of S, then $J \leqslant A(C^*{}_S)$, and so J is a large submodule of S, hence $A \subset J^{**}$, by (4) in the proof of 6.6. In view of 6.6, A is contained in the b-closure of J. Thus A is an atom of $L^b(S)$.

Now all atoms of $L^b(S)$ are contained in the socle C, hence $A \subset C \cap C^* = 0$, a contradiction. Therefore $L^b(C^*)$ has no atoms. To see that C^* is semi-prime, assume that N is a nilpotent ideal of C^*. Then N is also a nilpotent ideal of $S = C \oplus C^*$, and so $N = 0$, as required.

7.4. An immediate consequence of the preceding theorem is the following.

COROLLARY. *If S is a semi-prime b-complete ring whose Boolean algebra of annihilator ideals is atomic, then S is a complete direct product of b-complete prime rings.*

Theorem 7.3 reduces the study of all b-complete semi-prime rings S to three special cases.

Case 1. S is a b-complete prime ring with non-zero socle. This case is completely described in terms of dual vector spaces by 5.5 and 5.6.

Case 2. S is a b-complete prime ring with zero socle. Section 4 throws some light on prime rings with zero socle, but it is not clear whether this is helpful here.

Case 3. S is a b-complete semi-prime ring for which $L^b(S)$ has no atoms. Such rings are very common, as the following example shows.

7.5. *Example.* Let X be a compact Hausdorff space without isolated points. Following **(6)**, we consider the ring $C(X)$ of all continuous functions from X to the real line, under point-wise addition and multiplication. With every point $x \in X$ there is associated a maximal ideal $M_x = \{f \in C(X) | f(x) = 0\}$, and every maximal ideal has this form. Clearly $\bigcap_{x \in X} M_x = 0$, hence $C(X)$ is semi-prime (even semi-simple). Since X has no isolated points, also

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$\bigcap_{x \in X - \{x_0\}} M_x = 0$, for any point $x_0 \in X$. It is known **(6**, 2.11**)** that every prime ideal P is contained in a unique maximal ideal M_P. Let $\mathscr{P}$ be the set of all prime ideals, then

$$P^* = \bigcap \{P' \in \mathscr{P} | \ P \not\subset P'\} \subset \bigcap \{M_x \neq M_P | \ x \in X\} = 0.$$

Now it is not difficult to show, for any semi-prime ring, that every maximal proper annihilator ideal has the form $P = P^{**}$, where P is a prime ideal. Here $P^{**} = 0^* = C(X)$, hence there are no maximal proper annihilator ideals. Therefore the Boolean algebra of annihilator ideals has no atoms.

7.6. One can also obtain a kind of converse to Theorem 7.3. We shall here be content to remark one (probably well-known) fact.

LEMMA. *A complete direct product of semi-prime rings is semi-prime.*

Proof. First we observe that, if $S = C \oplus D$ as a direct sum of rings, then any prime ideal P of C gives rise to a prime ideal $P + D$ of S.

Now let $\{S_i\}_{i \in I}$ be a set of semi-prime rings, S their complete direct product. Let $s \in S$ and suppose that s lies in every prime ideal of S. In view of the above observation, the component s_i of s in S_i lies in every prime ideal of S_i. Since all S_i are semi-prime, $s_i = 0$, for all $i \in I$, hence $s = 0$.

COROLLARY. *A complete direct product of b-complete semi-prime rings is a b-complete semi-prime ring.*

REFERENCES

1. G. Birkhoff, *Lattice theory* (New York, 1948).

2. B. Brainerd and J. Lambek, *On the ring of quotients of a Boolean ring*, Can. Math. Bull., *2* (1959), 25–29.

3. J. Dieudonné, *Les idéaux minimaux dans les anneaux associatifs*, Proc. Inter. Congr. Math., vol. II (1950), 44–48.

4. B. Eckmann and A. Schopf, *Über injektive Moduln*, Arch. der Math., *4* (1956), 75–78.

5. G. D. Findlay and J. Lambek, *A generalized ring of quotients I, II*, Can. Math. Bull., *1* (1958), 77–85, 155–167.

6. L. Gillman and M. Jerison, *Rings of continuous functions* (New York, 1960).

7. A. W. Goldie, *Decompositions of semi-simple rings*, J. London Math. Soc., *31* (1956), 40–48.

8. ——— *The structure of prime rings under ascending chain conditions*, Proc. London Math. Soc. (3), *8* (1958), 589–608.

9. N. Jacobson, *Structure of rings* (Providence, 1956).

10. R. E. Johnson, *The extended centralizer of a ring over a module*, Proc. Amer. Math. Soc., *2* (1951), 891–895.

11. ——— *Semi-prime rings*, Trans. Amer. Math. Soc., *76* (1954), 375–388.

12. ——— *Structure theory of faithful rings I, II*, Trans. Amer. Math. Soc., *84* (1957), 508–522, 523–544.

13. P. R. Halmos, *Boolean algebra* (mimeographed, Chicago, 1959).

14. I. Kaplanski, *Infinite abelian groups* (Ann Arbor, 1954).
15. L. Lesieur and R. Croisot, *Anneaux premiers Noethériens à gauche*, Ann. Sci. Ec. Norm. Sup. (3), *76* (1959), 161–183.
16. N. H. McCoy, *Prime ideals in general rings*, Amer. J. Math., *71* (1949), 823–833.
17. Y. Utumi, *On quotient rings*, Osaka Math. J., *8* (1956), 1–18.
18. ——— *On a theorem on modular lattices*, Proc. Japan Acad., *25* (1959), 16–21.
19. E. T. Wong and R. E. Johnson, *Self-injective rings*, Can. Math. Bull., *2* (1959), 167–173.

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