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AN ORDER PROPERTY FOR FAMILIES OF SETS

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Abstract

We develop the idea of a θ-ordering (where θ is an infinite cardinal) for a family of infinite sets. A θ-ordering of the family A is a well ordering of A which decomposes A into a union of pairwise disjoint intervals in a special way, which facilitates certain transfinite constructions. We show that several standard combinatorial properties, for instance that of the family A having a θ-transversal, are simple consequences of A possessing a θ-ordering. Most of the paper is devoted to showing that under suitable restrictions, an almost disjoint family will have a θ-ordering. The restrictions involve either intersection conditions on A (the intersection of every λ-size subfamily of A has size at most κ) or a chain condition on A.

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1. Introduction

The family of sets $\mathcal{A}$ is said to be a (λ,κ)-family if $|\mathcal{A}| = \lambda$ and $|A| = \kappa$ for all A in $\mathcal{A}$. The family $\mathcal{A}$ is said to be *almost disjoint* if $|A \cap B| < |A|, |B|$ for all distinct, A, B in $\mathcal{A}$. Our interest in this paper is in almost disjoint (λ,κ)-families $\mathcal{A}$ which possess what we call a θ-ordering, for various values of θ with $\theta \leq \kappa$.

DEFINITION 1.1. A θ-ordering of the (λ,κ)-family $\mathcal{A}$ is a (strict) well order $\prec$ of $\mathcal{A}$ under which there is a family $\mathcal{I}$ of pairwise disjoint intervals with $\mathcal{A} = \bigcup \mathcal{I}$ such that $|I| \leq \kappa$ for each $I \in \mathcal{I}$, and for each $I \in \mathcal{I}$ and $A \in \mathcal{A}$:

$$(1) \qquad A \in I \Rightarrow \left| A \cap \bigcup \left\{ \bigcup J ; J \in \mathcal{I} \text{ and } J \prec I \right\} \right| < \theta,$$

$$(2) \qquad A \in I \Rightarrow \left| A \cap \bigcup \{B \in I; B \prec A\} \right| < \kappa,$$

(where in (1), $J \prec I$ means that $B \prec A$ for all $A \in I$ and $B \in J$).

A (λ, κ)-family clearly has a θ-ordering if $\theta > \kappa$ and so we always assume $\theta \leq \kappa$. Notice that a κ-ordering of the (λ, κ)-family $\mathcal{A}$ is just a well ordering $\prec$ of $\mathcal{A}$ such that for each $A \in \mathcal{A}$,

$$(3) \qquad \left| A \cap \bigcup \{B \in \mathcal{A}; B \prec A\} \right| < \kappa,$$

for we can take each interval in the family I to be a singleton set. The special case of an $\aleph_0$-ordering of a family of denumerable sets has appeared previously, see Davies and Erdös ([1], Lemma 3).

Obviously, any (λ, κ)-family which has a θ-ordering where $\theta \leq \kappa$ must be almost disjoint. Any almost disjoint (κ, κ)-family $\mathcal{A}$ where κ is regular has a θ-ordering, for any $\theta \leq \kappa$, since any well ordering of $\mathcal{A}$ of order type κ with $\mathcal{A}$ itself the only interval gives a θ-ordering. However, not every almost disjoint (κ^+, κ)-family possesses even a κ-ordering, for let $\mathcal{S}$ be a (κ, κ)-family of pairwise disjoint sets, and let $\mathcal{A}$ with $\mathcal{S} \subseteq \mathcal{A}$ be an almost disjoint (κ^+, κ)-family with $\bigcup \mathcal{A} = \bigcup \mathcal{S}$. Any $A \in \mathcal{A}$ coming after all the sets in $\mathcal{S}$ in a well ordering $\prec$ of $\mathcal{A}$ has $A \cap \bigcup \{B \in \mathcal{A}; B \prec A\} = A$, so $\mathcal{A}$ has no κ-ordering. If κ is singular, there are almost disjoint (κ, κ)-families that are maximal with respect to almost disjointness (see Erdös and Hechler [3]). No such maximal family possesses even a κ-ordering (for if $\prec$ is a κ-ordering of the (κ, κ)-family $\mathcal{A}$, choosing $x(A) \in A - \bigcup \{B; B \prec A\}$ and putting $T = \{x(A); A \in \mathcal{A}\}$ gives a set almost disjoint from each member of $\mathcal{A}$). If κ is singular, to ensure that the (κ, κ)-family $\mathcal{A}$ has a θ-ordering we need to assume the stronger condition that always $|A \cap B| < \eta$ for some fixed $\eta < \kappa$.

Before we explain our interest in θ-orderings, we need some more terminology. The family $\mathcal{A}$ is said to satisfy the *intersection condition* $C(\eta, \theta)$ if $|\bigcap \mathcal{B}| < \theta$ for all subfamilies $\mathcal{B}$ of $\mathcal{A}$ with $|\mathcal{B}| = \eta$. A set T is called a *θ-transversal* of the family $\mathcal{A}$ if $1 \leq |T \cap A| < \theta$ for all A in $\mathcal{A}$. The family $\mathcal{A}$ is said to be *sparse* if there is a function $f \colon \mathcal{A} \to P \bigcup \mathcal{A}$ with $f(A) \subseteq A$ and $|f(A)| < |A|$ for all A in $\mathcal{A}$, such that $\{A - f(A); A \in \mathcal{A}\}$ is a pairwise disjoint family.

It is a theorem of Erdös and Hajnal ([2], Theorem 7) that every (λ, κ)-family satisfying $C(2, \theta)$ has a θ^+-transversal, provided λ is not too large (and with some restriction on $\theta < \kappa$). It was recently shown by Komjáth ([5], Theorem 5) that, under similar conditions, every such family is sparse. The proofs of these two results are little involved. A similar inductive construction is used in both cases, though the details are different. Our interest in θ-orderings was aroused by the observation that the families in question possess a θ^+-order, and it is almost trivial to deduce from this that they are sparse and have a θ^+-transversal. Several further properties we looked at turned out to be an easy consequence of a θ-ordering. For instance, in ([8], Theorem 3.2) we showed that

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provided λ is not too large, every almost disjoint (λ, κ)-family satisfying certain chain conditions has a κ-transversal. We shall show in Section 3 that under these circumstances, the family possesses a κ-ordering. The existence of a κ-transversal follows easily from this. Here, by a chain condition we mean the following. The family $\mathcal{A}$ satisfies the *μ-chain condition* if there is no set $D \subseteq \bigcup \mathcal{A}$ and sequence $(A_\alpha; \alpha < \mu)$ of sets from $\mathcal{A}$ such that $D \cap A_\alpha \subset D \cap A_\beta$ whenever $\alpha < \beta < \mu$ (where $\subset$ means strict inclusion).

The main results which we prove concerning the existence of θ-orderings are the following. (These results appear in Theorems 2.4, 2.6, 3.3 and 3.4 below.)

THEOREM 1.2. *Let $\mathcal{A}$ be an almost disjoint (λ, κ)-family.*

(a) (*GCH*) *If $\theta^+ < \kappa$ and either $\mathcal{A}$ satisfies $C(2, \theta)$, or else κ is regular and $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$, then $\mathcal{A}$ has a θ^{++}-ordering.*

(b) (*GCH*) *If $\theta^+ \le \kappa$ and $cf(\mu) \ne cf(\theta)$ whenever $\kappa \le \mu < \lambda$, and $\mathcal{A}$ satisfies the same intersection conditions as in* (a), *then $\mathcal{A}$ has a θ^+-ordering.*

(c) (*$V = L$*) *If $\theta^+ \le \kappa$ and $\mathcal{A}$ satisfies the same intersection conditions as in* (a), *then $\mathcal{A}$ has a θ^+-ordering.*

(d) *If κ is regular and $\mathcal{A}$ satisfies the $\aleph_0$-chain condition, then $\mathcal{A}$ has a κ-ordering.*

(e) (*GCH*) *If κ is regular, and $cf(\mu) \ne \aleph_0$ whenever $\kappa < \mu < \lambda$, and $\mathcal{A}$ satisfies the κ-chain condition, then $\mathcal{A}$ has a κ-ordering.*

(f) (*$V = L$*) *If κ is regular and $\mathcal{A}$ satisfies the κ-chain condition then $\mathcal{A}$ has a κ-ordering.*

The paper is organized as follows. We continue this introduction with a couple of simple observations that provide our constructions for θ-orderings. In Section 2 we give the existence proofs when the family $\mathcal{A}$ satisfies the intersection conditions. Section 3 is devoted to the results under the chain conditions. And in Section 4 we give a number of applications.

Our notation is mostly standard. We use $[A]^\eta$ or $[A]^{<\eta}$ for the set of all subsets B of A with $|B| = \eta$ or $|B| < \eta$, respectively, and ${}^{<\eta}A$ for the set of all sequences of elements of A of length less than η. Weak cardinal exponentiation is indicated by $\kappa^{<\lambda}$. The cofinality of the cardinal κ is written $cf(\kappa)$. The cardinal successor of κ is κ^+, and $\kappa^{\alpha+}$ is the iteration of this α times. For cardinal κ, by a κ-sequence we mean a non-decreasing sequence $(\kappa_\sigma; \sigma < cf(\kappa))$ of cardinals $\kappa_\sigma < \kappa$ with $\kappa = \sum(\kappa_\sigma; \sigma < cf(\kappa))$. The letters $\eta, \theta, \kappa, \lambda, \mu$ will be used for infinite cardinal numbers, and other lower case Greek letters for ordinal numbers (with ω for the least infinite ordinal). The letters i, m, n will be used for finite ordinals.

We conclude this section with two constructions that will be used in the following sections.

LEMMA 1.3. *Let $\theta^+ \le \kappa$. Suppose the (λ,κ)-family $\mathcal{A}$ satisfies*
(i) *every $\mathcal{B}$ in $[\mathcal{A}]^{<\lambda}$ has a θ^+-ordering, and*
(ii) *for every $\mathcal{C} \in [\mathcal{A}]^{<\lambda}$ there is $\mathcal{C}^* \in [\mathcal{A}]^{<\lambda}$ with $\mathcal{C} \subseteq \mathcal{C}^*$ such that*

$$\forall A \in \mathcal{A}\left(\left|A \cap \bigcup \mathcal{C}^*\right| \ge \theta \Rightarrow A \in \mathcal{C}^*\right). \tag{4}$$

Then $\mathcal{A}$ possesses a θ^+-ordering.

PROOF. Write $\mathcal{A} = \bigcup\{\mathcal{A}_\sigma; \sigma < cf(\lambda)\}$ where always $|\mathcal{A}_\sigma| < \lambda$. Define recursively $\mathcal{B}_\sigma \in [\mathcal{A}]^{<\lambda}$ for $\sigma < cf(\lambda)$ by

$$\mathcal{B}_\sigma = \left(\mathcal{A}_\sigma \cup \bigcup\{\mathcal{B}_\tau; \tau < \sigma\}\right)^*.$$

Then $\mathcal{A} = \bigcup\{\mathcal{B}_\sigma; \sigma < cf(\lambda)\}$, and for each $A \in \mathcal{A}$ let $\sigma(A)$ be the least σ such that $A \in \mathcal{B}_\sigma$. Thus if $\tau < \sigma(A)$ then $A \notin \mathcal{B}_\tau$ so $|A \cap \bigcup \mathcal{B}_\tau| < \theta$ by (4), since $\mathcal{B}_\tau = \mathcal{C}^*$ for $\mathcal{C} = \mathcal{A}_\sigma \cup \bigcup\{\mathcal{B}_\tau; \tau < \sigma\}$.

Hence

$$\left|A \cap \bigcup\left\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\right\}\right| < \theta^+, \tag{5}$$

since $\bigcup \mathcal{B}_\rho \subseteq \bigcup \mathcal{B}_\tau$ for $\rho < \tau < \sigma(A)$.

By (i), there is a θ^+-ordering of $\mathcal{B}_\sigma$, say $\prec_\sigma$ with family of intervals $\mathcal{I}_\sigma$. Define $\prec$ on $\mathcal{A}$ by:

$$A \prec B \Leftrightarrow \sigma(A) < \sigma(B) \text{ or } [\sigma(A) = \sigma(B) \text{ and } A \prec_{\sigma(A)} B].$$

Clearly this is a well order of $\mathcal{A}$, and for each $I \in \mathcal{I}_\sigma$, if $I^* = I - \bigcup\{\bigcup \mathcal{B}_\tau; \tau < \sigma\}$ then I^* is an interval (possibly empty) of $\prec$. Put $\mathcal{I} = \{I^*; \exists\sigma < cf(\lambda)(I \in \mathcal{I}_\sigma)\}$, so $\mathcal{A} = \bigcup \mathcal{I}$. Take any $A \in \mathcal{A}$. If $A \in K$ for some $K \in \mathcal{I}$, then there must be $I \in \mathcal{I}_{\sigma(A)}$ with $K = I^*$ and $A \in I$. Since $\{B \in K; B \prec A\} \subseteq \{B \in I; B \prec_{\sigma(A)} A\}$, certainly $|A \cap \bigcup\{B \in K; B \prec A\}| < \kappa$ since $\prec_{\sigma(A)}$ is a θ^+-ordering. To establish that $\prec$ is a θ^+=ordering of $\mathcal{A}$, it remains to show that

$$\left|A \cap \bigcup\left\{\bigcup L; L \in \mathcal{I} \text{ and } L \prec K\right\}\right| < \theta^+. \tag{6}$$

Take $L \in \mathcal{I}$ with $L \prec K$. The $L = J^*$ for some J where either $J \in \mathcal{I}_\tau$ for some $\tau < \sigma(A)$, or $J \in \mathcal{I}_{\sigma(A)}$ with $J \prec_{\sigma(A)} I$. Since

$$\bigcup\left\{\bigcup J^*; J \in \mathcal{I}_{\sigma(A)} \text{ and } J \prec_{\sigma(A)} I\right\} \subseteq \bigcup\left\{\bigcup J; J \in \mathcal{I}_{\sigma(A)} \text{ and } J \prec_{\sigma(A)} I\right\}$$

and $\prec_{\sigma(A)}$ is a θ^+-ordering we have $\left|A \cap \bigcup\left\{\bigcup J^*; J \in \mathcal{I}_{\sigma(A)} \text{ and } J \prec_{\sigma(A)} I\right\}\right| < \theta^+$. Also $\bigcup\{\bigcup J^*; J \in \mathcal{I}_\tau \text{ and } \tau < \sigma(A)\} \subseteq \bigcup\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\}$ so by (5), $|A \cap \bigcup\{\bigcup J^*; J \in \mathcal{I}_\tau \text{ and } \tau < \sigma(A)\}| < \theta^+$. Hence (6) holds, and so $\prec$ is a θ^+-ordering of $\mathcal{A}$.

LEMMA 1.4. *Let $cf(\lambda) > \omega$. Suppose the (λ, κ)-family $\mathcal{A}$ satisfies*

(i) *every $\mathcal{B}$ in $[\mathcal{A}]^{<\lambda}$ has θ-ordering, and*

(ii) *there is a family $\{U_\rho; \rho < cf(\lambda)\}$ with $\bigcup \mathcal{A} = \bigcup\{U_\rho; \rho < cf(\lambda)\}$ and $U_\rho \subseteq U_\tau$ whenever $\rho < \tau < cf(\lambda)$, such that*

(a) $\forall A \in \mathcal{A} \exists \rho < cf(\lambda) \exists n \geq 1 (A \subseteq U_{\rho+n}$ *and* $|A \cap U_\rho| < \theta)$,

(b) $\forall \rho < cf(\lambda)(|\{A \in \mathcal{A}; A \subseteq U_\rho\}| < \lambda)$.

Then $\mathcal{A}$ possesses a θ-ordering.

PROOF. For each $\sigma < cf(\lambda)$ put $\mathcal{B}_\sigma = \{A \in \mathcal{A}; A \subseteq U_{\omega(\sigma+1)}\}$ (where $\omega(\sigma+1)$ means the ordinal product), so $\mathcal{B}_\sigma \subseteq \mathcal{B}_\tau$ if $\sigma < \tau < cf(\lambda)$, and $\mathcal{B}_\sigma \in [\mathcal{A}]^{<\lambda}$ with $\mathcal{A} = \bigcup\{\mathcal{B}_\sigma; \sigma < cf(\lambda)\}$. For each $A \in \mathcal{A}$ let $\sigma(A)$ be the least σ such that $A \in \mathcal{B}_\sigma$. Then by (a), $A \subseteq U_{\omega\sigma+m}$ for some $m \geq 1$, and $|A \cap U_{\omega\sigma(A)}| < \theta$. In particular, since

$$\bigcup\left\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\right\} = \bigcup\{U_{\omega(\tau+1)}; \tau < \sigma(A)\} \subseteq U_{\omega\sigma(A)}$$

we have $|A \cap \bigcup\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\}| < \theta$. By (i), there is a θ-ordering $\prec_\sigma$ of $\mathcal{B}_\sigma$. Define $\prec$ on $\mathcal{A}$ by

$$A \prec B \Leftrightarrow \sigma(A) < \sigma(B) \text{ or } [\sigma(A) = \sigma(B) \text{ and } A \prec_{\sigma(A)} B]. \tag{7}$$

Then just as in the proof of Lemma 1.3, $\prec$ is a θ-ordering of $\mathcal{A}$.

2. Intersection conditions

We shall make use of the following result, going back to Tarski (for example, see ([6], Lemma 3.2.3 and Corollary 3.2.4)).

LEMMA 2.1 (*GCH*). *Suppose $|S| = \mu$ and let $\mathcal{A}$ be a family of subsets of S satsifying $C(\mu^+, \theta)$. Then $|\mathcal{A}| \leq \mu$ provided either*

$$\theta^+ \leq \mu \text{ and } \forall A \in \mathcal{A}(|A| > \theta), \text{ or} \tag{8}$$

$$\theta \leq \mu \text{ and } cf(\theta) \neq cf(\mu) \text{ and } \forall A \in \mathcal{A}(|A| \geq \theta). \tag{9}$$

The following two lemmas, combined with Lemma 1.3, will enable us to prove parts (a) and (b) of Theorem 1.2.

LEMMA 2.2 (*GCH*). *Let $\theta^+ \leq \kappa$. Suppose the (λ, κ)-family $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$. Then for each $\mathcal{C} \in [\mathcal{A}]^{\geq\kappa}$ there is $\mathcal{C}^* \subseteq \mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{C}^*$ and $|\mathcal{C}^*| = |\mathcal{C}|$ such that*

$$\forall A \in \mathcal{A}\left(\left|A \cap \bigcup \mathcal{C}^*\right| \geq \theta^+ \Rightarrow A \in \mathcal{C}^*\right).$$

PROOF. Take $\mathcal{C} \in [\mathcal{A}]^{\geq\kappa}$. Recursively define families $\mathcal{C}_i$ for $i < \omega$ by putting $\mathcal{C}_0 = C$ and $\mathcal{C}_{i+1} = \{A \in \mathcal{A}; |A \cap \bigcup \mathcal{C}_i| > \theta\}$. Define $\mathcal{C}^* = \bigcup\{\mathcal{C}_i; i < \omega\}$. Always $\mathcal{C}_i \subseteq \mathcal{C}_{i+1}$ so $|\mathcal{C}_i| \geq \kappa$ and hence $|\bigcup \mathcal{C}_i| = |\mathcal{C}_i|$. Lemma 2.1 ensures that $|\{A \cap \bigcup \mathcal{C}_i; A \in \mathcal{C}_{i+1}\}| \leq |\bigcup \mathcal{C}_i| = |\mathcal{C}_i|$. For each $X \in [\bigcup \mathcal{C}_i]^{>\theta}$ we have $|\{A \in \mathcal{A}; A \cap \bigcup \mathcal{C}_i = X\}| \leq \kappa$ since $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$. Hence $|\mathcal{C}_{i+1}| \leq \kappa \times |\mathcal{C}_i| = |\mathcal{C}_i|$ so that $|\mathcal{C}_{i+1}| = |\mathcal{C}_i|$, for all $i < \omega$. Thus $|\mathcal{C}^*| = |\mathcal{C}_0| = |\mathcal{C}|$. Also if we have $A \in \mathcal{A}$ with $|A \cap \bigcup \mathcal{C}^*| > \theta$, then since $\bigcup \mathcal{C}^* = \bigcup\{\bigcup \mathcal{C}_i; i < \omega\}$, we must have $|A \cap \bigcup \mathcal{C}_i| > \theta$ for some $i < \omega$, so that $A \in \mathcal{C}_{i+1}$ and thus $A \in \mathcal{C}^*$. Thus $\mathcal{C}^*$ has the required properties.

LEMMA 2.3 (GCH). *Suppose $\theta^+ \leq \kappa < \lambda$ and $\lambda \neq \mu^+$ where $cf(\theta) = cf(\mu)$. Suppose the (λ, κ)-family $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$. Then for each $\mathcal{C} \in [\mathcal{A}]^{<\lambda}$ there is $\mathcal{C}^* \in [\mathcal{A}]^{<\lambda}$ with $\mathcal{C} \subseteq \mathcal{C}^*$ such that*

$$\forall A \in \mathcal{A} \left(\left| A \cap \bigcup \mathcal{C}^* \right| \geq \theta \Rightarrow A \in \mathcal{C}^* \right). \tag{10}$$

PROOF. Suppose first $cf(\theta) \neq \omega$. This case is similar to the proof of the previous lemma. Take $\mathcal{C} \in [\mathcal{A}]^{<\lambda}$, and we may suppose $|\mathcal{C}| \geq \kappa$. Recursively define families $\mathcal{C}_i$ for $i < \omega$ by putting $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_{i+1} = \{A \in \mathcal{A}; |A \cap \bigcup \mathcal{C}_i| \geq \theta\}$, and put $\mathcal{C}^* = \bigcup\{\mathcal{C}_i; i < \omega\}$. As before, $|\bigcup \mathcal{C}_i| = |\mathcal{C}_i|$. Let $|\mathcal{C}| = \mu$. We show by induction that always $|\mathcal{C}_i| = \mu$ if $cf(\mu) \neq cf(\theta)$, and always $|\mathcal{C}_i| \leq \mu^+$ if $cf(\mu) = cf(\theta)$. Put $\mathcal{D}_i = \{A \cap \bigcup \mathcal{C}_i; A \in \mathcal{C}_{i+1}\}$. Suppose $cf(\mu) \neq cf(\theta)$, and $|\mathcal{C}_i| = \mu$. Then $|\mathcal{D}_i| \leq \mu$ by Lemma 2.1. Since $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$, as in the proof of Lemma 2.2, this ensures that $|\mathcal{C}_{i+1}| = \mu$. Now suppose $cf(\mu) = cf(\theta)$. If $|\mathcal{C}_i| = \mu^+$, since $cf(\mu^+) \neq cf(\theta)$, just as above this ensures that $|\mathcal{C}_{i+1}| = \mu^+$. Whereas if $|\mathcal{C}_i| = \mu$, since $\mathcal{D}_i \subseteq [\bigcup \mathcal{C}_i]^{\geq\theta}$ certainly $|\mathcal{D}_i| \leq \mu^+$ so still $|\mathcal{C}_{i+1}| \leq \mu^+$. This completes the induction. Now $\mathcal{C}^* = \bigcup\{\mathcal{C}_i; i < \omega\}$, so $|\mathcal{C}^*| = \mu < \lambda$ if $cf(\mu) \neq cf(\theta)$. If $cf(\mu) = cf(\theta)$ then $|\mathcal{C}^*| \leq \mu^+$, and by hypothesis in this case $\mu^+ \neq \lambda$, so still $|\mathcal{C}^*| < \lambda$. To see that (10) holds, take any A in $\mathcal{A}$ with $|A \cap \bigcup \mathcal{C}^*| \geq \theta$. Since $\bigcup \mathcal{C}^* = \bigcup\{\bigcup \mathcal{C}_i; i < \omega\}$ and $cf(\theta) \neq \omega$, we must have $|A \cap \bigcup \mathcal{C}_i| \geq \theta$ for some $i > \omega$, so that $A \in \mathcal{C}_{i+1}$, and hence $A \in \mathcal{C}^*$.

Now consider the case $cf(\theta) = \omega$. Take $\mathcal{C} \in [\mathcal{A}]^{\mu}$ where we may assume $\kappa \leq \mu < \lambda$. This time we recursively define families $\mathcal{C}_\alpha$ for $\alpha < \omega_1$ by putting $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_{\alpha+1} = \{A \in \mathcal{A}; |A \cap \bigcup \mathcal{C}_\alpha| \geq \theta\}$, and $\mathcal{C}_\alpha = \bigcup\{\mathcal{C}_\beta; \beta < \alpha\}$ when α is a limit ordinal. Put $\mathcal{C}^* = \bigcup\{\mathcal{C}_\alpha; \alpha < \omega_1\}$. Similarly to the previous case, we show by induction that always $|\mathcal{C}_\alpha| = \mu$ if $cf(\mu) \neq cf(\theta)$ or $|\mathcal{C}_\alpha| \leq \mu^+$ if $cf(\mu) = cf(\theta)$. If follows that $|\mathcal{C}^*| < \lambda$ (noting that $\lambda > \aleph_1$, for this case to hold). To see that (10) holds, take any $A \in \mathcal{A}$ with $|A \cap \bigcup \mathcal{C}^*| \geq \theta$. Still $\bigcup \mathcal{C}^* = \bigcup\{\bigcup \mathcal{C}_\alpha; \alpha < \omega_1\}$. Take an increasing θ-sequence $(\theta_n; n < \omega)$. For each n there is $\alpha_n < \omega_1$ such that $|A \cap \bigcup\{\bigcup \mathcal{C}_\alpha; \alpha < \alpha_n\}| \geq \theta_n$. Let β be the least limit ordinal larger than all the α_n, so $\beta < \omega_1$ and $|A \cap \bigcup\{\bigcup \mathcal{C}_\alpha; \alpha < \beta\}| \geq \theta$; thus $|A \cap \bigcup \mathcal{C}_\beta| \geq \theta$ so $A \in \mathcal{C}_{\beta+1}$ and hence $A \in \mathcal{C}^*$.

THEOREM 2.4 (*GCH*). *Let A be an almost disjoint* (λ, κ)*-family. Suppose either* $\mathcal{A}$ *satisfies* $C(2, \theta)$, *or else* κ *is regular and* $\mathcal{A}$ *satisfies* $C(\kappa^+, \theta)$.

(a) *If* $\theta^+ < \kappa$, *then* $\mathcal{A}$ *has a* θ^{++}*-ordering.*

(b) *If* $\theta^+ \leq \kappa$, *and* $(\kappa \leq \mu < \lambda \Rightarrow cf(\mu) \neq cf(\theta))$, *then* $\mathcal{A}$ *possesses a* θ^+*-ordering.*

PROOF. For $\lambda \leq \kappa$, if $\mathcal{A}$ satisfies $C(2, \theta)$ any well order of $\mathcal{A}$ of order type λ is suitable, and the same is true if κ is regular since $\mathcal{A}$ is almost disjoint. For $\lambda > \kappa$ we proceed by induction on λ. Take a suitable (λ, κ)-family $\mathcal{A}$. Consider (b) first. We shall use Lemma 1.3. By the inductive hypothesis, (i) of Lemma 1.3 holds, and (ii) holds by Lemma 2.3, noting that under the conditions in (b), $\lambda \neq \mu^+$ with $cf(\mu) = cf(\theta)$. So Lemma 1.3 ensures that $\mathcal{A}$ possesses a θ^+-ordering.

Similarly for (a), Lemmas 2.2 and 1.3 show that $\mathcal{A}$ has a θ^{++}-ordering.

The least λ for which (b) in Lemma 2.4 does not apply is when $\lambda = \kappa^{cf(\theta)+}$. This method of proof fails for larger λ, though the result may still be true. Indeed, under stronger set theoretic hypotheses the restriction on λ may be lifted. We can continue the transfinite induction if we assume Jensen's principle $\square_\mu$ whenever $cf(\mu) = cf(\theta)$. It is well known that if the axiom of constructibility $(V = L)$ is assumed, then $\square_\mu$ holds for all μ. The statement $\square_\mu$ asserts: for each limit ordinal $\alpha < \mu^+$ there is a closed unbounded set $C_\alpha \subseteq \alpha$ such that $|C_\alpha| < \mu$ whenever $cf(\alpha) < \mu$, and $C_\beta = C_\alpha \cap \beta$ whenever β is a limit point of C_α. It is convenient first to isolate the construction from $\square_\mu$ that we require.

LEMMA 2.5. *Suppose* μ *is singular, and assume* $\square_\mu$. *For each limit ordinal* $\alpha < \mu^+$ *there is a decomposition* $\alpha = \bigcup\{T(\alpha, \sigma); \sigma < cf(\mu)\}$ *where each* $T(\alpha, \sigma)$ *is cofinal in* α *with* $|T(\alpha, \sigma)| < \mu$ *and there is a subset* $D(\alpha) \subseteq \alpha$ *with* $|D(\alpha)| < \mu$ *such that* $T(\gamma, \sigma) \subseteq T(\beta, \sigma)$ *whenever* $\beta, \gamma \in D(\alpha)$ *with* $\gamma < \beta$. *If* $cf(\alpha) > \omega$ *then* $D(\alpha)$ *is cofinal in* α *and* $T(\alpha, \sigma) = \bigcup\{T(\beta, \sigma); \beta \in D(\alpha)\}$.

PROOF. (See Komjáth [5].) Let $(\mu_\sigma; \sigma < cf(\mu))$ be an increasing μ-sequence. For each limit $\alpha < \mu^+$, take sets C_α as provided by $\square_\mu$ where we may suppose $0 \in C_\alpha$, and let $(c_{\alpha\xi}; \xi < ot(C_\alpha))$ be the increasing enumeration of C_α, where $ot(C_\alpha)$ means the order type of C_α. Put $D(\alpha) = \{\beta \in C_\alpha; \beta \text{ is a limit point of } C_\alpha\}$. For $\beta \in D(\alpha)$ we have $C_\beta = C_\alpha \cap \beta$ so $c_{\beta\xi} = c_{\alpha\xi}$ whenever $\xi < ot(C_\beta)$. For $\gamma < \delta < \mu^+$, fix a decomposition $\{\xi; \gamma \leq \xi < \delta\} = \bigcup\{S(\gamma, \delta, \sigma); \sigma < cf(\mu)\}$ where always $1 \leq |S(\gamma, \delta, \sigma)| \leq \mu_\sigma$. For $\sigma < cf(\mu)$ and limit $\alpha < \mu^+$, put $T(\alpha, \sigma) = \bigcup\{S(c_{\alpha\xi}, c_{\alpha\xi+1}, \sigma); \xi < ot(C_\alpha)\}$, so $\alpha = \bigcup\{T(\alpha, \sigma); \sigma < cf(\mu)\}$. Whenever $\beta \in D(\alpha)$ we have $T(\beta, \sigma) = T(\alpha, \sigma) \cap \beta$. Hence $T(\gamma, \sigma) \subseteq T(\beta, \sigma)$ if $\beta, \gamma \in D(\alpha)$ with $\gamma < \beta$. If $cf(\alpha) > \omega$, then $D(\alpha)$ is cofinal in α, and so $T(\alpha, \sigma) = \bigcup\{T(\beta, \sigma); \beta \in D(\alpha)\}$. Also $|D(\alpha)| \leq |C_\alpha|$ and $|C_\alpha| < \mu$ since μ is singular. Finally $|T(\alpha, \sigma)| \leq \sum(|S(c_{\alpha\xi}, c_{\alpha\xi+1}, \sigma)|; \xi < ot(C_\alpha)) \leq \mu_\sigma \times |C_\alpha| < \mu$.

THEOREM 2.6 ($V = L$). *Suppose $\theta^+ \leq \kappa$. Let $\mathcal{A}$ be an almost disjoint (λ, κ)-family satisfying either $C(2, \theta)$, or, if κ is regular, $C(\kappa^+, \theta)$. Then $\mathcal{A}$ possesses a θ^+-ordering.*

PROOF. As in the proof of Theorem 2.4(b), we may suppose $\lambda > \kappa$ and proceed by induction on λ. The previous argument holds unless $\lambda = \mu^+$ where $cf(\mu) = cf(\theta)$. So suppose indeed that $\lambda = \mu^+$, with $cf(\mu) = cf(\theta)$. Let $\mathcal{A} = \{A_\delta; \delta < \mu^+\}$, and we may suppose $\bigcup \mathcal{A} = \mu^+$. We have the sets $D(\alpha)$ and $T(\alpha, \sigma)$ for $\alpha < \mu^+$ and $\sigma < cf(\mu)$, as in Lemma 2.5.

Define limit ordinals $l_\varepsilon < \mu^+$ by transfinite recursion for $\varepsilon < \mu^+$ as follows. Let l_0 be the least limit ordinal $\gamma > \bigcup A_0$, and for ε a limit ordinal, put $l_\varepsilon = \bigcup\{l_\delta; \delta < \varepsilon\}$. At successor stages, put

$$\mathcal{B}_\varepsilon = \{A \in \mathcal{A}; \exists \text{ limit } \alpha < l_\varepsilon \exists \sigma < cf(\mu)(|A \cap T(\alpha, \sigma)| \geq \theta)\},$$

and define $l_{\varepsilon+1}$ to be the least ordinal $\gamma > l_\varepsilon \cup \bigcup A_\varepsilon \cup \bigcup \bigcup \mathcal{B}_\varepsilon$ with $cf(\gamma) > \omega$. To see that then $l_{\varepsilon+1} < \mu^+$, note that since $|T(\alpha, \sigma)| < \mu$ we have $|[T(\alpha, \sigma)]^\theta| \leq \mu$, and for each $X \in [T(\alpha, \sigma)]^\theta$ since $\mathcal{A}$ satisfies $C(\kappa^+, \theta)$ we have $|\{A \in \mathcal{A}; A \cap T(\alpha, \sigma) = X\}| \leq \kappa$, so that $|\mathcal{B}_\varepsilon| \leq \mu$ and hence $l_{\varepsilon+1} < \mu^+$.

Take $A \in \mathcal{A}$, and we show by induction on $\varepsilon < \mu^+$ that if $|A \cap l_\varepsilon| \geq \theta^+$ then $A \in \mathcal{B}_\varepsilon$. If $cf(l_\varepsilon) > \omega$, by Lemma 2.5, $l_\varepsilon = \bigcup\{T(l_\varepsilon, \sigma); \sigma < cf(\mu)\}$ so there must be $\sigma < cf(\mu)$ such that $|A \cap T(l_\varepsilon, \sigma)| \geq \theta^+$, since $cf(\mu) = cf(\theta) < \theta^+$. Now $T(l_\varepsilon, \sigma) = \bigcup\{T(\beta, \sigma); \beta \in D(l_\varepsilon)\}$ and $T(\gamma, \sigma) \subseteq T(\beta, \sigma)$ for $\beta, \gamma \in D(l_\varepsilon)$ with $\gamma < \beta$, so there must be $\beta \in D(l_\varepsilon)$ such that $|A \cap T(\beta, \sigma)| \geq \theta$. And $\beta \in D(l_\varepsilon) \subseteq l_\varepsilon$, so $A \in \mathcal{B}_\varepsilon$ as claimed. If $cf(l_\varepsilon) = \omega$ there are $\varepsilon(n) < \varepsilon$ such that $l_\varepsilon = \bigcup\{l_{\varepsilon(n)}; n < \omega\}$ and since $|A \cap l_\varepsilon| \geq \theta^+$ there must be $n < \omega$ with $|A \cap l_{\varepsilon(n)}| \geq \theta^+$. Then the inductive hypothesis gives that $A \in \mathcal{B}_{\varepsilon(n)}$, so $A \in \mathcal{B}_\varepsilon$.

For $A \in \mathcal{A}$, define $\varepsilon(A)$ to be the least ε such that $A \in \mathcal{B}_\varepsilon$. (Such $\varepsilon(A)$ exists, for if $A = A_\delta$ we have $A \subseteq l_{\delta+1}$, so $A \in \mathcal{B}_{\delta+1}$ by the previous observation.) Now $\varepsilon(A)$ is not a limit ordinal, since for limit ε we have $l_\varepsilon = \bigcup\{l_\delta; \delta < \varepsilon\}$ so $\mathcal{B}_\varepsilon = \bigcup\{\mathcal{B}_\delta : \delta < \varepsilon\}$. And if $\rho < \varepsilon(A)$ we must have $|A \cap l_\rho| < \theta^+$, for if $|A \cap l_\rho| \geq \theta^+$ then $A \in \mathcal{B}_\rho$ by the observation above. Also since $A \in \mathcal{B}_{\varepsilon(A)}$ and $\bigcup \bigcup \mathcal{B}_{\varepsilon(A)} \subseteq l_{\varepsilon(A)+1}$ we have $A \subseteq l_{\varepsilon(A)+1}$.

We show that the conditions of Lemma 1.4 are satsified for $\mathcal{A}$ to possess a θ^+-ordering. Every $\mathcal{B}$ in $[\mathcal{A}]^{<\lambda}$ has a θ^+-ordering by the inductive hypothesis, so (i) of Lemma 1.4 holds. For (ii), define $U_\rho = l_\rho$, for $\rho < \mu^+ = cf(\lambda)$. Certainly $\bigcup \mathcal{A} = \mu^+ = \bigcup\{U_\rho; \rho < \mu^+\}$ and $U_\rho \subseteq U_\tau$ if $\rho < \tau < \mu^+$. Take any $A \in \mathcal{A}$. Since $\varepsilon(A)$ is not a limit ordinal, we can put $\rho = \varepsilon(A) - 1$. Since $A \subseteq l_{\varepsilon(A)+1}$, we have $A \subseteq U_{\rho+2}$, and since $\rho < \varepsilon(A)$ we have $|A \cap U_\rho| < \theta^+$. Hence (iia) holds. For (iib), suppose $\rho < \mu^+$ is given. If $A \subseteq U_\rho = l_\rho$ then $A \in \mathcal{B}_\rho$, but $|\mathcal{B}_\rho| \leq \mu$ and so (iib) is satisfied. Hence by Lemma 1.4, $\mathcal{A}$ possesses a θ^+-ordering.

3. Chain conditions

We shall need the following lemma, from Williams [8].

LEMMA 3.1. *Let κ be regular and suppose $\mathcal{A}$ is an almost disjoint (λ, κ)-family which satisfies either*

(a) *the $\aleph_0$-chain condition, or*

(b) *(GCH) the κ-chain condition.*

Then $\lambda \leq |\bigcup \mathcal{A}|$.

PROOF. For (b), suppose $\mathcal{A} = \{A_\alpha; \alpha < \lambda\}$ is such a family, satisfying the κ-chain condition. For a contradiction, suppose $|\bigcup \mathcal{A}| = \mu$ where $\mu < \lambda$, so $\kappa < \mu^+ \leq \lambda$. We consider two cases.

Case 1. $cf(\mu) < \kappa$. Write $\bigcup \mathcal{A} = \bigcup\{X_\sigma; \sigma < cf(\mu)\}$ where alway $|X_\sigma| < \mu$. For each A in $\mathcal{A}$ there must be $\sigma(A) < cf(\mu)$ such that $|A \cap X_{\sigma(A)}| = \kappa$, and there must be $\sigma < cf(\mu)$ and $\mathcal{B} \subseteq \mathcal{A}$ with $\mathcal{B}| \geq \mu^+$ such that $\sigma(A) = \sigma$ for all $A \in \mathcal{B}$. Since $\mathcal{A}$ is almost disjoint, then $|\{A \cap X_\sigma; A \in \mathcal{B}\}| = |\mathcal{B}| \geq \mu^+$ which is impossible since $|[X_\sigma]^\kappa| \leq |X_\sigma|^+ \leq \mu$.

Case 2. $\kappa \leq cf(\mu)$. For sequences $s, t \in {}^{<\kappa}\mu$, write $t < \cdot\, s$ if t is an initial segment of s. For each sequence $s \in {}^{<\kappa}\mu$ we define an ordinal $\alpha(s) < \mu^+$ and, provided the length of s is a successor ordinal, an element $x(s) \in \bigcup \mathcal{A}$, by recursion on the length of s as follows. Put $\bigcup \mathcal{A} - \bigcup\{A_{\alpha(t)}; t < \cdot\, s\} = \{x(s^\frown\gamma); \gamma < \mu\}$, (noting that this set has cardinality μ since $\mathcal{A}$ is almost disjoint), where by $s^\frown\gamma$ we mean that sequence extending s by one place and having value γ at its last place. If there is $\alpha \in \mu^+ - \{\alpha(t); t < \cdot\, s\}$ such that $x(t) \in A_\alpha$ for all $t \leq \cdot\, s$ of successor length, then $\alpha(s)$ is to be the least such α, and otherwise $\alpha(s) = 0$.

By GCH, $|{}^{<\kappa}\mu| = \mu$ and we can choose $\beta \in \mu^+ - \{\alpha(s); s \in {}^{<\kappa}\mu\}$. For every $s \in {}^{<\kappa}\mu$, since $\mathcal{A}$ is almost disjoint $|A_\beta \cap \bigcup\{A_{\alpha(t)}; t < \cdot\, s\}| < \kappa$ so $A_\beta \cap \left(\bigcup \mathcal{A} - \bigcup\{A_{\alpha(t)}; t < \cdot\, s\}\right)$ is non-empty. Hence there is $\gamma < \mu$ such that $x(s^\frown\gamma) \in A_\beta$. This means we can define a sequence $r \in {}^\kappa\mu$ by recursively defining $r(\delta)$ for each $\delta < \kappa$ to be the least γ such that $x((r|\delta)^\frown\gamma) \in A_\beta$, where by $r|\delta$ we mean the sequence $(r(\varepsilon); \varepsilon < \delta)$. Put $x_\delta = x(r|(2\delta + 1))$ and $\alpha(\delta) = \alpha(r|(2\delta+1))$. The definition of $\alpha(\sigma)$ ensures that $x_\delta \in A_{\alpha(\sigma)}$ whenever $\delta \leq \sigma < \kappa$, whereas the definition of x_δ ensures that $x_\delta \notin A_{\alpha(\sigma)}$ whenever $\sigma < \delta < \kappa$. Hence $A_{\alpha(\sigma)} \cap \{x_\delta; \delta < \kappa\} = \{x_\delta; \delta \leq \sigma\}$. Thus if $D = \{x_\delta; \delta < \kappa\}$, we have $D \cap A_{\alpha(\sigma)} \subset D \cap A_{\alpha(\tau)}$ whenever $\sigma < \tau < \kappa$, contradicting that $\mathcal{A}$ satisfies the κ-chain condition.

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The proof of (a) proceeds as Case 2 above, defining $\alpha(s)$ and $x(s)$ for sequences $s \in {}^{<\omega}\mu$.

LEMMA 3.2. *Let κ be regular and suppose $\mathcal{A}$ is an almost disjoint (λ, κ)-family.*

(a) *If $\mathcal{A}$ satisfies the $\aleph_0$-chain condition, then for every $\mathcal{C} \in [\mathcal{A}]^{\geq\kappa}$ there is $\mathcal{C}^* \subseteq \mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{C}^*$ and $|\mathcal{C}^*| = |\mathcal{C}|$ such that*

$$\forall A \in \mathcal{A} \left(\left| A \cap \bigcup \mathcal{C}^* \right| = \kappa \Rightarrow A \in \mathcal{C}^* \right). \tag{11}$$

(b) (*GCH*) *If $\mathcal{A}$ satisfies the κ-chain condition, then for every $\mathcal{C} \in [\mathcal{A}]^{\geq\kappa}$ with $cf(|\mathcal{C}|) \geq \kappa$ there is $\mathcal{C}^* \subseteq \mathcal{A}$ with $\mathcal{C} \subseteq \mathcal{C}^*$ and $|\mathcal{C}^*| = |\mathcal{C}|$ such that* (11) *holds.*

PROOF. Take $\mathcal{C} \in [\mathcal{A}]^\mu$ where $\kappa \leq \mu < \lambda$, and $cf(\mu) \geq \kappa$ if (b) holds. (Obviously if $\mu = \lambda$, we can take $\mathcal{C}^* = \mathcal{A}$.) To cover both (a) and (b) at once, put $\eta = \aleph_0$ if (a) holds, and $\eta = \kappa$ if (b) holds. Recursively define families $\mathcal{C}_\alpha \in [\mathcal{A}]^\mu$ for $\alpha < \eta$ as follows. Put $Z_\alpha = \bigcup\{\bigcup \mathcal{C}_\beta; \beta < \alpha\}$, so $|Z_\alpha| = \mu$. For each $S \in [Z_\alpha]^{<\eta}$, choose $A(\alpha, S) \in \mathcal{A} - \bigcup\{\mathcal{C}_\beta; \beta < \alpha\}$ with $S \subseteq A(\alpha, S)$ if such a set $A(\alpha, S)$ exists; otherwise let $A(\alpha, S)$ be empty. Put $X_\alpha = \bigcup \mathcal{C} \cup Z_\alpha \cup \bigcup\{A(\alpha, S); S \in [Z_\alpha]^{<\eta}\}$. Our assumptions ensure that $\mu^{<\eta} = \mu$, so that $|X_\alpha| = \mu$. Now define $\mathcal{C}_\alpha = \{A \in \mathcal{A}; |A \cap X_\alpha| = \kappa\}$. Since $\{A \cap X_\alpha; A \in \mathcal{C}_\alpha\}$ is an an almost disjoint decomposition of X_α satisfying the η-chain condition, by Lemma 3.1 we have $|\{A \cap X_\alpha; A \in \mathcal{C}_\alpha\}| \leq |X_\alpha| = \mu$, and hence since $\mathcal{A}$ is almost disjoint we have $|\mathcal{C}_\alpha| \leq \mu$, so $|\mathcal{C}_\alpha| = \mu$. Note that the definition of $\mathcal{C}_\alpha$ ensures that $\mathcal{C} \subseteq \mathcal{C}_\alpha$ and $\bigcup\{\mathcal{C}_\beta; \beta < \alpha\} \subseteq \mathcal{C}_\alpha$. Also $Z_\alpha \subseteq X_\alpha \subseteq Z_{\alpha+1}$. Define $\mathcal{C}^* = \{\mathcal{C}_\alpha; \alpha < \eta\}$ so $\mathcal{C} \subseteq \mathcal{C}^* \subseteq \mathcal{A}$ and $|\mathcal{C}^*| = \mu$.

We show $\mathcal{C}^*$ has the required property. Take $A \in \mathcal{A}$ such that $|A \cap \bigcup \mathcal{C}^*| = \kappa$. Now $\bigcup \mathcal{C}^* = \bigcup\{Z_\alpha; \alpha < \eta\}$, and we claim that there is $\delta < \eta$ such that $|A \cap Z_\delta| = \kappa$. If so, $|A \cap X_\delta| = \kappa$ so $A \in \mathcal{C}_\delta \subseteq \mathcal{C}^*$ and the proof would be complete. So for a contradiction, suppose $|A \cap Z_\delta| < \kappa$ for all $\delta < \eta$. There must then be an increasing sequence $(\alpha(\sigma); \sigma < \eta)$ such that $A \cap (Z_{\alpha(\sigma+1)} - Z_{\alpha(\sigma)})$ is non-empty for each $\sigma < \eta$, and (by deleting every second term if necessary) we may in fact suppose $A \cap (Z_{\alpha(\sigma+1)} - Z_{\alpha(\sigma)+1})$ is non-empty. Choose $x_\sigma \in A \cap (Z_{\alpha(\sigma+1)} - Z_{\alpha(\sigma)+1})$, and put $S_\sigma = \{x_\tau; \tau \leq \sigma\}$. Then $S_\sigma \in [Z_{\alpha(\sigma+1)}]^{<\eta}$. Now $S_\sigma \subseteq A$ and $A \notin \bigcup\{\mathcal{C}_\beta; \beta < \alpha(\sigma+1)\}$ so $S_\sigma \subseteq A(\alpha(\sigma+1), S_\sigma) \in \mathcal{A}$. Put $A_\sigma = A(\alpha(\sigma+1), S_\sigma)$. Then $A_\sigma \subseteq X_{\alpha(\sigma+1)} \subseteq Z_{\alpha(\sigma+1)+1}$. If $\sigma < \tau < \kappa$ then $\alpha(\sigma+1) \leq \alpha(\tau) < \alpha(\tau)+1$ and $x_\tau \notin Z_{\alpha(\tau)+1}$ so $x_\tau \notin Z_{\alpha(\sigma+1)+1}$, and hence $x_\tau \notin A_\sigma$. Thus $A_\sigma \cap \{x_\tau; \tau < \eta\} = \{x_\tau; \tau \leq \sigma\}$. Put $D = \{x_\tau; \tau < \eta\}$, so $D \cap A_\sigma \subset D \cap A_\tau$ whenever $\sigma < \tau < \eta$, contradicting that $\mathcal{A}$ satisfies the η-chain condition.

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THEOREM 3.3. *Let κ be regular and suppose $\mathcal{A}$ is an almost disjoint (λ,κ)-family. Suppose either*

(a) *$\mathcal{A}$ satisfies the $\aleph_0$-chain condition, or*

(b) *$\mathcal{A}$ satisfies the κ-chain condition and $\lambda \leq \kappa^{\omega+}$.*

Then $\mathcal{A}$ has a κ-ordering.

PROOF. If $\lambda \leq \kappa$, then $\mathcal{A}$ has a κ-ordering since $\mathcal{A}$ is almost disjoint. For $\lambda > \kappa$ we proceed by induction on λ. Let $(\lambda_\sigma; \sigma < cf(\lambda))$ be a λ-sequence of cardinals with $\kappa \leq \lambda_0 \leq \lambda_1 \leq \cdots$ and $\sum(\lambda_\sigma; \sigma < \tau) \leq \lambda_\tau$ for each $\tau < cf(\lambda)$, with always $cf(\lambda_\sigma) \geq \kappa$ if (b) holds. (Note such a sequence can be found except when $\lambda = \mu^+$ where $cf(\mu) < \kappa$.) Put $\eta = \aleph_0$ if (a) holds, and $\eta = \kappa$ if (b) holds. Take an almost disjoint (λ,κ)-family $\mathcal{A}$ with the η-chain condition. Write $\mathcal{A} = \bigcup\{\mathcal{A}_\sigma; \sigma < cf(\lambda)\}$ where always $|\mathcal{A}_\sigma| = \lambda_\sigma$. Use Lemma 3.2 to define families $\mathcal{B}_\sigma \subseteq \mathcal{A}$ for $\sigma < cf(\lambda)$ by $\mathcal{B}_\sigma = (\mathcal{A}_\sigma \cup \bigcup\{\mathcal{B}_\tau; \tau < \sigma\})^*$, and it follows from Lemma 3.2 that always $|\mathcal{B}_\sigma| = \lambda_\sigma$. Thus $\mathcal{B}_\rho \subseteq \beta_\sigma$ if $\rho < \sigma < cf(\lambda)$, and $\mathcal{A} = \bigcup\{\mathcal{B}_\sigma, \sigma < cf(\lambda)\}$.

We proceed as in the proof of Lemma 1.3. For each A let $\sigma(A)$ be the least σ such that $A \in \mathcal{B}_\sigma$. By the inductive hypothesis, there is a κ-ordering $\prec_\sigma$ of $\mathcal{B}_\sigma$. Define $\prec$ on $\mathcal{A}$ by

$$A \prec B \Leftrightarrow \sigma(A) < \sigma(B) \text{ or } [\sigma(A) = \sigma(B) \text{ and } A \prec_{\sigma(A)} B].$$

Just as in the proof of Lemma 1.3, this will be a κ-ordering of $\mathcal{A}$ provided that

$$\left|A \cap \bigcup\left\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\right\}\right| < \kappa. \tag{12}$$

If $\tau < \sigma(A)$ then $A \notin \mathcal{B}_\tau = \mathcal{C}^*$ where $\mathcal{C} = \mathcal{A}_\tau \cup \bigcup\{\mathcal{B}_\rho; \rho < \tau\}$, so by (11) we have $|A \cap \bigcup \mathcal{B}_\tau| < \kappa$. So if $cf(\lambda) \leq \kappa$, certainly (12) holds. And if $\sigma(A)$ is a successor ordinal, $\sigma(A) = \xi + 1$, then $\bigcup\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\} = \bigcup \mathcal{B}_\xi$ so there is no difficulty. We are left with the case that $\sigma(A)$ is a limit ordinal (and $\kappa < cf(\lambda)$, though we won't make use of this condition). Suppose for a contradiction that (12) is false, so $|A \cap \bigcup\{\bigcup \mathcal{B}_\tau; \tau < \sigma(A)\}| = \kappa$. Since $\bigcup \mathcal{B}_\rho \subseteq \bigcup \mathcal{B}_\tau$ if $\rho < \tau < \sigma(A)$, there must be an increasing sequence $(\rho(\sigma); \sigma < \kappa)$ of ordinals below $\sigma(A)$ such that $A \cap \left(\bigcup \mathcal{B}_{\rho(\sigma+1)} - \bigcup \mathcal{B}_{\rho(\sigma)}\right)$ is non-empty for each $\sigma < \kappa$. Choose $x_\sigma \in A \cap \left(\bigcup \mathcal{B}_{\rho(\sigma+1)} - \bigcup \mathcal{B}_{\rho(\sigma)}\right)$, and put $S_\sigma = \{x_\tau; \tau \leq \sigma\}$, so $S_\sigma \subseteq \bigcup \mathcal{B}_{\rho(\sigma+1)}$. Put $\mathcal{C} = \mathcal{A}_{\rho(\sigma+1)} \cup \bigcup\{\mathcal{B}_\tau; \tau < \rho(\sigma+1)\}$, so $\mathcal{B}_{\rho(\sigma+1)} = \mathcal{C}^*$, and $S_\sigma \subseteq \bigcup \mathcal{C}^*$. Consider the construction of $\mathcal{C}^*$ in the proof of Lemma 3.2. We have $\mathcal{C}^* = \bigcup\{\mathcal{C}_\alpha; \alpha < \eta\}$ so $\bigcup \mathcal{C}^* = \bigcup\{\bigcup \mathcal{C}_\alpha; \alpha < \eta\}$, and $\bigcup \mathcal{C}_\beta \subseteq \bigcup \mathcal{C}_\alpha$ whenever $\beta < \alpha < \eta$. When $\sigma < \eta$ we have $S_\sigma \in [\bigcup \mathcal{C}^*]^{<\eta}$ so there must be $\alpha < \eta$ such that $S_\sigma \subseteq \bigcup \mathcal{C}_\alpha$. Thus $S_\sigma \in [Z_{\alpha+1}]^{<\eta}$ and so $S_\sigma \subseteq A(\alpha+1, S_\sigma)$ and $A(\alpha+1, S_\sigma) \in \mathcal{C}^*$. (Note that $A(\alpha+1, S_\sigma)$ can't be empty since $S_\sigma \subseteq A$ and $A \notin \bigcup\{\mathcal{C}_\beta; \beta < \alpha\}$ because $A \notin \mathcal{B}_{\rho(\sigma+1)} = \mathcal{C}^*$.) Put $A_\sigma = A(\alpha+1, S_\sigma)$. Thus for $\sigma < \eta$, we have $A_\sigma \in \mathcal{B}_{\rho(\sigma+1)}$ with $S_\sigma \subseteq A_\sigma$. And if $\sigma < \tau < \eta$ then $x_\tau \notin \bigcup \mathcal{B}_{\rho(\tau)}$ so $x_\tau \notin \bigcup \mathcal{B}_{\rho(\sigma+1)}$

and hence $x_\tau \notin A_\sigma$. Thus $A_\sigma \cap \{x_\tau; \tau < \eta\} = \{x_\tau; \tau \leq \sigma\}$. Put $D = \{x_\tau; \tau < \eta\}$, so $D \cap A_\sigma \subset D \cap A_\tau$ whenever $\sigma < \tau < \eta$, contradicting that $\mathcal{A}$ satisfies the η-chain condition. Hence (12) holds, and the proof is complete.

The transfinite induction in Theorem 3.3(b) breaks down first when $\lambda = \kappa^{(\omega+1)+}$. As was the case with Theorem 2.4(b) we can continue the induction if $\square_\mu$ holds for appropriate μ.

THEOREM 3.4($V = L$). *Let κ be regular and suppose $\mathcal{A}$ is an almost disjoint (λ, κ)-family satisfying the κ-chain condition. Then $\mathcal{A}$ has a κ-ordering.*

PROOF. We proceed by transfinite induction on λ, as in the proof of Theorem 3.3(b). The previous argument works unless $\aleph_0 < \kappa < \lambda = \mu^+$ and $cf(\mu) < \kappa$, so suppose this to be the case. Let $\mathcal{A} = \{A_\delta; \delta < \mu^+\}$, where we may suppose $\bigcup \mathcal{A} = \mu^+$. We have the sets $D(\alpha)$ and $T(\alpha, \sigma)$ for $\alpha < \mu^+$ and $\sigma < cf(\mu)$ as in Lemma 2.5. Define families $\mathcal{B}_\varepsilon \in [\mathcal{A}]^{\leq \mu}$ and limit ordinals l_ε with $\mu \leq l_\varepsilon < \mu^+$ by transfinite recursion for $\varepsilon < \mu^+$ as follows. Put $\mathcal{B}_0 = \{A_0\}$ and let l_0 be the least limit ordinal $\gamma > \bigcup A_0$. Suppose $\mathcal{B}_\delta$ defined for all $\delta < \varepsilon$ (for $\varepsilon > 0$). If ε is a limit ordinal, define $l_\varepsilon = \bigcup\{l_\delta; \delta < \varepsilon\}$. If ε is a successor, define l_ε to be the least ordinal $\gamma > l_{\varepsilon-1} \cup \bigcup\{\mathcal{B}_\delta; \delta < \varepsilon\}$. So $l_\varepsilon < \mu^+$. For each limit $\alpha < l_\varepsilon$ and for each $\sigma < cf(\mu)$, for each $S \in [T(\alpha, \sigma)]^{<\kappa}$ choose $A(\varepsilon, \alpha, \sigma, S) \in \mathcal{A} - \bigcup\{\mathcal{B}_\delta; \delta < \varepsilon\}$ with $S \subseteq A(\varepsilon, \alpha, \sigma, S)$ if such a set $A(\varepsilon, \alpha, \sigma, S)$ exists; otherwise let $A(\varepsilon, \alpha, \sigma, S)$ be empty. Put

$$X_\varepsilon = l_\varepsilon \cup A_\varepsilon \cup \bigcup\{A(\varepsilon, \alpha, \sigma, S); \exists \text{ limit } \alpha < l_\varepsilon \exists \sigma < cf(\mu)(S \in [T(\alpha, \sigma)]^{<\kappa})\}$$

and finally put

$$\mathcal{B}_\varepsilon = \{A \in \mathcal{A}; |A \cap X_\varepsilon| = \kappa\}.$$

We have to check that $|\mathcal{B}_\varepsilon| \leq \mu$. Note if $\delta < \varepsilon$ then $\bigcup \mathcal{B}_\delta \subseteq l_\varepsilon \subseteq X_\varepsilon$, so $\mathcal{B}_\delta \subseteq \mathcal{B}_\varepsilon$. Always $|T(\alpha, \sigma)| < \mu$, so $|[T(\alpha, \sigma)]^{<\kappa}| \leq \mu$ and hence $|X_\varepsilon| \leq \mu$. Since $\{A \cap X_\varepsilon; A \in \mathcal{B}_\varepsilon\}$ is an almost disjoint decomposition of X_ε with the κ-chain condition, $|\{A \cap X_\varepsilon; A \in \mathcal{B}_\varepsilon\}| \leq \mu$ by Lemma 3.1. Hence, since $\mathcal{A}$ is almost disjoint, we have $|\mathcal{B}_\varepsilon| \leq \mu$.

For $A \in \mathcal{A}$, define $\varepsilon(A)$ to be the least ε such that $|A \cap l_\varepsilon| = \kappa$. (Such $\varepsilon(A)$ exists, for if $A = A_\delta$ we have $A \in \mathcal{B}_\delta$ so $A \subseteq \bigcup \mathcal{B}_\delta \subseteq l_{\delta+1}$.) We claim that $\varepsilon(A)$ is not a limit ordinal. For suppose on the contrary that $\varepsilon(A)$ is a limit. Write $\xi = l_{\varepsilon(A)}$, so $\xi = \bigcup\{l_\delta; \delta < \varepsilon(A)\}$ and $cf(\xi) = cf(\varepsilon(A))$ since the l_δ increase with δ. There must be an increasing sequence $(\delta(\sigma); \sigma < cf(\xi))$ with $\delta(\sigma) < \varepsilon(A)$ such that $\{l_{\delta(\sigma)}; \sigma < cf(\xi)\}$ is cofinal in ξ. Since $|A \cap l_\delta| < \kappa$ for $\delta < \varepsilon(A)$, yet $|A \cap \xi| = \kappa$, we must have $cf(\xi) = \kappa$ and we may suppose $A \cap (l_{\delta(\sigma+1)} - l_{\delta(\sigma)+1})$ is non-empty for each $\sigma < \kappa$. Also since $cf(\xi) = \kappa > \omega$, we have $D(\xi)$ cofinal in ξ. Define recursively $\delta(\sigma) \in D(\xi)$ and $\rho(\sigma) < \varepsilon(A)$ for $\sigma < \kappa$ as follows. Let $\gamma(0)$ be the least element of $D(\xi)$, and if σ is a limit let $\gamma(\sigma)$

be the least γ in $D(\xi)$ with $\gamma \geq \bigcup\{\gamma(\tau); \tau < \sigma\}$. Suppose $\gamma(\sigma)$ is defined. Let $\rho(\sigma)$ be the least $\rho < \varepsilon(A)$ such that $\gamma(\sigma) < l_{\delta(\rho)}$, and then define $\gamma(\sigma+1)$ to be the least γ in $D(\xi)$ with $\gamma > l_{\delta(\rho(\sigma)+1)}$. Choose $x_\sigma \in A \cap (l_{\delta(\rho(\sigma)+1)} - l_{\delta(\rho(\sigma))+1})$, so $x_\sigma \in A \cap (\gamma(\sigma+1) - \gamma(\sigma))$ since $\gamma(\sigma+1) > l_{\delta(\rho(\sigma)+1)}$ and $\gamma(\sigma) < l_{\delta(\rho(\sigma))}$. Now $\gamma(\sigma+1) = \bigcup\{T(\gamma(\sigma+1), \varsigma); \varsigma < cf(\mu)\}$ so there is $\varsigma(\sigma) < cf(\mu)$ such that $x_\sigma \in T(\gamma(\sigma+1), \varsigma(\sigma))$. Because $cf(\mu) < \kappa$, there are $H \in [\kappa]^\kappa$ and $\varsigma < cf(\mu)$ such tht $\varsigma(\sigma) = \varsigma$ for all $\sigma \in H$. By re-indexing, we may suppose $\varsigma(\sigma) = \varsigma$ for all $\sigma < \kappa$. For each $\sigma < \kappa$, put $S_\sigma = \{x_\tau; \tau < \sigma\}$. Since all $\gamma(\sigma) \in D(\xi)$, from Lemma 2.5 $T(\gamma(\tau+1), \varsigma) \subseteq T(\gamma(\sigma), \varsigma)$ whenever $\tau < \sigma < \kappa$, so $S_\sigma \subseteq T(\gamma(\sigma), \varsigma)$. Put $A_\sigma = A(\delta(\rho(\sigma)), \gamma(\sigma), \varsigma, S_\sigma)$. Note $S_\sigma \subseteq A_\sigma \in \mathcal{A}$, since $S_\sigma \subseteq A$ and $A \notin \bigcup\{\mathcal{B}_\delta; \delta < \delta(\rho(\sigma))\}$ (for if $A \in \mathcal{B}_\delta$ then $A \subseteq l_{\delta+1}$ so $|A \cap l_{\delta+1}| = \kappa$, yet $\delta(\rho(\sigma)) < \varepsilon(A)$). Thus $A_\sigma \subseteq X_{\delta(\rho(\sigma))}$, so $A_\sigma \in \mathcal{B}_{\delta(\rho(\sigma))}$, and hence $A_\sigma \subseteq l_{\delta(\rho(\sigma))+1}$. And if $\tau \leq \sigma$ then $x_\sigma \notin l_{\delta(\rho(\tau))+1}$ so $x_\sigma \notin A_\tau$. Thus $A_\sigma \cap \{x_\tau; \tau < \kappa\} = S_\sigma = \{x_\tau; \tau < \sigma\}$. Put $D = \{x_\tau; \tau < \kappa\}$, so $D \cap A_\sigma \subset D \cap A_\tau$ whenever $\sigma < \tau < \kappa$ contradicting that $\mathcal{A}$ satisfies the κ-chain condition. This establishes our claim that $\varepsilon(A)$ is not a limit ordinal.

We complete the proof that $\mathcal{A}$ has a κ-ordering by appealing to Lemma 1.4. Every $\mathcal{B}$ in $[\mathcal{A}]^{<\lambda}$ has a κ-ordering by the inductive hypothesis, so (i) of Lemma 1.4 holds. For (ii), define $U_\rho = l_\rho$, for $\rho < \mu^+ = cf(\lambda)$. Certainly $\bigcup \mathcal{A} = \mu^+ = \bigcup\{U_\rho; \rho < \mu^+\}$ and $U_\rho \subseteq U_\tau$ if $\rho < \tau < \mu^+$. Take any $A \in \mathcal{A}$. Since $\varepsilon(A)$ is not a limit, we can put $\rho = \varepsilon(A) - 1$. Now $|A \cap l_{\varepsilon(A)}| = \kappa$, so $A \in \mathcal{B}_{\varepsilon(A)}$ and hence $A \subseteq l_{\varepsilon(A)+1} = U_{\rho+2}$. Since $\rho < \varepsilon(A)$ we have $|A \cap U_\rho| < \kappa$. Hence (iia) holds. To verify that (iib) holds, suppose $A \subseteq U_\rho = l_\rho$. Then $A \in \mathcal{B}_\rho$. Since $|\mathcal{B}_\rho| \leq \mu$, this means that (iib) holds. Hence by Lemma 1.4, $\mathcal{A}$ has a κ-ordering.

4. Applications

In this section we present several applications of the idea of a θ-ordering. The first is a trivial observation, but when combined with Theorems 2.4 and 2.6, it gives a proof of Komjáth's theorem mentioned in the introduction ([5], Theorem 5).

THEOREM 4.1. *If $\mathcal{A}$ is a (λ, κ)-family which possesses a κ-ordering, then $\mathcal{A}$ is sparse.*

PROOF. Let $\prec$ be a κ-ordering of $\mathcal{A}$. Define $f\colon \mathcal{A} \to P\bigcup\mathcal{A}$ by

$$f(A) = A \cap \bigcup\{B \in \mathcal{A}; B \prec A\}.$$

By (3), $f(A) \in [A]^{<\kappa}$ and clearly $\{A - f(A); A \in \mathcal{A}\}$ is a pairwise disjoint family, so f shows $\mathcal{A}$ is sparse.

The next couple of results concern transversals of the family $\mathcal{A}$. The case $\kappa = \aleph_0$ of Theorem 4.2 is essentially due to Davies and Erdös ([1], Proposition A), and their construction carries over to larger κ.

THEOREM 4.2. *Every (λ, κ)-family $\mathcal{A}$ which possesses a κ-ordering can be split into κ subfamilies, $\mathcal{A} = \bigcup\{\mathcal{A}_\xi; \xi < \kappa\}$, where each subfamily $\mathcal{A}_\xi$ has a 2-transversal T_ξ, and moreover $\bigcup \mathcal{A} = \bigcup\{T_\xi; \xi < \kappa\}$.*

PROOF. Let $\prec$ be a κ-ordering of $\mathcal{A}$. For each $A \in \mathcal{A}$, write $A - \bigcup\{B \in \mathcal{A}; B \prec A\} = \{a(A,\alpha); \alpha < \kappa\}$, where $a(A,\alpha) \neq a(A,\beta)$ if $\alpha \neq \beta$. By transfinite recursion on $\prec$, for each $A \in \mathcal{A}$ use induction to choose $\xi(A,\alpha)$ for $\alpha < \kappa$ so that

$$\xi(A,\alpha) \in \kappa - (\{\xi(B,\beta); B \prec A \text{ and } \beta < \kappa \text{ and } a(B,\beta) \in A\} \cup \{\xi(A,\gamma); \gamma < \alpha\}).$$

(Since $|A \cap \bigcup\{B; B \prec A\}| < \kappa$ and $a(B,\beta) \neq a(C,\gamma)$ if $(B,\beta) \neq (C,\gamma)$, such a choice is possible.) For each $\xi < \kappa$, put

$$T_\xi = \{a(A,\alpha); \xi(A,\alpha) = \xi\}, \text{ and}$$

$$\mathcal{A}_\xi = \{A \in \mathcal{A}; \xi(A,\alpha) = \xi, \text{ for some } \alpha < \kappa\}.$$

Clearly $\mathcal{A} = \bigcup\{\mathcal{A}_\xi; \xi < \kappa\}$ and $\bigcup \mathcal{A} = \bigcup\{T_\xi; \xi < \kappa\}$. We show that $|T_\xi \cap A| = 1$ for each A in $\mathcal{A}_\xi$, so T_ξ is a 2-transversal of $\mathcal{A}_\xi$. If $A \in \mathcal{A}_\xi$, then $a(A,\alpha) \in A \cap T_\xi$ for that α with $\xi(A,\alpha) = \xi$. Take any $x \in A \cap T_\xi$ with $x \neq a(A,\alpha)$ for this α. Then either (i) $x = a(B,\beta)$ and $\xi(B,\beta) = \xi$ for some B with $B \prec A$, which is contrary to the choice of $\xi(A,\alpha)$, or (ii) $x = a(B,\beta)$ and $\xi(B,\beta) = \xi$ for some B with $A \prec B$, which is contrary to the choice of $\xi(B,\alpha)$, or (iii) $x = a(A,\beta)$ and $\xi(A,\beta) = \xi$ for some $\beta \neq \alpha$, contrary to $\xi(A,\alpha) = \xi$. Hence there is no such x, and thus $|A \cap T_\xi| = 1$ as required.

THEOREM 4.3. *Let $\mathcal{A}$ be a (λ, κ)-family with a θ-ordering, and suppose for every subfamily $\mathcal{B} \in [\mathcal{A}]^\kappa$ the family $\{B - R(B); B \in \mathcal{B}\}$ has a θ-transversal, for every choice of $R(B) \in [B]^{<\theta}$. Then $\mathcal{A}$ has θ-transversal.*

PROOF. Let $\prec$ be a θ-ordering of $\mathcal{A}$, with family of intervals $\mathcal{I}$. For each $I \in \mathcal{I}$, put $I^* = \bigcup I - \bigcup\{\bigcup J; J \in \mathcal{I} \text{ and } J \prec I\}$. Take $A \in \mathcal{A}$, and suppose $A \in I$. Put $R(A) = A \cap \bigcup\{\bigcup J; J \in \mathcal{I} \text{ and } J \prec I\}$ so $R(A) \in [A]^{<\theta}$ by (1), and $A \cap I^* = A - R(A)$. Since $|I| \leq \kappa$, by assumption there is a θ-transversal, say $T(I)$, for $\{A \cap I^*; A \in I\}$, and we may assume $T(I) \subseteq I^*$. But then $T = \bigcup\{T(I); I \in \mathcal{I}\}$ is a θ-transversal for $\mathcal{A}$, since for each A in $\mathcal{A}$, if $A \in I$ then $A = (A \cap I^*) \cup R(A)$ so $A \cap T \subseteq (A \cap I^* \cap T(I)) \cup R(A)$, and consequently $1 \leq |A \cap T| < \theta$ as required.

Combining Theorems 3.3 and 4.3, together with the observation that for regular κ, every almost disjoint (κ, κ)-family has a κ-transversal provides a proof of ([8], Theorem 3.2) (which can be extended by using Theorem 3.4 as well).

Results on the existence of θ-transversals for families $\mathcal{A}$ when $\mathcal{A}$ satisfies intersection conditions were first studied extensively by Erdös and Hajnal [2]. (Having a θ-transversal was there referred to as possessing property $\mathcal{B}(\theta)$.) We can deduce their results from Theorem 4.3, as follows.

COROLLARY 4.4. *Let $\mathcal{A}$ be an almost disjoint (λ,κ)-family. Take $\theta < \kappa$ and suppose either $\mathcal{A}$ satisfies $C(2,\theta)$, or else κ is regular and $\mathcal{A}$ satisfies $C(\kappa,\theta)$. Then*

(a) *(GCH) $\mathcal{A}$ has a θ^{++}-transversal.*

(b) *(GCH) Suppose either $\theta^+ = \kappa$ or else $\theta^+ < \kappa$ but $\kappa \neq \mu^+$ where $cf(\theta) = cf(\mu)$, and suppose $cf(\eta) \neq cf(\theta)$ whenever $\kappa \leq \eta < \lambda$. Then $\mathcal{A}$ has a θ^+-transversal.*

(c) *$(V = L)$ $\mathcal{A}$ has a θ^+-transversal.*

PROOF. The result will follow from Theorems 4.3, 2.4 and 2.6 once we show that every (κ,κ)-family $\mathcal{B}$ satisfying these conditions has a θ^{++}-transversal or a θ^+-transversal, respectively. For (b), take the appropriate (κ,κ)-family $\mathcal{B} = \{B_\alpha; \alpha < \kappa\}$ and we show that $\mathcal{B}$ has a θ^+-transversal. (This is essentially ([2], Result 4.9).) If $\theta^+ = \kappa$, the result is immediate since κ is regular and $\mathcal{A}$ is almost disjoint. So suppose $\theta^+ < \kappa$. Recursively define elements $x_\beta \in \bigcup \mathcal{B}$ for $\beta < \kappa$, as follows. Choose $x_0 \in B_0$. For $\beta > 0$, put $X_\beta = \{x_\gamma; \gamma < \beta\}$ and let $\mathcal{C}_\beta = \{B \in \mathcal{B}; |B \cap X_\beta| \geq \theta\}$. Choose $x_\beta \in B_\beta - \bigcup \mathcal{C}_\beta$ if $B_\beta - \bigcup \mathcal{C}_\beta$ is non-empty, and otherwise put $x_\beta = x_0$.

We claim that $|\mathcal{C}_\beta| < \kappa$ for all $\beta < \kappa$. Certainly $|\{B \cap X_\beta; B \in \mathcal{C}_\beta\}| < \kappa$, for this is immediate if $|X_\beta|^+ < \kappa$, and if $|X_\beta|^+ = \kappa$ it follows from Lemma 2.1 since in this case $\{B \cap X_\beta; B \in \mathcal{C}_\beta\}$ satisfies $C(|X_\beta|^+, \theta)$ and $cf(|X_\beta|) \neq cf(\theta)$ by hypothesis. Also, for any $Z \in [X_\beta]^\theta$, we have $|\{B \in \mathcal{B}; Z \subseteq B \cap X_\beta\}| < \kappa$ since $\mathcal{B}$ satisfies $C(\kappa,\theta)$, and in fact $|\{B \in \mathcal{B}; Z \subseteq B \cap X_\beta\}| < 2$ if κ is singular since then $\mathcal{B}$ satisfies $C(2,\theta)$. Hence $|\mathcal{C}_\beta| < \kappa$. Thus, if $B_\beta \notin \mathcal{C}_\beta$ then $|B_\beta - \bigcup \mathcal{C}_\beta| = \kappa$, and so then $x_\beta \in B_\beta$.

Put $T = \{x_\beta; \beta < \kappa\}$, so always $|T \cap B_\beta| \geq 1$. And if for any $B \in \mathcal{B}$ we have $|B \cap X_\beta| = \theta$ then for all $\gamma \geq \beta$ it follows that $B \in \mathcal{C}_\gamma$ so either $x_\gamma = x_0$ or $x_\gamma \notin B$, and hence $|T \cap B| = \theta$. Thus for all $B \in \mathcal{B}$ we have $1 \leq |T \cap B| \leq \theta$, so T is a θ^+-transversal of $\mathcal{B}$.

The argument for (a) is similar, putting $\mathcal{C}_\beta = \{B \in \mathcal{B}; |B \cap X_\beta| > \theta\}$.

Case (b) also covers case (c), except when $\kappa = \mu^+ > \theta^+$ where $cf(\mu) = cf(\theta)$. For this situation we use $\Box_\mu$. As in case (b), write $\mathcal{B} = \{B_\alpha; \alpha < \kappa\}$, and we may suppose $\bigcup \mathcal{B} = \mu^+$. We have the sets $D(\alpha)$ and $T(\alpha,\sigma)$ for $\alpha < \mu^+$ and $\sigma < cf(\mu)$, as in Lemma 2.5. Recursively define elements $x_\beta \in \bigcup \mathcal{B}$ for $\beta < \kappa$, ensuring that $x_\gamma < x_\beta$ whenever $\gamma < \beta$ (unless $x_\beta = x_0$). Choose $x_0 \in B_0$. For $\beta > 0$, put $X_\beta = \{x_\gamma; \gamma < \beta\}$ and let l_β be the least ordinal ξ with $cf(\xi) > \omega$

and $\xi > \bigcup X_\beta$. Define

$$\mathcal{C}_\beta = \{B \in \mathcal{B}; \exists \text{ limit } \alpha < l_\beta \exists \sigma < cf(\mu)(|B \cap X_\beta \cap T(\alpha, \sigma)| \geq \theta)\}.$$

Since $|[T(\alpha,\sigma)]^\theta| \leq \mu$ and $|\{B \in \mathcal{B}; B \cap T(\alpha, \sigma) = X\}| < \kappa$ for each $X \in [T(\alpha,\sigma)]^\theta$, we have $|\mathcal{C}_\beta| \leq \mu$. Hence if $B_\beta \notin \mathcal{C}_\beta$ then $|B_\beta - \bigcup \mathcal{C}_\beta| = \kappa$ since $\mathcal{B}$ is almost disjoint, and we can choose $x_\beta \in B_\beta - \bigcup \mathcal{C}_\beta$ with $x_\beta > x_\gamma$ for all $\gamma < \beta$. If $B_\beta \in \mathcal{C}_\beta$, put $x_\beta = x_0$.

Put $T = \{x_\beta; \beta < \kappa\}$, so always $|B_\beta \cap T| \geq 1$. We claim that $|B \cap T| \leq \theta$ for each $B \in \mathcal{B}$, so T is a θ^+-transversal of $\mathcal{B}$. Suppose for a contradiction that there is $B \in \mathcal{B}$ with $|B \cap T| \geq \theta^+$. There must be $\delta < \kappa$ such that $|B \cap X_\delta| \geq \theta^+$, for otherwise we could choose $\beta(\xi)$ for $\xi < \theta^+$ such that $\{x_{\beta(\xi)}; \xi < \theta^+\}$ was cofinal in B, and hence in κ (since $B \in [\kappa]^\kappa$), which is impossible with $cf(\kappa) = \mu^+ > \theta^+$. Hence $|B \cap X_\delta \cap l_\delta| \geq \theta^+$. Since $l_\delta = \bigcup\{T(l_\delta; \sigma); \sigma < cf(\mu)\}$ and $cf(\mu) = cf(\theta) < \theta^+$ there must be $\sigma < cf(\mu)$ such that $|B \cap X_\delta \cap T(l_\delta, \sigma)| \geq \theta^+$. And $T(l_\delta, \sigma) = \bigcup\{T(\beta, \sigma); \beta \in D(l_\delta)\}$ with $T(\gamma, \sigma) \subseteq T(\beta, \sigma)$ whenever $\beta, \gamma \in D(l_\delta)$ with $\gamma < \beta$, so there must be $\beta \in D(l_\delta)$ such that $|B \cap X_\delta \cap T(\beta, \sigma)| \geq \theta$. Let β^* be the least limit ordinal such that $\exists \sigma < cf(\mu)(|B \cap X_\delta \cap T(\beta^*, \sigma)| = \theta)$. Let γ be least such that $x_\gamma \geq \beta^*$, so $|B \cap X_\gamma \cap T(\beta^*, \sigma)| = \theta$. Also $B \in \mathcal{C}_\beta$ whenever $\beta > \gamma$, since then $X_\gamma \subseteq X_\beta$ and $\beta^* \leq x_\gamma \leq l_{\gamma+1} \leq l_\beta$, so that $x_\beta \notin B$ unless $x_\beta = x_0$. Hence $B \cap T \subseteq X_\gamma$. By the choice of β^*, we have $|B \cap X_\gamma \cap T(\beta^*, \tau)| \leq \theta$ for all $\tau < cf(\mu)$ and hence $|B \cap X_\gamma| \leq \theta$ since $B \cap X_\gamma \subseteq \beta^* = \bigcup\{T(\beta^*, \tau); \tau < cf(\mu)\}$. Hence $|B \cap T| \leq \theta$, contradicting that $|B \cap T| \geq \theta^+$. Thus T is a θ^+-transversal of $\mathcal{B}$, and the proof is complete.

The final result concerns the existence of Δ-families. The family $\mathcal{B}$ is said to be a Δ-family if there is a fixed set Z such that $B \cap C = Z$ for all distinct $B, C \in \mathcal{B}$.

THEOREM 4.5. *Suppose $\kappa^{<\kappa} = \kappa$, and let $\mathcal{A}$ be a (κ^+, κ)-family which possesses a κ-ordering. Then there is a Δ-family $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \kappa^+$.*

PROOF. Let $\prec$ be a κ-ordering of $\mathcal{A}$, and we may suppose $\mathcal{A} = \{A_\alpha; \alpha < \kappa^+\}$ is the enumeration of $\mathcal{A}$ in increasing $\prec$-order, so always

$$|A_\alpha \cap \bigcup\{A_\beta; \beta < \alpha\}| < \kappa. \tag{13}$$

Recursively choose subsets $X_\gamma \subseteq \kappa^+$ (possibly empty) for $\gamma \leq \kappa$ as follows: put $X(\gamma) = \bigcup\{X_\beta; \beta < \gamma\}$ and $A(\gamma) = \bigcup\{A_\alpha; \alpha \in X(\gamma)\}$, and choose $X_\gamma \subseteq \kappa^+ - X(\gamma)$ maximal such that the family $\{A_\alpha - A(\gamma); \alpha \in X_\gamma\}$ is pairwise disjoint. We claim there is $\gamma < \kappa$ with $|X_\gamma| = \kappa^+$. For if not, $|X_\beta| \leq \kappa$ for all $\beta < \kappa$, and so $|X(\kappa)| \leq \kappa$. Take $\delta \in \kappa^+$ with $\delta > \alpha$ for all $\alpha \in X(\kappa)$, then by the maximality of X_β, for each $\beta < \kappa$ there must be $\alpha(\beta) \in X_\beta$ such that $(A_\delta - A(\beta)) \cap (A_{\alpha(\beta)} - A(\beta))$ is non-empty, so we can choose $x_\beta \in (A_\delta \cap A_{\alpha(\beta)}) - A(\beta)$. Now $x_\beta \neq x_\gamma$ if $\gamma < \beta < \kappa$, since $x_\gamma \in A_{\alpha(\gamma)} \subseteq A(\beta)$. Since

$$\{x_\beta; \beta < \kappa\} \subseteq A_\delta \cap \bigcup \{A_{\alpha(\beta)}; \beta < \kappa\} \subseteq A_\delta \cap \bigcup \{A_\alpha; \alpha < \delta\},$$

this contradicts (13), and proves the claim.

Let γ be least such that $|X_\gamma| = \kappa^+$. Then $|X(\gamma)| \leq \kappa$ and $|A(\gamma)| = \kappa$, and if $X = \{\beta \in X_\gamma; \forall \alpha \in X(\gamma)(\alpha < \beta)\}$ then $|X| = \kappa^+$. For $\beta \in X$, since $A_\beta \cap A(\gamma) \subseteq A_\beta \cap \bigcup\{A_\alpha; \alpha < \beta\}$, by (13) $A_\beta \cap A(\gamma) \in [A(\gamma)]^{<\kappa}$. Since $\kappa^{<\kappa} = \kappa$, there must be Z in $[A(\gamma)]^{<\kappa}$ and $Y \in [X]^{\kappa^+}$ such that $A_\beta \cap A(\gamma) = Z$ for all $\beta \in Y$. However, $\{A_\beta - A(\gamma); \beta \in Y\}$ is pairwise disjoint, so $\{A_\beta; \beta \in Y\}$ is a Δ-family of size κ^+.

Combining Theorems 4.5 and 2.4 proves a result of Erdös, Milner and Rado ([4], Theorem 1), that for κ regular, every almost disjoint (κ^+, κ)-family satisfying $C(\kappa^+, \theta)$ where $\theta < \kappa$ contains a Δ-family of size κ^+. Combining Theorems 4.5 and 3.3 gives a result of Williams ([7], Corollary 2.9), that for κ regular, every almost disjoint (κ^+, κ)-family with the κ-chain condition contains a Δ-family of size κ^+.

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