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EXTENSIONS OF POLYNOMIALS ON PREDUALS OF LORENTZ SEQUENCE SPACES

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Abstract. We show that there is a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on the predual $d_*(w, 1)$ of a complex Lorentz sequence space $d(w, 1)$ to $d^*(w, 1)$, but there is no unique norm-preserving extension from $\mathcal{P}(^n d_*(w, 1))$ to $\mathcal{P}(^n d^*(w, 1))$ for $n \geq 3$.

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1. Introduction. A bounded linear functional on a Banach space E has clearly a norm preserving extension to its bidual E^{**} by the Hahn-Banach theorem. In particular, when the Banach space E is an M-ideal in E^{**}, the extension is unique.

Aron and Berner [**2**] first studied "Hahn-Banach type theorems" for spaces of polynomials on Banach spaces in 1978. They proved that every continuous n-homogeneous polynomial P on a Banach space E can be extended to a continuous n-homogeneous polynomial $\widehat{P}$ to its bidual E^{**}. In 1989 Davie and Gamelin [**4**] showed that the Aron-Berner extension $\widehat{P}$ is a norm-preserving extension of P. These facts lead us to the following question. What classes of continuous n-homogeneous polynomials on a Banach space E can have a unique norm-preserving extension to its bidual E^{**}, when the Banach space E is an M-ideal in E^{**} ? For example, c_0 and the predual $d_*(w, 1)$ of a Lorentz sequence space $d(w, 1)$ is an M-ideal in its bidual l_∞ and $d^*(w, 1)$, respectively [**5**].

Aron, Boyd and Choi [**3**] proved that every norm-attaining 2-homogeneous polynomial on complex c_0 has a unique norm-preserving extension to l_∞. They also showed that for $n \geq 3$ there exists a norm-attaining n-homogeneous polynomial on c_0 whose norm-preserving extension to l_∞ is not unique. However, it is still an open problem whether every continuous 2-homogeneous polynomial on complex c_0 has a unique norm-preserving extension. For real c_0 they showed that there exists a norm-attaining n-homogeneous polynomial on c_0 whose norm-preserving extension is not unique.

Since $d_*(w, 1)$ contains a subspace isomorphic to c_0, we became interested in the same problems on $d_*(w, 1)$ as studied on c_0 in [**3**]. Both cases show the same results about the uniqueness of norm-preserving extension, but there is a different property between those polynomials. The main results of this article are the following.

(1) In the real case, for $n \geq 2$ we construct an n-homogeneous polynomial on $d_*(w, 1)$ with two distinct norm-preserving extensions to its bidual $d^*(w, 1)$.

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(2) In the complex case, every norm-attaining 2-homogeneous polynomial on $d_*(w,1)$ is finite, but this is not true for n-homogeneous polynomials, $n \geq 3$. Furthermore, we show that every norm-attaining 2-homogeneous polynomial on $d_*(w,1)$ has a unique norm-preserving extension to $d^*(w,1)$, but for $n \geq 3$ there exists a norm-attaining n-homogeneous polynomial whose norm-preserving extension is not unique.

(3) It was proved in [**3**] that if an n-homogeneous polynomial P on l_∞ satisfies $\|P\| = \|P|_{c_0}\|$, then it is w^*-continuous on bounded sets at 0. Differently from that, for $w \in l_2 \setminus l_1$ there is an n-homogeneous polynomial P on $d^*(w,1)$ with $\|P\| = \|P|_{d_*(w,1)}\|$, but P is not w^*-continuous on bounded sets at 0.

2. Main results. Let $w = (w_i)_{i=1}^\infty$ be a decreasing sequence of positive numbers such that $w \in c_0 \setminus l_1$, which is called an *admissible* sequence. Given a sequence $x = (x_i)$ of scalars, let $[x] = ([x]_i)_{i=1}^\infty$ be the rearrangement of $(|x_i|)_{i=1}^\infty$ so that $[x]_i \geq [x]_{i+1}$ for all $i \in \mathbb{N}$. The Lorentz sequence space $d(w,1)$ is defined to be the Banach space of all sequences of scalars $x = (x_1, x_2, \ldots)$ for which $\|x\| = \sum_{n=1}^\infty [x]_n w_n < \infty$. Recall that its dual space

$$d^*(w,1) = \left\{ (x_i)_{i=1}^\infty; \quad \left(\frac{\sum_{i=1}^k [x]_i}{\sum_{i=1}^k w_i} \right)_{k=1}^\infty \in l_\infty \right\}$$

has the norm defined by

$$\|x\| = \sup_k \frac{\sum_{i=1}^k [x]_i}{\sum_{i=1}^k w_i}, \qquad x = (x_k)_{k=1}^\infty \in d^*(w,1),$$

and its predual space

$$d_*(w,1) = \left\{ (x_i)_{i=1}^\infty; \quad \left(\frac{\sum_{i=1}^k [x]_i}{\sum_{i=1}^k w_i} \right)_{k=1}^\infty \in c_0 \right\}$$

has the norm induced by $d^*(w,1)$. Let B_E be the closed unit ball of a Banach space E.

We can easily verify that $x = (x_i)_{i=1}^\infty \in B_{d^*(w,1)}$ if and only if given a positive integer n

$$\sum_{i \in I} |x_i| \leq \sum_{i=1}^n w_i$$

for any finite subset $I = \{i_1, \ldots, i_n\}$ of $\mathbb{N}$.

Since $d_*(w,1)$ is an M-ideal in $d^*(w,1)$, it is clear that each $f \in (d_*(w,1))^*$ has a unique norm preserving extension $\widetilde{f} \in (d^*(w,1))^*$. We now consider the problem of a unique norm-preserving extension for n-homogeneous polynomials on $d_*(w,1)$ with $n \geq 2$.

In the real case, for $n \geq 2$ there exists a norm-attaining n-homogeneous polynomial on $d_*(w,1)$ with two different norm-preserving extensions to $d^*(w,1)$. To construct such polynomials let us first define a bounded linear operator $T : d^*(w,1) \longrightarrow l_\infty$ by $T((x_i)_{i=1}^\infty) = (y_k)_{k=1}^\infty$, where

$$y_k = \frac{\sum_{i=1}^k x_i}{\sum_{i=1}^k w_i} \quad (k \in \mathbb{N}).$$

Clearly $T(d_*(w,1)) \subset c_0$, $T(w) = (1,1,1,\ldots) \in \ell_\infty$ and $\|T\| = \|T(w)\| = 1$. Let ϕ be a Banach limit functional on l_∞, and define $\widetilde{\phi} = \phi \circ T \in (d^*(w,1))^*$. Since

$\phi(x) = \lim_i x_i$ for a convergent sequence $x = (x_i) \in c$, we have that $\|\widetilde{\phi}\| = 1$, $\widetilde{\phi}(w) = 1$, and $\widetilde{\phi}|_{d_*(w,1)} = 0$.

Now consider the n-homogeneous polynomial $P(x) = {x_1}^n$ on $d_*(w, 1)$ with norm one. Then $P_1(x) = {x_1}^n$ and $P_2(x) = {x_1}^n - {x_1}^{n-2}\widetilde{\phi}^2(x)$ are two distinct norm-preserving extensions of P to its bidual $d^*(w, 1)$.

In the complex case, for $n \geq 3$ there also exists a norm-attaining n-homogeneous polynomial on $d_*(w, 1)$ with two distinct norm-preserving extensions to $d^*(w, 1)$. For this we need the following lemma.

Lemma 1. *Suppose that* $0 < t < 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, *and* $|\alpha| + |\beta| \leq 1 + t$. *For a positive integer* $n \geq 3$,

$$|t\alpha - \beta|^n + (1 + t^2)|\alpha + t\beta|^{n-1} \leq (1 + t^2)^n.$$

Proof. For $0 < t < 1$, $|\alpha| \leq 1$, $|\beta| \leq 1$, and $|\alpha| + |\beta| \leq 1 + t$, it is easily checked that $|t\alpha - \beta| \leq 1 + t^2$, $|\alpha|^2 + |\beta|^2 \leq (1 + t^2)$, and

$$|t\alpha - \beta|^2 + |\alpha + t\beta|^2 \leq (1 + t^2)(|\alpha|^2 + |\beta|^2) \leq (1 + t^2)^2.$$

Since $(a + b)^p \geq a^p + b^p$ for $a > 0$, $b > 0$ and $p \geq 1$,

$$\begin{aligned}|t\alpha - \beta|^n + (1 + t^2)|\alpha + t\beta|^{n-1} &\leq (1 + t^2)(|t\alpha - \beta|^{n-1} + |\alpha + t\beta|^{n-1})\\ &\leq (1 + t^2)(|t\alpha - \beta|^2 + |\alpha + t\beta|^2)^{\frac{n-1}{2}} \leq (1 + t^2)^n.\end{aligned}$$

□

Let an n-homogeneous polynomial P on complex $d_*(w, 1)$ be defined by

$$P(x) = (w_2x_1 - x_2)^n.$$

Clearly $\|P\| = (1 + {w_2}^2)^n$ and P attains its norm. Consider the following two n-homogeneous polynomials P_1 and P_2 on complex $d^*(w, 1)$ defined by

$$P_1(x) = (w_2x_1 - x_2)^n$$

and

$$P_2(x) = (w_2x_1 - x_2)^n + \left(1 + w_2^2\right)(x_1 + w_2x_2)^{n-1}\widetilde{\phi}(x).$$

Clearly $P_1|_{d_*(w,1)} = P$, and $P_2|_{d_*(w,1)} = P$ because $\widetilde{\phi}|_{d_*(w,1)} = 0$. It follows from Lemma 1 and the fact $\|\widetilde{\phi}\| = 1$ that $\|P_2\| \leq (1 + {w_2}^2)^n = P_2(w) \leq \|P_2\|$. Hence $\|P_1\| = \|P\| = \|P_2\|$. Note that $P_1(w) = 0$, which implies that P_1 and P_2 are distinct norm preserving extensions of P to $d^*(w, 1)$. Therefore, for $n \geq 3$ there exists a norm-attaining n-homogeneous polynomial on complex $d_*(w, 1)$ with two distinct norm-preserving extensions to $d^*(w, 1)$.

We recall that a 2-homogeneous polynomial P on $d_*(w, 1)$ is called *finite* if there exists a positive integer n such that

$$P(x) = \sum_{i=1}^{n}\left(\sum_{j=1}^{i} a_{ij}x_ix_j\right),$$

for all $x = (x_i)_{i=1}^{\infty} \in d_*(w, 1)$. We note that the closed unit ball of $d_*(w, 1)$ has no extreme points, like c_0. See Lemma 2 of [**1**].

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THEOREM 2. *A 2-homogeneous polynomial P on complex $d_*(w, 1)$ attains its norm if and only if it is finite.*

Proof. If P is a finite 2-homogeneous polynomial on $d_*(w, 1)$, it can be regarded as a polynomial defined on an n-dimensional subspace of $d_*(w, 1)$ for some $n \in \mathbb{N}$ and hence P attains its norm.

Conversely, suppose a 2-homogeneous polynomial P attains its norm at $x_0 = (\lambda_i)_{i=1}^{\infty} \in B_{d_*(w,1)}$. Without loss of generality we may assume that $\|P\| = 1 = P(x_0)$. By change of variable and rearrangement of indices, we may assume that $x_0 = (\lambda_i)_{i=1}^{\infty}$ satisfies $\lambda_i \geq \lambda_{i+1}$, $\lambda_i \geq 0$ for all $i \in \mathbb{N}$. Obviously

$$\|x_0\| = \sup_k \frac{\sum_{i=1}^k \lambda_i}{\sum_{i=1}^k w_i} = 1.$$

Since

$$\lim_{k\to\infty} \frac{\sum_{i=1}^k \lambda_i}{\sum_{i=1}^k w_i} = 0,$$

we can choose the largest positive integer n such that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} w_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i < \sum_{i=1}^{k} w_i \text{ for all } k \geq n+1.$$

Let

$$a = 1 - \sup\left\{\frac{\sum_{i=1}^k \lambda_i}{\sum_{i=1}^k w_i};\ k \geq n+1\right\} > 0.$$

Clearly $\lambda_n > \lambda_{n+1}$. Choose $\delta > 0$ such that $\lambda_n > \lambda_{n+1} + \delta$ and let

$$b = \min\{a, \delta\} > 0.$$

Let $y = (0, \ldots, 0, y_{n+1}, y_{n+2}, \ldots) \in B_{d_*(w,1)}$ and λ with $|\lambda| \leq b$ be given. Then we have

$$x_0 + \lambda y = (\lambda_1, \ldots, \lambda_n, \lambda_{n+1} + \lambda y_{n+1}, \lambda_{n+2} + \lambda y_{n+2}, \ldots)$$

and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > \lambda_{n+1} + \delta \geq \lambda_i + |\lambda||y_i| \geq |\lambda_i + \lambda y_i|$$

for all $i \geq n+1$.

Let $\mathbb{N}_0 = \mathbb{N} \setminus \{1, 2, \ldots, n\}$. Given $k \geq n+1$ and a finite subset J of $\mathbb{N}_0$ with $|J| = k - n$, we have

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i + \sum_{i\in J} |\lambda_i + \lambda y_i| &\leq \sum_{i=1}^{n} \lambda_i + \sum_{i\in J} \lambda_i + a \sum_{i=1}^{k-n} w_i \\ &\leq \sum_{i=1}^{n} \lambda_i + \sum_{i\in J} \lambda_i + \left(1 - \frac{\sum_{i=1}^k \lambda_i}{\sum_{i=1}^k w_i}\right) \sum_{i=1}^{k} w_i \\ &= \sum_{i=1}^{k} w_i + \left(\sum_{i\in J} \lambda_i - \sum_{i=n+1}^{k} \lambda_i\right) \leq \sum_{i=1}^{k} w_i,\end{aligned}$$

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which implies that

$$x_0 + \lambda y \in B_{d_*(w,1)}.$$

Hence we obtain

$$|P(x_0 \pm \lambda y)| = |1 \pm 2\lambda \check{P}(x_0, y) + \lambda^2 P(y)| \leq |P(x_0)| = 1,$$

where $\check{P}$ is the unique symmetric bilinear form associated with P. It follows from a phase manipulation that

$$P(y) = 0, \quad \check{P}(x_0, y) = 0.$$

Taking $y_0 = (0, \ldots, 0, \lambda_{n+1}, \lambda_{n+2}, \ldots)$ we clearly have

$$P(y_0) = 0, \quad \check{P}(x_0, y_0) = 0,$$

which implies that

$$P(\lambda_1, \lambda_2, \ldots, \lambda_n, 0, \ldots) = P(x_0 - y_0) = P(x_0) + P(y_0) - 2\check{P}(x_0, y_0) = P(x_0) = 1.$$

Define

$$\begin{aligned} z_1 &= (\lambda_1, \lambda_2, \ldots, \lambda_n) \\ z_2 &= (\lambda_1, \lambda_2 - n\lambda_2, \ldots, \lambda_n) \\ &\vdots \\ z_n &= (\lambda_1, \lambda_2, \ldots, \lambda_n - n\lambda_n). \end{aligned}$$

Repeating the argument given above we see that $\check{P}(\tilde{z}_1, y) = 0$, where $\tilde{z}_j = (z_j, 0, 0, \ldots)$, $j = 1, \ldots, n$. Since

$$(x_1, x_2, \ldots, x_n) = \frac{1}{n}\left(\frac{x_1}{\lambda_1} + \frac{x_2}{\lambda_2} + \cdots + \frac{x_n}{\lambda_n}\right)z_1 + \frac{1}{n}\sum_{j=2}^{n}\left(\frac{x_1}{\lambda_1} - \frac{x_j}{\lambda_j}\right)z_j,$$

we have

$$P(x_1, x_2, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots) = P(x_1, x_2, \ldots, x_n, 0, \ldots) + \frac{2}{n}\sum_{j=2}^{n}\left(\frac{x_1}{\lambda_1} - \frac{x_j}{\lambda_j}\right)\check{P}(\tilde{z}_j, y).$$

Applying the same computation as in Proposition 2 in [**3**] we have $\check{P}(\tilde{z}_j, y) = 0$, for all $j,\ 2 \leq j \leq n$ and hence P depends only on finitely many variables $x_1, x_2, \ldots, x_n$. □

Remark 3. Sevilla and Payá [**6**] proved that every norm-attaining n-homogeneous polynomial P on complex $d_*(w, 1)$ satisfies $P(e_k) = 0$, for sufficiently large $k \in \mathbb{N}$, where $\{e_k\}_{k=1}^{\infty}$ is the standard unit vector basis of $d_*(w, 1)$. Theorem 2 is stronger than this for 2-homogeneous polynomials.

Remark 4. For $n \geq 3$, there exists a norm attaining n-homogeneous polynomial P on $d_*(w, 1)$ that is not finite. Let

$$P(x) = (w_2 x_1 - x_2)^n + \left(1 + w_2^2\right)(x_1 + w_2 x_2)^{n-1}\sum_{j=3}^{\infty}\frac{x_j}{2^j}.$$

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Clearly P is not finite. By Lemma 1 we have

$$|P(x)| \leq |w_2x_1 - x_2|^n + \left(1 + w_2^2\right)|x_1 + w_2x_2|^{n-1} \leq \left(1 + w_2^2\right)^n.$$

Since $|P(x_0)| = (1 + w_2^2)^n$ for $x_0 = (w_2, -1, 0, 0, \dots) \in B_{d_*(w,1)}$, P attains its norm.

THEOREM 5. *Every 2-homogeneous norm-attaining polynomial on complex $d_*(w, 1)$ has a unique norm-preserving extension to $d^*(w, 1)$.*

Proof. Suppose that P is a norm-attaining 2-homogeneous polynomial on $d_*(w, 1)$ that attains its norm at $x_0 = (\lambda_i)_{i=1}^{\infty} \in B_{d_*(w,1)}$ and suppose that Q is its norm-preserving extension to $d^*(w, 1)$. Let n be the largest positive integer such that

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} w_i \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i < \sum_{i=1}^{k} w_i \text{ for all } k \geq n+1.$$

As in the proof of Theorem 2, we may assume that P depends only on the first n variables $x_1, \dots, x_n$ and also that $\|P\| = 1 = P(z_0)$ for some $z_0 = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n > 0$. Let

$$a = \min\left\{\lambda_n, 1 - \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{n+1} w_i}\right\} > 0.$$

Let $y = (0, \dots, 0, y_{n+1}, y_{n+2}, \dots) \in B_{d^*(w,1)}$ and let λ with $|\lambda| \leq a$ be given. Then we have

$$z_0 + \lambda y = (\lambda_1, \lambda_2, \dots, \lambda_n, \lambda y_{n+1}, \lambda y_{n+2}, \dots)$$

and

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq a \geq |\lambda y_i|,$$

for all $i \geq n+1$. Let $\mathbb{N}_0 = \mathbb{N} \setminus \{1, 2, \dots, n\}$. Given $k \geq n+1$ and a finite subset J of $\mathbb{N}_0$ with $|J| = k - n$, we have

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i + \sum_{i \in J} |\lambda y_i| &\leq \sum_{i=1}^{n} \lambda_i + a \sum_{i=1}^{k-n} w_i \\
&\leq \sum_{i=1}^{n} \lambda_i + \left(1 - \frac{\sum_{i=1}^{n} \lambda_i}{\sum_{i=1}^{k} w_i}\right) \sum_{i=1}^{k} w_i \\
&= \sum_{i=1}^{k} w_i,
\end{aligned}$$

which implies that

$$z_0 + \lambda y \in B_{d^*(w,1)}.$$

Let $\check{Q}$ be the unique symmetric bilinear form associated with Q. Since

$$|Q(z_0 \pm \lambda y)| = |1 \pm 2\lambda \check{Q}(z_0, y) + \lambda^2 Q(y)| \leq |Q(z_0)| = 1,$$

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we can see that $Q(y) = 0$, $\check{Q}(z_0, y) = 0$, for all $y = (0, \ldots, 0, y_{n+1}, \ldots) \in B_{d^*(w,1)}$. As in the proof of Theorem 2, we conclude again that Q depends only on the first n variables. If Q_1 and Q_2 are norm-preserving extensions of P to $d^*(w, 1)$, then

$$Q_1(x_1, \ldots, x_n, x_{n+1}, \ldots) = P(x_1, \ldots, x_n, 0, \ldots) = Q_2(x_1, \ldots, x_n, x_{n+1}, \ldots),$$

for all $x = (x_n)_{n=1}^{\infty} \in d^*(w, 1)$. Hence P has a unique norm-preserving extension to $d^*(w, 1)$. □

We can see that n-homogeneous polynomials on $d_*(w, 1)$ have the same properties concerning the uniqueness of norm-preserving extensions as those on c_0. However, they don't always share the same properties as polynomials on c_0. For instance, every continuous polynomial on c_0 is weakly continuous on bounded sets and Proposition 4 in [**3**] shows that every n-homogeneous polynomial P on l_∞ with $\|P\| = \|P|_{c_0}\|$ is w^*-continuous on bounded sets at 0. In Example 7 we can find a continuous 2-homogeneous polynomial P on $d^*(w, 1)$ such that $\|P\| = \|P|_{d_*(w,1)}\|$, but P is not w^*-continuous on bounded sets at 0 and $P|_{d_*(w,1)}$ is not weakly continuous on bounded sets at 0.

LEMMA 6. *Let (x_i) and (y_i) be decreasing sequences of nonnegative real numbers. If $\sum_{i=1}^{n} y_i \le \sum_{i=1}^{n} x_i$ for every positive integer n, then $\sum_{i=1}^{n} y_i^2 \le \sum_{i=1}^{n} x_i^2$ for every positive integer n.*

Proof. We are going to prove the result by induction. It is clear for $n = 1$. Suppose that it is true for the positive integer $n = k - 1$. If $y_k \le x_k$, clearly $\sum_{i=1}^{k} y_i^2 \le \sum_{i=1}^{k} x_i^2$ by the induction hypothesis; hence we might as well assume $x_k < y_k$. Let $J = \{j : x_j < y_j, 1 < j \le k\}$. If the cardinality of the set J is l, then we write $J = \{j_1 < j_2 < \cdots < j_l = k\}$. Put $\alpha_i = x_i - y_i \ge 0$ for $i \notin J$, $1 \le i < k$, and $\beta_j = y_j - x_j > 0$ for $j \in J$. Since $\sum_{i=1}^{n} y_i \le \sum_{i=1}^{n} x_i$ for every positive integer $1 \le n \le k$, we have

$$\begin{aligned}
&\beta_{j_1} \le \alpha_1 + \cdots + \alpha_{j_1-1},\\
&\beta_{j_1} + \beta_{j_2} \le \left(\alpha_1 + \cdots + \alpha_{j_1-1}\right) + \left(\alpha_{j_1+1} + \cdots + \alpha_{j_2-1}\right),\\
&\vdots\\
&\sum_{j\in J} \beta_j \le \sum_{i=1, i\notin J}^{k-1} \alpha_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{k} x_i^2 &= \left[\sum_{i=1}^{j_1-1} (y_i + \alpha_i)^2 + \left(y_{j_1} - \beta_{j_1}\right)^2\right] + \left[\sum_{i=j_1+1}^{j_2-1} (y_i + \alpha_i)^2 + \left(y_{j_2} - \beta_{j_2}\right)^2\right]\\
&\quad + \cdots + \left[\sum_{i=j_{l-1}+1}^{k-1} (y_i + \alpha_i)^2 + (y_k - \beta_k)^2\right]\\
&\ge \sum_{i=1}^{k} y_i^2 + 2\left\{\left(\sum_{i=1}^{j_1-1} \alpha_i\right) - \beta_{j_1}\right\} y_{j_1} + 2\left\{\left(\sum_{i=j_1+1}^{j_2-1} \alpha_i\right) - \beta_{j_2}\right\} y_{j_2}\\
&\quad + \cdots + 2\left\{\left(\sum_{i=j_{l-1}+1}^{k-1} \alpha_i\right) - \beta_k\right\} y_k
\end{aligned}$$

$$\geq \sum_{i=1}^{k} y_i^2 + 2\left\{\left(\sum_{i=1, i\neq j_1}^{j_2-1} \alpha_i\right) - (\beta_{j_1} + \beta_{j_2})\right\} y_{j_2} + \cdots + 2\left\{\left(\sum_{i=j_{l-1}+1}^{k-1} \alpha_i\right) - \beta_k\right\} y_k$$

$$\vdots$$

$$\geq \sum_{i=1}^{k} y_i^2 + 2\left\{\left(\sum_{i=1, i\notin J}^{k-1} \alpha_i\right) - \left(\sum_{j\in J} \beta_j\right)\right\} y_k \geq \sum_{i=1}^{k} y_i^2,$$

where the inequalities follow from the above inequalities and the fact that the sequence (y_i) is decreasing. □

EXAMPLE 7. Let $w = (w_i)_{i=1}^{\infty} \in l_2 \setminus l_1$ and define the 2-homogeneous polynomial P on $d^*(w, 1)$ by

$$P(x) = \sum_{i=1}^{\infty} x_i{}^2, \quad x = (x_i)_{i=1}^{\infty} \in d^*(w, 1).$$

It follows from Lemma 6 that $\|P\| = \sum_{i=1}^{\infty} w_i^2 = \|P|_{d_*(w,1)}\|$. However, the sequence $(e_i)_{i=1}^{\infty}$ converges weak-star (weakly) to 0 in $d^*(w, 1)$ ($d_*(w, 1)$), and $P(e_i) = 1$ for all i. Therefore, P is not w^*-continuous on bounded sets at 0, and $P|_{d_*(w,1)}$ is not weakly continuous on bounded sets at 0.

Let $\mathbf{i}^n = (i_1, \ldots, i_n) \in \mathbb{N}^n$. We denote by $B_{\mathbf{i}^n}$ the closed unit ball of the n-dimensional subspace of $d^*(w, 1)$ spanned by $\{e_{i_1}, \ldots, e_{i_n}\}$. By the Krein-Milman theorem, $B_{\mathbf{i}^n}$ is the (closed) convex hull of its extreme points, that is, $B_{\mathbf{i}^n} = co(ext(B_{\mathbf{i}^n}))$. It is worthwhile to characterize its extreme points.

PROPOSITION 8. *Given* $\mathbf{i}^n = (i_1, \ldots, i_n) \in \mathbb{N}^n$, *the extreme points* (x_i) *of* $B_{\mathbf{i}^n}$ *are the points with coordinates* $|x_{i_j}| = w_{\sigma(j)}$, $1 \leq j \leq n$, *for some permutation* σ *on* $\{1, 2, \ldots, n\}$ *and* $x_i = 0$, *otherwise.*

Proof. We might as well assume $\mathbf{i}^n = (1, \ldots, n)$. An easy computation shows that the points with coordinates $|x_i| = w_{\sigma(i)}$, $1 \leq i \leq n$ for some permutation σ on $\{1, 2, \ldots, n\}$ and $x_i = 0$ otherwise, are extreme points of $B_{\mathbf{i}^n}$.

We shall prove that the other points x in $B_{\mathbf{i}^n}$ are not extreme points. Without loss of generality we may assume that $x = (x_i)$ is rearranged so that $|x_1| \geq |x_2| \geq \cdots \geq |x_n|$. Let k be the smallest positive integer i with $|x_i| \neq w_i$. If $k = n$, then $|x_{n-1}| = w_{n-1} > w_n > |x_n|$. Choose $\delta > 0$ so that $|x_n| + \delta < w_n$. Set u and v to be the points in $B_{\mathbf{i}^n}$ such that

$$u = (x_1, \ldots, x_{n-1}, sgn(x_n)(|x_n| + \delta))$$

and

$$v = (x_1, \ldots, x_{n-1}, sgn(x_n)(|x_n| - \delta)).$$

Then $x = 1/2(u + v)$, and hence x is not an extreme points.

Suppose that $1 < k < n$. Let $p = \max\{l : |x_l| = |x_k|,\ k \leq l \leq n\}$. If $p = k$, then $|x_{k-1}| = w_{k-1} > w_k > |x_k| > |x_{k+1}|$. Let $q = \max\{l : |x_l| = |x_{k+1}|,\ k+1 \leq l \leq n\}$. If $q < n$, choose $\delta > 0$ so that $w_k > |x_k| + \delta$, $|x_k| - \delta > |x_{k+1}| + \delta$, $|x_q| - \delta > |x_{q+1}|$ and

$$|x_k| + |x_{k+1}| + \cdots + |x_{k+j}| + \delta < w_k + w_{k+1} + \cdots + w_{k+j},$$

for all j, $1 \le j \le q-k-1$. We note that

$$|x_k| + |x_{k+1}| + \cdots + |x_{k+j}| < w_k + w_{k+1} + \cdots + w_{k+j},$$

for all j, $1 \le j \le q-k-1$. In the case where $q = n$, the condition $|x_q| - \delta > |x_{q+1}|$ is omitted for the choice of δ. Set $u = (u_i)$ and $v = (v_i)$ to be the points in $B_{\mathbf{i}^n}$ such that

$$u_k = sgn(x_k)(|x_k| - \delta), \quad v_k = sgn(x_k)(|x_k| + \delta),$$
$$u_{k+1} = sgn(x_{k+1})(|x_{k+1}| + \delta), \quad v_{k+1} = sgn(x_{k+1})(|x_{k+1}| - \delta)$$

and $u_i = x_i = v_i$ for $i \ne k,\ k+1$. Then $x = (u+v)/2$, and hence x is not an extreme point.

If $k < p < n$, choose $\delta > 0$ so that $w_k > |x_k| + \delta$, $|x_p| - \delta > |x_{p+1}|$ and

$$|x_k| + |x_{k+1}| + \cdots + |x_{k+j}| + \delta < w_k + w_{k+1} + \cdots + w_{k+j},$$

for all j, $1 \le j \le p-k-1$. In the case where $p = n$, the condition $|x_p| - \delta > |x_{p+1}|$ is omitted for the choice of δ. Set $u = (u_i)$ and $v = (v_i)$ to be the points in $B_{\mathbf{i}^n}$ such that

$$u_k = sgn(x_k)(|x_k| + \delta), \quad v_k = sgn(x_k)(|x_k| - \delta),$$
$$u_p = sgn(x_p)(|x_p| - \delta), \quad v_p = sgn(x_p)(|x_p| + \delta)$$

and $u_i = x_i = v_i$ for $i \ne k, p$. Then $x = (u+v)/2$, and hence x is not an extreme point.

By a similar argument to the above the same conclusion can be drawn for the case remaining where $k = 1$. □

The proof of Lemma 6 also follows from Proposition 8. Given a positive integer n, let $\mathbf{i}^n = (1, \ldots, n)$ and $(w_1, \ldots, w_n) = (x_1, \ldots, x_n)$. Then $y = (y_1, \ldots, y_n) \in B_{\mathbf{i}^n}$, and it is a convex combination of extreme points of $B_{\mathbf{i}^n}$. For simplicity, suppose that $y = \lambda e_1 + (1-\lambda)e_2$, where $0 \le \lambda \le 1$ and $e_k = (w_{\sigma_k(j)})_{j=1}^n$, for some permutation σ_k on $\{1, 2, \ldots, n\}$, $k = 1,\ 2$. Then $\sum_{i=1}^n y_i^2 = a\lambda^2 + b\lambda + c$ for some real numbers $a > 0$, b and c. Since it is always positive on the interval $0 \le \lambda \le 1$ and $a > 0$, its maximum on $0 \le \lambda \le 1$ occurs at $\lambda = 0$ or 1. Therefore, $\sum_{i=1}^n y_i^2 \le \sum_{i=1}^n w_i^2$.

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