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ON A CHARACTERISTIC PROPERTY OF POINT PROCESSES

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1. Abstract

This note is concerned with a certain property of point processes. We prove that if N_1, N_2 and N_3 are three independent point processes, then the bivariate point process $(N_1 + N_3, N_2 + N_3)$ uniquely determines the point processes N_1, N_2 and N_3.

2. Main result

It is known that every point process $N(\cdot)$ corresponds to a triple $(\Omega, \mathcal{F}, P_N)$ where Ω is the set of all countable sequences of real numbers $\{t_i\}$ without limit points and $\mathcal{F}$ is the σ-algebra generated by cylinder sets and P_N is a probability measure (cf. Harris (1963)). We say that the point process $N(\cdot)$ is *degenerate* if P_N is concentrated at a single point $(r_1, r_2, \cdots)$ in Ω. Let V denote the set of measurable functions ξ such that $0 \leqq \xi(t) \leqq 1$ for all real t and $\xi(t) = 1$ outside a bounded interval.

DEFINITION 2.1. *The probability generating functional (p.g.fl) of a point process $N(\cdot)$ is defined by*

$$G(\xi) = E\left\{\exp \int \log \xi(t) dN(t)\right\} \tag{1}$$

for $\xi \in V$. (If $\xi(t) \equiv 0$ over some set A the exponential in (1) is taken as zero unless $N(A) = 0$ when it equals one).

DEFINITION 2.2. *The p.g.fl of a bivariate point process $(M_1(\cdot), M_2(\cdot))$ is defined by*

(2) $$H(\xi_1, \xi_2) = E\left\{\exp\left[\int \log \xi_1(t)\, dM_1(t) + \int \log \xi_2(t)\, dM_2(t)\right]\right\}$$

for $\xi_1 \in V$, $\xi_2 \in V$.

THEOREM 2.1. *Let* N_1, N_3 *and* N_2 *be three independent point processes and let* $M_1 = N_1 + N_3$ *and* $M_2 = N_3 + N_2$. *Then the bivariate point process* (M_1, M_2) *uniquely determines the point processes* N_1, N_3 *and* N_2.

PROOF. Let $G_1(\xi)$, $G_3(\xi)$, $G_2(\xi)$ and $H(\xi_1, \xi_2)$ denote the p.g.fl's of N_1, N_3, N_2 and (M_1, M_2) respectively. It is easy to see that

(3) $$\begin{aligned} H(\xi_1, \xi_2) &= E\left\{\exp\left[\int \log \xi_1(t)\, dM_1(t) + \int \log \xi_2(t)\, dM_2(t)\right]\right\} \\ &= E\left\{\exp\left[\int \log \xi_1(t)\, dN_1(t) + \int \log \xi_2(t)\, dN_2(t) \right.\right. \\ &\qquad \left.\left. + \int \log(\xi_1(t)\, \xi_2(t))\, dN_3(t)\right]\right\} \\ &= G_1(\xi_1)\, G_2(\xi_2)\, G_3(\xi_1 \xi_2) \end{aligned}$$

for $\xi_1 \in V$, $\xi_2 \in V$ since N_1, N_2 and N_3 are independent point processes. Suppose now that R_1, R_3, R_2 are independent point processes such that the bivariate point process (S_1, S_2) has the same probability structure as (M_1, M_2) where $S_1 = R_1 + R_3$ and $S_2 = R_3 + R_2$. Let $K_1(\xi)$, $K_3(\xi)$ and $K_2(\xi)$ be the p.g.fl's of R_1, R_3 and R_2 respectively. It is now easy to see that

(4) $$H(\xi_1, \xi_2) = K_1(\xi_1) K_2(\xi_2) K_3(\xi_1 \xi_2).$$

Let A_j, $1 \leqq j \leqq K$ be disjoint Borel sets in R^1 and let $G_i(z)$ and $K_i(z)$ denote the p.g.fl's of $(N_i(A_1), \cdots, N_i(A_K))$ and $(R_i(A_1), \cdots, R_i(A_K))$ respectively. Then (3) and (4) imply that

(5) $$G_1(z_1) G_2(z_2) G_3(z_1 z_2) = K_1(z_1) K_2(z_2) K_3(z_1 z_2)$$

for all $z \in [0, 1]^K$ where $z_1 z_2$ is the vector obtained by multiplying z_1 and z_2 coordinate wise. $G_i(z)$ and $K_i(z)$, $1 \leqq i \leqq 3$ are non zero in the set $D^* = \{0 < z_j \leqq 1, 1 \leqq j \leqq K\}$ where $z = (z_1, \cdots, z_K)$. Let $J_i(z) = G_i(z)/K_i(z)$, $1 \leqq i \leqq 3$. Then $J_i(z)$ is non zero in D^* and $J_1(z_1) J_2(z_2) J_3(z_1 z_2) = 1$ for all z_1, z_2 in D^*. Substituting $z_2 = \mathbf{1}$, it can be seen that $J_1(z_1) J_3(z_1) = 1$ for all $z_1 \in D^*$. Similarly we get that $J_2(z_2) J_3(z_2) = 1$ for all $z_2 \in D^*$. Hence $J_3(z_1) J_3(z_2) = J_3(z_1 z_2)$ for all

$z_1, z_2 \in D^*$. Further J_3 is continuous in D^*. But the only continuous solution of this equation are functions of the type $\Pi_{j=1}^K z_j^{c_j}$ where c_j are constants by results of Aczel (1966), p. 215. Hence $G_3(z) = K_3(z)\, \Pi_{j=1}^K z_j^{c_j}$ for all $z \in D^*$. Splitting the product $\Pi_{j=1}^K z_j^{c_j}$ into two parts consisting of the positive and negative c_j's respectively, it can be seen that

$$(*) \qquad \prod_{j=1}^{K} z_j^{b_j} G_3(z) = \prod_{j=1}^{K} z_j^{d_j} K_3(z), b_j \geqq 0, d_j \geqq 0$$

for all $z \in D^*$. Since both sides of (*) are analytic in $D = \{|z_j| < 1, 1 \leqq j \leqq K\}$ and they agree on a subset D^* which has a limit point in D, they agree on D by analytic continuation. In other words $G_3(\xi) = K_3(\xi)J_3(\xi)$ for every ξ which is of the form

$$1 - \sum_{j=1}^{K} (1 - z_j)\chi_{A_j}(t),\ 0 \leqq z_j \leqq 1,\ 1 \leqq j \leqq K,$$

where $J_3(\xi) = \Pi_{j=1}^K z_j^{c_j}$, c_j real.

χ_A is the indicator function of set A. Define $J_3(\xi) = \lim J_3(\xi_n)$ for any $\xi \in V$. This is possible since every $\xi \in V$ can be uniformly approximated by an increasing sequence of simple functions of the above type. Note that G_3 and K_3 are continuous. Hence it follows that $G_3(\xi) = K_3(\xi)\, J_3(\xi)$ for all $\xi \in V$ where $J_3(\xi)$ is the p.g.fl of a degenerate point process by Westcott (1972). But the p.g.fl uniquely determines the point process by Proposition 1 of Vere-Jones (1968). Hence N_3 and R_3 differ by a degenerate point process. Similar argument shows that N_1, R_1 and N_2, R_2 differ by degenerate point processes. But the structure of the bivariate process (M_1, M_2) shows that we cannot add a degenerate process to one component without subtracting it from another. Hence N_1, N_2 and N_3 are unique to the process (M_1, M_2).

3. Remarks

Milne (1970) considers a system in which each point of a stationary Poisson process is subjected to a random displacement, the displacements being independent and identically distributed. He proved that the displacement distribution is identifiable if a complete input-output record is given but not the linkage between the two. Theorem 2.1 is similar to his result except that no assumptions are made about the distributions of the processes involved.

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