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FUGLEDE'S COMMUTATIVITY THEOREM AND $\cap R(T-\lambda)$

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ABSTRACT. Fuglede's commutativity theorem for normal operators is an easy consequence of the result that: For T normal, denoting the range of $T-\lambda$ by $R(T-\lambda)$, $\cap\{R(T-\lambda):\ \text{all}\ \lambda\}=\{0\}$:

Fuglede's commutativity theorem for normal operators is an easy consequence of the elegant intersection of ranges theorem: If T is normal, then the intersection of the ranges $R(T-\lambda)$, for all λ, is zero

(1) $$\cap\{R(T-\lambda):\ \text{all}\ \lambda\}=\{0\}$$

Equation (1), with $(T-\lambda)$ replaced by $(T-\lambda)^2$, was established by Johnson in [3]. Equation (1) was proved in [5], with reference to Johnson's work, and independently in [6]; proofs can also be found in [8, lemma 5.1] and [1, lemma 3.5]. Equation (1) can be extended to T hyponormal, for which see [1], by the use of Stampfli's powerful local spectral theory [1, 9, 10, 11, 12, 13].

Lemma 1 and corollary 2 below give a simple proof of Fuglede's theorem using (1). Lemma 3 gives an easy proof of a special case of (1) which is sufficient to establish Fuglede's theorem.

LEMMA 1. *Let T be a normal operator. For each λ there is a unitary operator U_λ with*

(2) $$(T-\lambda)=U_\lambda(T-\lambda)^*$$

This U_λ commutes with both T and T^*.

PROOF. Define U_λ on the range $R(T-\lambda)^*$ of $(T-\lambda)^*$ by $U_\lambda(T-\lambda)^*x=(T-\lambda)x$. Because T is normal, U_λ is an isometry and so has a unique extension to the closure of $R(T-\lambda)^*$. Extend U_λ to all of the Hilbert space as the identity on $[R(T-\lambda)^*]^\perp = N(T-\lambda)=N(T-\lambda)^*$. Equation (2) holds by construction. Since U_λ is unitary, equation (2) implies that

(3) $$U_\lambda^*(T-\lambda)=(T-\lambda)^*.$$

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Taking adjoints in (2) shows that $\cup_\lambda^*$ commutes with $(T-\lambda)$ and thus $\cup_\lambda$ commutes with $(T-\lambda)^*$. Then

$$\cup_\lambda(T-\lambda) = \cup_\lambda^2(T-\lambda)^* = \cup_\lambda(T-\lambda)^*\cup_\lambda = (T-\lambda)\cup_\lambda$$

and $\cup_\lambda$ commutes with $(T-\lambda)$ and $\cup_\lambda^*$ with $(T-\lambda)^*$. □

COROLLARY 2. *Fuglede's Theorem: Let T be normal and suppose that B commutes with T. Then B commutes with T^*.*

PROOF. Using the lemma, write

$$T^*B - BT^* = (T-\lambda)^*B - B(T-\lambda)^* = (T-\lambda)(\cup_\lambda^* B - B\cup_\lambda^*) \tag{4}$$

By the intersection of the ranges theorem, $T^*B = BT^*$. □

For a normal operator T, use the spectral theorem to represent T as multiplication on $L^2(S, \Sigma, \nu)$ by an $L^\infty(S, \Sigma, \nu)$ function φ. Assume that g belongs to the $\cap R(T-\lambda)$ so that for all $\lambda = x + iy$

$$f(x, y) = \int_S \frac{|g(s)|^2}{|\varphi(s)-\lambda|^2}\, \nu(ds) < \infty \tag{5}$$

Define $\mu(E) = \int_E |g(s)|^2 \nu(ds)$, a finite measure, and rewrite equation (5) as

$$f(x, y) = \int_S \frac{1}{|\varphi(s)-\lambda|^2}\, \mu(ds) < \infty \tag{6}$$

Equation (1) will hold if it can be shown that $\mu(S) = 0$. Note that the example of constant φ shows that (6) must hold for *all* λ before one can, in general, conclude that $\mu = 0$.

To establish Fuglede's theorem the full strength of (1) is not required. From equation (4), if g belongs to the range of $T^*B - BT^*$, then $g = (T-\lambda)(\cup_\lambda^* B - B\cup_\lambda^*)g$, so $f(x, y)$ can be chosen to be bounded with

$$f(x, y) = \|(\cup_\lambda^* B - B\cup_\lambda^*)g\|^2 \leqq 4\|B\|^2\|g\|^2$$

In this case where f is bounded, the measure μ can be shown to be zero by complex variable methods, as in the proof of [1, theorem 3.4]. Lemma 3 gives a simple real variable proof.

LEMMA 3. *If the function $f(x, y)$ of equation (5) is bounded then $\mu = 0$.*

PROOF. For $z \neq 0$, define

$$F(x, y, z) = \int_S \frac{1}{|\varphi(s)-\lambda|^2 + z^2}\, \mu(ds) \tag{7}$$

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By the Monotone Convergence Theorem, $F(x, y, z)$ increases to $f(x, y)$ as z tends to zero: hence $f(x, y)$ is lower semicontinuous, therefore measurable and so has a finite intergral over any compact subset of R^2.

If necessary, change φ on a set of measure zero so that $|\varphi(s)| \leqq M$ for all s in S. Set $a(s) = \mathrm{Re}\varphi(s)$, $b(s) = \mathrm{Im}\varphi(s)$, $\lambda = x + iy$, and consider:

$$\int_{-2M}^{2M}\int_{-2M}^{2M} f(x, y)dxdy = \int_S \int_{-2M}^{2M}\int_{-2M}^{2M} \frac{1}{|\varphi(s)-\lambda|^2}\, dxdy\ \mu(ds)$$

$$= \int_S \int_{-2M-a(s)}^{2M-a(s)}\int_{-2M-b(s)}^{2M-b(s)} \frac{1}{x^2+y^2}\, dxdy\ \mu(ds)$$

$$\geqq \int_S \int_{-M}^{M}\int_{-M}^{M} \frac{1}{x^2+y^2}\, dxdy\ \mu(ds) \geqq 2\pi\mu(S)\int_0^M (1/r)dr$$

and the last term is infinite unless $\mu(S) = 0$. □

Lemma 3, or the stronger equation (1), can be extended to more general φ. Note that if $\varphi(s_0) = \infty$, then μ can be a non-zero point mass at s_0 and still have (6) hold. However, one can extend the result to the case where φ is a measurable function which is finite μ-almost everywhere as follows: Let $S_n = \{s : |\varphi(s)| \leqq n\}$. Then for the measure μ_n defined by $\mu_n(E) = \mu(s_n \cap E)$,

$$f_n(x, y) = \int_S \frac{1}{|\varphi(s)-\lambda|^2}\, \mu_n(ds) < \infty$$

and φ belongs to $L^\infty(S, \Sigma, \mu_n)$. By the theorem for essentially bounded φ, $\mu_n = 0$. Since this is true for all n, $\mu = 0$.

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