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ARTICLE

Poset Ramsey numbers: large Boolean lattice versus a fixed poset

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Abstract

Given partially ordered sets (posets) $(P, \leq_P)$ and $(P', \leq_{P'})$, we say that P' contains a copy of P if for some injective function $f : P \to P'$ and for any $X, Y \in P$, $X \leq_P Y$ if and only if $f(X) \leq_{P'} f(Y)$. For any posets P and Q, the poset Ramsey number $R(P, Q)$ is the least positive integer N such that no matter how the elements of an N-dimensional Boolean lattice are coloured in blue and red, there is either a copy of P with all blue elements or a copy of Q with all red elements. We focus on a poset Ramsey number $R(P, Q_n)$ for a fixed poset P and an n-dimensional Boolean lattice Q_n, as n grows large. We show a sharp jump in behaviour of this number as a function of n depending on whether or not P contains a copy of either a poset V, that is a poset on elements A, B, C such that $B > C$, $A > C$, and A and B incomparable, or a poset Λ, its symmetric counterpart. Specifically, we prove that if P contains a copy of V or Λ then $R(P, Q_n) \geq n + \frac{1}{15}\frac{n}{\log n}$. Otherwise $R(P, Q_n) \leq n + c(P)$ for a constant $c(P)$. This gives the first non-marginal improvement of a lower bound on poset Ramsey numbers and as a consequence gives $R(Q_2, Q_n) = n + \Theta\left(\frac{n}{\log n}\right)$.

Keywords: poset Ramsey number; Boolean lattice; combinatorics of posets

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1. Introduction

Ramsey-type problems for partially ordered sets, or shortly posets, were considered first in great generality by Nešetřil and Rödl [24], who determined all posets U such that for any poset P there is a ground poset W with the property that any colouring of induced copies of U in W with a fixed number of colours results in a monochromatic induced copy of P. See also a paper by Paoli, Trotter, and Walker [25] on this topic. If U is the poset consisting of a single element, we arrive at a natural special case, where one writes $W \to P$ if any two-colouring of the elements of W contains a monochromatic induced copy of P. Kierstead and Trotter [18] considered this setting for general posets with the goal of minimising $p(W)$ for all P with a fixed $p(P)$, where p is a poset parameter such as size, height, or width.

In this paper, we consider a closely related poset Ramsey problem, where the ground poset is a Boolean lattice Q_N, a poset whose elements are all subsets of an N-element set equipped with set inclusion relation. For posets P and Q, let the *Boolean poset Ramsey number* or simply *poset Ramsey number* $R(P, Q)$ be the least integer N such that in any colouring of elements of the N-dimensional Boolean lattice Q_N in blue and red, there is an induced copy of P with all blue elements or an induced copy of Q with all red elements.

This function was first studied in detail relatively recently by Axenovich and Walzer [1]. In the diagonal case $P = Q = Q_n$, the bounds $2n + 1 \leq R(Q_n, Q_n) \leq n^2 - n + 2$ are the best currently known, see listed chronologically Walzer [29], Axenovich and Walzer [1], Cox and Stolee [10], Lu and Thompson [20], Bohman and Peng [3]. Falgas-Ravry, Markström, Treglown, and Zhao [12] showed that $R(Q_3, Q_3) = 7$. Since any posets P and Q are induced subposets of Q_n for sufficiently large n, we see that $R(P, Q)$ is well-defined. Boolean poset Ramsey numbers and their rainbow variants were considered for some special classes of posets by Chen, Chen, Cheng, Li, and Liu [7, 8] as well as by Chang et al. [6]. In the off-diagonal setting $R(Q_m, Q_n)$ with m fixed and n large, an exact result is only known if $m = 1$. It is easy to see that $R(Q_1, Q_n) = n + 1$. For $m = 2$, it was shown in [1] that $R(Q_2, Q_n) \leq 2n + 2$. This was improved by Lu and Thompson to $R(Q_2, Q_n) \leq (5/3)n + 2$. Finally, it was further improved by Grósz, Methuku, and Tompkins [15]:

Theorem 1. (Grósz-Methuku-Tompkins [15]) *For any $\epsilon > 0$ there is n_0 such that for any integer $n \geq n_0$, we have $n + 3 \leq R(Q_2, Q_n) \leq n + (2 + \epsilon)n/\log n$.*

In this paper, we focus on $R(P, Q_n)$ for an arbitrary fixed poset P and large n. Note that $R(P, Q_n) \geq n$ for any non-empty poset P as witnessed by a colouring of all elements of Q_{n-1} in red. We prove that a central role is played by a small, three-element poset $\Lambda = (\Lambda, <)$, with elements Z_1, Z_2, and Z_3, such that $Z_1 < Z_3$, $Z_2 < Z_3$, and Z_1 and Z_2 incomparable. A poset V is the symmetric counterpart of Λ, having elements Z_1, Z_2, and Z_3, such that $Z_1 > Z_3$, $Z_2 > Z_3$, and Z_1 and Z_2 not comparable. Our main result shows a sharp jump in the behaviour of $R(P, Q_n)$ as a function of n depending whether or not P contains a copy Λ or V.

Theorem 2. *For every poset P there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the following holds.*

- *If P contains a copy of Λ or V, then $R(P, Q_n) \geq n + \frac{1}{15}\frac{n}{\log n}$.*
- *If P contains neither a copy of Λ nor a copy of V, then $R(P, Q_n) \leq n + f(P)$, for some function f.*

The first part of Theorem 2 relies on the lower bound on $R(\Lambda, Q_n)$ that we provide in the next theorem along with an asymptotically matching upper bound.

Theorem 3. *Let $\epsilon > 0$. There exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$n + \frac{1}{15} \cdot \frac{n}{\log n} \leq R(\Lambda, Q_n) \leq n + \big(1 + \epsilon\big) \cdot \frac{n}{\log n}.$$

More precisely, it can be seen that the lower bound holds for $\log n_0 \geq 535$, while the upper bound requires $\log n_0 \geq \frac{36}{\epsilon^2}$. Note that $R(\Lambda, Q_n) \leq R(Q_2, Q_n)$, so Theorem 1 already implies a bound for $R(\Lambda, Q_n)$ which is weaker but asymptotically equal to the upper bound of Theorem 3. Our main tool used to prove both the lower and the improved upper bound on $R(\Lambda, Q_n)$ is a structural duality statement, Theorem 12. The upper bound is obtained from that using a counting argument, while for the lower bound we give a probabilistic construction to find a desired colouring. This is the first of a kind non-marginal improvement of a trivial lower bound for poset Ramsey numbers. Most other known lower bounds correspond to so-called layered colourings of Boolean lattices, where any two elements of the same size have the same colour. The only two previously given non-layered constructions are those from Theorem 1 by Grósz, Methuku, and Tompkins [15], improving the trivial lower bound $R(Q_2, Q_n) \geq n + 2$ to $n + 3$ and by Bohman and Peng [3] improving the trivial lower bound for the diagonal case $R(Q_n, Q_n) \geq 2n$ to $2n + 1$.

Theorems 1 and 2 also give a lower bound for $R(Q_2, Q_n)$ which is asymptotically tight not only in the first but also in the second summand.

Corollary 4. $R(Q_2, Q_n) = n + \Theta\left(\frac{n}{\log(n)}\right)$.

Note that the Ramsey variant we consider is related to extremal problems on posets and their induced subposets. Carroll and Katona [5] introduced a Turán-type function $La^{\#}(n, P)$ as the largest number of elements in Q_n that do not induce a copy of the poset P. Most notable is a result

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by Methuku and Pálvölgyi [22], who showed that $La^{\#}(n, P) \le f(P)\binom{n}{\lfloor n/2 \rfloor}$, thus proving a conjecture of Katona, Lu, and Milans [19]. Their statement has been refined for several special cases, see for example Lu and Milans [19] and Méroueh [21], as well as an earlier result by Boehlein and Jiang [2]. Further Turán-type results were given by Methuku and Tompkins [23] and Tomon [28]. Note that the corresponding function $La(n, P)$ for non-induced, so-called weak subposet, P, was extensively studied, see for example [4, 9, 11, 13, 14, 16, 26, 27]. In addition, saturation-type extremal problems have been addressed for induced and weak subposets, see a recent paper of Keszegh, Lemons, Martin, Pálvölgyi, and Patkós [17].

The structure of the paper is as follows. In Section 2, we give the formal definitions and notations, define special posets we call factorial trees and shrubs, and prove some basic properties. In Section 2.4, we provide an alternative proof of the upper bound in Theorem 1. This makes our paper self-contained since we need this result for Corollary 4. In Section 3, we provide a structural duality statement, Theorem 12, which is the key tool for the main proofs. In Section 4, we use a probabilistic construction to find a colouring with 'good' properties. Lastly, in Section 5 we complete the proofs of Theorems 3 and 2.

2. Preliminaries

2.1. Basic notations and definitions

A *poset* is a set P equipped with a relation $\le_P$ that is transitive, reflexive, and antisymmetric. For any non-empty set $\mathcal{X}$, let $\mathcal{Q}(\mathcal{X})$ be the *Boolean lattice* of dimension $|\mathcal{X}|$ on a *ground set* $\mathcal{X}$, that is the poset consisting of all subsets of $\mathcal{X}$ equipped with the inclusion relation $\subseteq$. We use Q_N to denote a Boolean lattice of dimension N, that is a set of all subsets of an N-element set with set inclusion order. We refer to a poset either as a pair $(P, \le_P)$, or, when it is clear from context, simply as a set P. The elements of P are often called *vertices*.

For two posets $(P_1, \le_{P_1})$ and $(P_2, \le_{P_2})$, an *embedding* $\phi : P_1 \to P_2$ of P_1 into P_2 is an injective function such that for every $X_1, X_2 \in P_1$, $X_1 \le_{P_1} X_2$ if and only if $\phi(X_1) \le_{P_2} \phi(X_2)$. A poset P_1 is an *induced subposet* of P_2 if $P_1 \subseteq P_2$ and for every $X_1, X_2 \in P_1$, $X_1 \le_{P_1} X_2$ if and only if $X_1 \le_{P_2} X_2$. A *copy* of a poset P_1 in P_2 is an induced subposet P' of P_2, isomorphic to P_1.

Consider an assignment of two colours, blue and red, to the vertices of posets. Such a colouring $c{:}P \to \{blue, red\}$ is a *blue/red colouring* of P. A coloured poset is *monochromatic* if all of its vertices share the same colour. A monochromatic poset whose vertices are blue is called a *blue poset*. Similarly defined is a *red poset*. Using this terminology, the *poset Ramsey number* of two posets P and Q is

$$R(P, Q) = \min\{N \in \mathbb{N} : \text{ every blue/red colouring of } Q_N \text{ contains either a blue copy of } P \text{ or a red copy of } Q\}.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets. Then the vertices of the Boolean lattice $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$, that is the unordered subsets of $\mathcal{X} \cup \mathcal{Y}$, can be partitioned with respect to $\mathcal{X}$ and $\mathcal{Y}$ in the following manner. Every $Z \subseteq \mathcal{X} \cup \mathcal{Y}$ has an $\mathcal{X}$-part $X_Z = Z \cap \mathcal{X}$ and a $\mathcal{Y}$-part $Y_Z = Z \cap \mathcal{Y}$. In this setting, we refer to Z alternatively as the pair (X_Z, Y_Z). Conversely, for all $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, the pair (X, Y) has a 1-to-1 correspondence to the vertex $X \cup Y \in \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$. One can think of such pairs as elements of the Cartesian product $2^{\mathcal{X}} \times 2^{\mathcal{Y}}$ which has a canonical bijection to $2^{\mathcal{X} \cup \mathcal{Y}} = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$. Observe that for $X_i \subseteq \mathcal{X}$, $Y_i \subseteq \mathcal{Y}$, $i \in [2]$, we have $(X_1, Y_1) \subseteq (X_2, Y_2)$ if and only if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$.

For any poset, we refer to vertices Z_1, Z_2 which are incomparable as $Z_1 \nsim Z_2$. For a positive integer $n \in \mathbb{N}$, we use $[n]$ to denote the set $\{1, \ldots, n\}$. Given an integer $n \in \mathbb{N}$ and a set $\mathcal{X}$, let $\binom{\mathcal{X}}{n}$ be the set of all n-element subsets of $\mathcal{X}$. Throughout the paper, 'log' always refers to the logarithm with base 2, while 'ln' refers to the natural logarithm. For sets A and B, we write $A \subset B$ to denote that A is a *proper* subset of B. We omit floors and ceilings where appropriate.

2.2. Structure of posets with forbidden Λ or V

A poset $\mathcal{T}$ is an *up-tree* if there is a unique minimal vertex in $\mathcal{T}$ and for every vertex $X \in \mathcal{T}$, the set $\{Y \in \mathcal{T} : Y \leq X\}$ is a chain, that is its vertices are pairwise comparable. We say that two subposets of a given poset are *independent* if they are vertexwise incomparable. Furthermore, a collection of subposets is *independent* if they are pairwise independent. We use this notation to describe posets which don't contain a copy of Λ (or V).

Lemma 5. *Let P be a poset. There is no copy of Λ in P if and only if P is an independent collection of up-trees.*

Proof. Observe that a poset P is an independent collection of up-trees if and only if for every vertex $X \in P$, $\{Y \in P : Y \leq X\}$ forms a chain.

Suppose that there is a copy of Λ in P on vertices Z_i, $i \in [3]$ with $Z_1 < Z_3, Z_2 < Z_3$ and $Z_1 \nsim Z_2$. Then Z_1, Z_2 witness that $\{Y \in P : Y < Z_3\}$ is not a chain, so P is not an independent collection of up-trees.

Now assume that P is not an independent collection of up-trees. Then there exist some $X \in P$ and $Z_1, Z_2 \in \{Y \in P : Y \leq X\}$ such that $Z_1 \nsim Z_2$. Since X is comparable to all vertices in $\{Y \in P : Y \leq X\}$, $X > Z_1, X > Z_2$. Now X, Z_1, Z_2 form a copy of Λ. □

By symmetry an analogous statement holds for posets with forbidden induced copy of V. If we forbid both V and Λ simultaneously we obtain the following structure.

Corollary 6. *Let P be a poset such that there is neither a copy of V nor of Λ. Then P is an independent collection of chains.*

2.3. Embeddings of Q_n

When considering an embedding ϕ of a Boolean lattice Q_n into a larger Boolean lattice $\mathcal{Q}(\mathcal{Z})$, we can partition $\mathcal{Z}$ such that it has the following nice property. This result is due to Axenovich and Walzer [1], here we state an alternative proof.

Lemma 7. (Axenovich-Walzer [1]). *Let $n \in \mathbb{N}$. Let $\mathcal{Z}$ be a set with $|\mathcal{Z}| > n$ and let $Q = \mathcal{Q}(\mathcal{Z})$. If there is an embedding $\phi : Q_n \to Q$, then there exist a subset $\mathcal{X} \subset \mathcal{Z}$ with $|\mathcal{X}| = n$, and an embedding $\phi' : \mathcal{Q}(\mathcal{X}) \to Q$ with the same image as ϕ such that $\phi'(X) \cap \mathcal{X} = X$ for all $X \subseteq \mathcal{X}$.*

Proof. Let the ground set of Q_n be $\mathcal{X}'$. We consider the embedding of singletons of Q_n, that is $\phi(\{a\})$, $a \in \mathcal{X}'$. If $\phi(\{a\}) \subseteq \bigcup_{X' \subseteq \mathcal{X}' \setminus \{a\}} \phi(X')$, then $\phi(\{a\}) \subseteq \bigcup_{X' \subseteq \mathcal{X}' \setminus \{a\}} \phi(X') \subseteq \phi(\mathcal{X}' \setminus \{a\})$. But $\{a\} \nsubseteq \mathcal{X}' \setminus \{a\}$ and ϕ is an embedding, a contradiction. Thus $\phi(\{a\}) \nsubseteq \bigcup_{X' \subseteq \mathcal{X}' \setminus \{a\}} \phi(X')$. For every $a \in \mathcal{X}'$, pick an arbitrary

$$b(a) \in \phi(\{a\}) \setminus \bigcup_{X' \subseteq \mathcal{X}' \setminus \{a\}} \phi(X').$$

Note that $b(a_1) \notin \phi(\{a_2\})$ for any $a_1, a_2 \in \mathcal{X}'$, $a_1 \neq a_2$, so all representatives are distinct. Let $\mathcal{X} = \{b(a) : a \in \mathcal{X}'\}$. We see that the map $b : \mathcal{X}' \to \mathcal{X}$ is a bijection. For every $B \subseteq \mathcal{X}$, let $A_B \subseteq \mathcal{X}'$ be such that $B = \{b(a) : a \in A_B\}$. We define $\phi' : \mathcal{Q}(\mathcal{X}) \to Q$ as follows: $\phi'(B) = \phi(A_B)$, $B \in \mathcal{X}$. Then ϕ' is an embedding. Observe that for $X \subseteq \mathcal{X}$ and $b \in \mathcal{X}$, $b \in \phi'(X)$ if and only if $b \in X$. Thus $\phi'(X) \cap \mathcal{X} = X$ for all $X \subseteq \mathcal{X}$. This concludes the proof. □

We call an embedding ϕ of $\mathcal{Q}(\mathcal{X})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ for disjoint $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X}$-*good* if $\phi(X) \cap \mathcal{X} = X$ for all $X \in \mathcal{X}$. We also call a copy Q of $\mathcal{Q}(\mathcal{X})$ in $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ $\mathcal{X}$-*good* if there is an $\mathcal{X}$-good embedding of $\mathcal{Q}(\mathcal{X})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$.

Lemma 7 claims in particular that for any copy of Q_n in a larger Boolean lattice Q, there is a subset $\mathcal{X}$ of the ground set of Q with $|\mathcal{X}| = n$ such that there is an $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$ in Q.

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2.4. Red copy of Q_n vs. blue chain

The main goal of this subsection is to present an alternative proof for the upper bound of Theorem 1 and thus of Corollary 4. Grósz, Methuku, and Tompkins [15] stated the following lemma using a different formulation. While they used algorithmic tools in their proof, we prove the statement recursively. Recall that for a given partition $\mathcal{X} \cup \mathcal{Y}$ of the ground set of a Boolean lattice we denote a vertex $X \cup Y \in \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$, where $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, as (X, Y).

Lemma 8. *Let $\mathcal{X}, \mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$, for some $n, k \in \mathbb{N}$. Let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ be a blue/red coloured Boolean lattice. Fix some linear ordering $\pi = (y_1, \dots, y_k)$ of $\mathcal{Y}$ and define $Y(0), \dots, Y(k)$ by $Y(0) = \varnothing$ and $Y(i) = \{y_1, \dots, y_i\}$ for $i \in [k]$. Then there exists at least one of the following in Q:*

(1) a red $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$, or

(2) a blue chain of length $k+1$ of the form $(X_0, Y(0)), \dots, (X_k, Y(k))$ where $X_0 \subseteq X_1 \subseteq \dots \subseteq X_k \subseteq \mathcal{X}$.

Proof. Suppose that there is no blue chain as described in (b). For every $X \subseteq \mathcal{X}$, we recursively define a label $\ell_X \in \{0, \dots, k\}$ such that $\phi : \mathcal{Q}(\mathcal{X}) \to Q$, $\phi(X) = (X, Y(\ell_X))$, is an embedding with monochromatic red image. We require ℓ_X to fulfil three properties:

(1) For any $X' \subseteq X$, $\ell_{X'} \leq \ell_X$.

(2) There is a blue chain of length ℓ_X contained in the Boolean lattice with ground set $X \cup Y(\ell_X)$, which we denote by Q^X.

(3) $(X, Y(\ell_X))$ is red.

First, consider the vertex $\varnothing$. Let $\ell_\varnothing$ be the minimum ℓ, $0 \leq \ell \leq k$, such that $(\varnothing, Y(\ell))$ is red. If such an ℓ does not exist, then $(\varnothing, Y(0)), \dots, (\varnothing, Y(k))$ form a blue chain, a contradiction. It is clear to see that Properties (1) and (3) hold. If $\ell_\varnothing = 0$, (2) is trivially true. If $\ell_\varnothing \geq 1$, $(\varnothing, Y(0)), \dots, (\varnothing, Y(\ell_\varnothing - 1))$ form a blue chain of length $\ell_\varnothing$ and (2) holds as well.

Consider an arbitrary $X \subseteq \mathcal{X}$ and suppose that for all $X' \subset X$ we already defined $\ell_{X'}$ with Properties (1)–(3). Let $\ell'_X = \max_{\{U \subset X\}} \ell_U$. Then let ℓ_X be the minimumℓ, $\ell'_X \leq \ell \leq k$ such that $(X, Y(\ell))$ is coloured in red. If there is no such ℓ, then $\left(X, Y\left(\ell'_X\right)\right), \dots, (X, Y(k))$ is a blue chain of length $k - \ell'_X + 1$. By definition of ℓ'_X there is some $U \subset X$ with $\ell_U = \ell'_X$. In particular, (2) holds for U, so there is a blue chain of length ℓ'_X in Q^U. Note that $(U, Y(\ell_U)) \subset \left(X, Y\left(\ell'_X\right)\right)$, so we obtain a blue chain of length $k+1$. This is a contradiction, thus ℓ_X is well-defined and fulfils Property (3).

If $\ell_X = \ell'_X$, consider the aforementioned blue chain of length ℓ'_X in Q^U, and otherwise consider this chain together with $\left(X, Y\left(\ell'_X\right)\right), \dots, (X, Y(\ell_X - 1))$. In both cases, we obtain a blue chain of length ℓ_X, which proves (2). For $X' \subset X \subseteq \mathcal{X}$, $\ell_{X'} \leq \ell'_X \leq \ell_X$, thus (1) holds.

We define $\phi : \mathcal{Q}(\mathcal{X}) \to Q$, $\phi(X) = (X, Y(\ell_X))$. Note that $\phi(X) \cap \mathcal{X} = X$ for every $X \subseteq \mathcal{X}$ and Property (3) implies that $\phi(X)$ is red. Let $X_1, X_2 \subseteq \mathcal{X}$. If $\phi(X_1) \subseteq \phi(X_2)$, it is immediate that $X_1 \subseteq X_2$. Conversely, if $X_1 \subseteq X_2$, then by Property (1) we have $\ell_{X_1} \leq \ell_{X_2}$. Thus $\left(X_1, Y\left(\ell_{X_1}\right)\right) \subseteq \left(X_2, Y\left(\ell_{X_2}\right)\right)$. As a consequence, ϕ is an $\mathcal{X}$-good embedding of $\mathcal{Q}(\mathcal{X})$. □

This Lemma implies the following corollary which is already given in an alternative form by Axenovich and Walzer, see Lemma 4 of [1].

Corollary 9. *Let $\mathcal{X}, \mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$. Let $\mathcal{P}$ be a subposet of a Boolean lattice $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ such that there is no chain of length $k+1$ in $\mathcal{P}$. Then there exists a copy of Q_n in Q which contains no vertex of $\mathcal{P}$.*

Proof. Fix an arbitrary linear ordering of $\mathcal{Y}$. Furthermore, let $c: Q \to \{blue, red\}$ be the colouring such that

$$c(X) = \begin{cases} \text{blue}, & \text{if } X \in \mathcal{P}, \\ \text{red}, & \text{otherwise.} \end{cases}$$

There is no blue chain of length $k+1$ in c, so by Lemma 8 there is a monochromatic red copy of Q_n in Q. This copy does not contain any vertex of $\mathcal{P}$. □

With the help of Lemma 8, we can now prove an upper bound for $R(Q_2, Q_n)$. The concluding arguments are due to Grósz, Methuku, and Tompkins [15].

Proof of Corollary 4. The lower bound follows from Theorem 2. For the upper bound, let $k \in \mathbb{N}$ with $k = \frac{(2+\epsilon)n}{\log(n)}$. Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$. Consider a blue/red colouring of $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ with no monochromatic red copy of Q_n. Let $\pi = \left(y_1^\pi, \ldots, y_k^\pi\right)$ be a linear ordering of $\mathcal{Y}$. By Lemma 8, there exists a blue chain of length $k+1$ of the form $\left(X_0^\pi, \varnothing\right), \left(X_1^\pi, \{y_1^\pi\}\right), \left(X_2^\pi, \{y_1^\pi, y_2^\pi\}\right), \ldots, \left(X_k^\pi, \mathcal{Y}\right)$ where $X_i^\pi \subseteq \mathcal{X}$.

Note that there are $k!$ distinct orderings of $\mathcal{Y}$. For each linear ordering π of $\mathcal{Y}$ we consider X_0^π and X_k^π, that is the minimal and maximal vertex of the aforementioned chain restricted to $\mathcal{X}$. By the choice of k, we obtain $k! > 2^{2n}$. In particular by pigeonhole principle, there are distinct π_1, π_2 with $X_0^{\pi_1} = X_0^{\pi_2}$ and $X_k^{\pi_1} = X_k^{\pi_2}$. Since π_1, π_2 are distinct, there exists an index $1 \le i \le k-1$ with $\left\{y_1^{\pi_1}, \ldots, y_i^{\pi_1}\right\} \neq \left\{y_1^{\pi_2}, \ldots, y_i^{\pi_2}\right\}$. Then the four vertices

$$\left(X_0^{\pi_1}, \varnothing\right), \left(X_i^{\pi_1}, \{y_1^{\pi_1}, \ldots, y_i^{\pi_1}\}\right), \left(X_i^{\pi_2}, \{y_1^{\pi_2}, \ldots, y_i^{\pi_2}\}\right), \left(X_k^{\pi_1}, \mathcal{Y}\right)$$

form a blue copy of Q_2. □

2.5. Factorial trees and shrubs

Besides the Boolean lattice, there is another poset which plays a major role in this paper, which we call the *factorial tree*.

Consider the set of ordered subsets of a fixed non-empty set $\mathcal{Y}$, that also could be thought of as a set of strings with non-repeated letters over the alphabet $\mathcal{Y}$. Note that we also allow the empty set as such an ordered subset. Occasionally, if it is clear from the context, we refer to the empty ordered set $(\varnothing, \le)$ simply as $\varnothing$. For an ordered subset S of $\mathcal{Y}$, we refer to its underlying unordered set as $\underline{S}$. Let $|S| = |\underline{S}|$ be the *size* of S. We also say that S is an *ordering* of $\underline{S}$.

Let S be an ordered subset of $\mathcal{Y}$. A *prefix* of S is an ordered subset T of $\mathcal{Y}$ consisting of the first $|T|$ elements of S in the ordering induced by S. If T is a prefix of S, we write $T \le_{\mathcal{O}} S$. Note that the empty ordered set is a prefix of every ordered set. If $T \neq S$, we say that a prefix T of S is *strict*, denoted by $T <_{\mathcal{O}} S$. Observe that the prefix relation $\le_{\mathcal{O}}$ is transitive, reflexive, and antisymmetric. Let $\mathcal{O}(\mathcal{Y})$ be the poset of all ordered subsets of $\mathcal{Y}$ equipped with $\le_{\mathcal{O}}$. We say that this poset is the *factorial tree* on ground set $\mathcal{Y}$.

In a factorial tree $\mathcal{O}(\mathcal{Y})$ for every vertex $S \in \mathcal{O}(\mathcal{Y})$, the set of prefixes $\{T \in \mathcal{O}(\mathcal{Y}) : T \le_{\mathcal{O}} S\}$ induces a chain. Furthermore, the vertex $\varnothing$ is the unique minimal vertex of $\mathcal{Y}$, thus $\mathcal{O}(\mathcal{Y})$ is an up-tree.

Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets. Let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ and $\mathcal{O}(\mathcal{Y})$ be the factorial tree with ground set $\mathcal{Y}$. An embedding τ of $\mathcal{O}(\mathcal{Y})$ into Q is *$\mathcal{Y}$-good* if for every $S \in \mathcal{O}(\mathcal{Y})$, $\tau(S) \cap \mathcal{Y} = \underline{S}$. We say that a subposet $\mathcal{P}$ of Q is a *$\mathcal{Y}$-good copy* of $\mathcal{O}(\mathcal{Y})$ if there exists a $\mathcal{Y}$-good embedding $\tau : \mathcal{O}(\mathcal{Y}) \to Q$ with image $\mathcal{P}$. We refer to such a copy also as a *$\mathcal{Y}$-shrub*.

Besides that, we also consider a related subposet with slightly weaker conditions. A *weak $\mathcal{Y}$-shrub* is a subposet $\mathcal{P}$ of Q such that there is a function $\tau : \mathcal{O}(\mathcal{Y}) \to Q$ with image $\mathcal{P}$ such that for every $S \in \mathcal{O}(\mathcal{Y})$, $\tau(S) \cap \mathcal{Y} = \underline{S}$ and for every $S, T \in \mathcal{O}(\mathcal{Y})$ with $S <_{\mathcal{O}} T$, $\tau(S) \subset \tau(T)$. In particular, a weak $\mathcal{Y}$-shrub might not correspond to an injective embedding of $\mathcal{O}(\mathcal{Y})$.

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Clearly a $\mathcal{Y}$-shrub is also a weak $\mathcal{Y}$-shrub. Surprisingly, the converse statement is also true for subposets of a Boolean lattices which do not contain a copy of Λ.

Proposition 10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets, let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$. Let $\mathcal{P}$ be a weak $\mathcal{Y}$-shrub in Q such that $\mathcal{P}$ contains no copy of Λ. Then $\mathcal{P}$ is a $\mathcal{Y}$-shrub.*

Proof. Let $\tau : \mathcal{O}(\mathcal{Y}) \to Q$ be a map such that for every $S, T \in \mathcal{O}(\mathcal{Y})$ with $S <_{\mathcal{O}} T$, we have $\tau(S) \subset \tau(T)$ and $\tau(S) \cap \mathcal{Y} = \underline{S}$, and let $\mathcal{P}$ be its image. For all $S \in \mathcal{O}(\mathcal{Y})$, let $X_S = \tau(S) \cap \mathcal{X}$, that is $\tau(S) = (X_S, \underline{S})$. We shall show that τ is an embedding, thus proving that $\mathcal{P}$ is a $\mathcal{Y}$-shrub. For that we need to prove that the condition $\tau(S) \subseteq \tau(T)$ implies that $S \leq_{\mathcal{O}} T$ for any ordered subsets S and T of $\mathcal{Y}$.

Let $\tau(S) \subseteq \tau(T)$, that is $(X_S, \underline{S}) \subseteq (X_T, \underline{T})$. In particular, $\underline{S} \subseteq \underline{T}$ and so $|S| \leq |T|$. Let R be the largest common prefix of S and T. Such a prefix exists since $\varnothing$ is a prefix of every ordered set. If $|R| = |S|$, then $S = R \leq_{\mathcal{O}} T$ and we are done. So we can assume that $|S| \geq |R| + 1$.

If $|T| \leq |R| + 1$, then $|R| + 1 \leq |S| \leq |T| \leq |R| + 1$. This implies $|S| = |T|$ and since $\underline{S} \subseteq \underline{T}$, we have $\underline{S} = \underline{T}$. Let $\{y\} = \underline{S} \backslash \underline{R} = \underline{T} \backslash \underline{R}$. Then both S, T have R as prefix of size $|S| - 1 = |T| - 1$ and y as final vertex. Thus $S = T$ and we are done as well.

From now on, we assume that $|S| \geq |R| + 1$ and $|T| > |R| + 1$. Consider prefixes $S' \leq_{\mathcal{O}} S$ and $T' \leq_{\mathcal{O}} T$ of size $|R| + 1$. Then R is a prefix of both S' and T'. Let y_S such that $\underline{S'} \backslash \underline{R} = \{y_S\}$ and let y_T with $\underline{T'} \backslash \underline{R} = \{y_T\}$.

If $y_S = y_T$, we obtain $S' = T'$, which implies that R is not the largest common prefix of S and T, a contradiction.

If $y_S \neq y_T$, the unordered sets $\underline{S'}$ and $\underline{T'}$ are not comparable. In particular, $(X_{T'}, \underline{T'})$ and $(X_{S'}, \underline{S'})$ are incomparable. Because $S' \leq_{\mathcal{O}} S$, $T' <_{\mathcal{O}} T$ and by our initial assumption, we know that $(X_{S'}, \underline{S'}) \subseteq (X_S, \underline{S}) \subseteq (X_T, \underline{T})$ and $(X_{T'}, \underline{T'}) \subseteq (X_T, \underline{T})$. Since $|S'| = |T'| = |R| + 1 < |T|$, we obtain that both $(X_{S'}, \underline{S'})$ and $(X_{T'}, \underline{T'})$ are proper subsets of $(X_T, \underline{T})$. Then the three vertices $(X_T, \underline{T})$, $(X_{T'}, \underline{T'})$ and $(X_{S'}, \underline{S'})$ form a copy of Λ in Q, so we reach a contradiction. □

2.6. Construction of an almost optimal shrub

Let $\mathcal{Y}$ be a k-element set. Note that a $\mathcal{Y}$-shrub has $k!$ maximal vertices corresponding to all permutations of $\mathcal{Y}$. These maximal vertices form an antichain, that is are pairwise incomparable. Sperner's theorem implies that a ground set of any $\mathcal{Y}$-shrub must have size at least q, where $\binom{q}{\lfloor q/2 \rfloor} \geq k!$, so $q \geq k(\log k + \log e) + o(k)$. Next, we shall construct a $\mathcal{Y}$-shrub which is almost optimal in the sense that $\mathcal{Y}$ has ground set of size almost matching the lower bound above.

Proposition 11. *Let $\mathcal{Y}$ be a k-element set. Let A be a set disjoint from $\mathcal{Y}$ such that $|A| \geq k \cdot \min\{\log k + \log\log k, 11\}$. Then there is a $\mathcal{Y}$-shrub in $\mathcal{Q}(A \cup \mathcal{Y})$.*

Proof. Let $\mathcal{Y} = \{y_0, \ldots, y_{k-1}\}$ and let $Q = \mathcal{Q}(A \cup \mathcal{Y})$. We use addition of indices modulo k. Let $A_0, \ldots, A_{k-1}$ be pairwise disjoint subsets of A such that $|A_i| = \ell$ for the smallest integer ℓ satisfying $\binom{\ell}{\lfloor \ell/2 \rfloor} \geq k$. Since $\ell \leq \log k + \log\log k$ for $k \geq 256$ and $\ell \leq 11$ for $k \leq 256$, such subsets A_i's can be chosen. In each $\mathcal{Q}(A_i)$, $i \in \{0, \ldots, k-1\}$, the elements of size $\lfloor \ell/2 \rfloor$ form an antichain of size k. Let $\{A_i^j : j \in \{0, \ldots, k-1\}\}$ be this antichain enumerated arbitrarily.

Consider the factorial tree $\mathcal{O}(\mathcal{Y})$. We shall construct an embedding τ of $\mathcal{O}(\mathcal{Y})$ into Q as follows. Let $\tau(\varnothing) = \varnothing$. Consider any non-empty ordered subset of $\mathcal{Y}$, say $(y_{i_1}, y_{i_2}, \ldots, y_{i_j})$, $1 \leq j \leq k$. If $j = 1$, let $\tau((y_{i_1})) = A_{i_1} \cup \{y_{i_1}\}$. If $j > 1$, let

$$\tau\big((y_{i_1}, \ldots, y_{i_j})\big) = A_{i_1} \cup A_{i_1+1}^{i_2} \cdots \cup A_{i_1+j-1}^{i_j} \cup \big\{y_{i_1}, \ldots, y_{i_j}\big\}.$$

For example for $k = 4$, $\tau((y_0, y_1, y_2)) = A_0 \cup A_1^1 \cup A_2^2 \cup \{y_0, y_1, y_2\}$, $\tau((y_2, y_3, y_1)) = A_2 \cup A_3^3 \cup A_0^1 \cup \{y_1, y_2, y_3\}$, and $\tau((y_3, y_1)) = A_3 \cup A_0^1 \cup \{y_1, y_3\}$.

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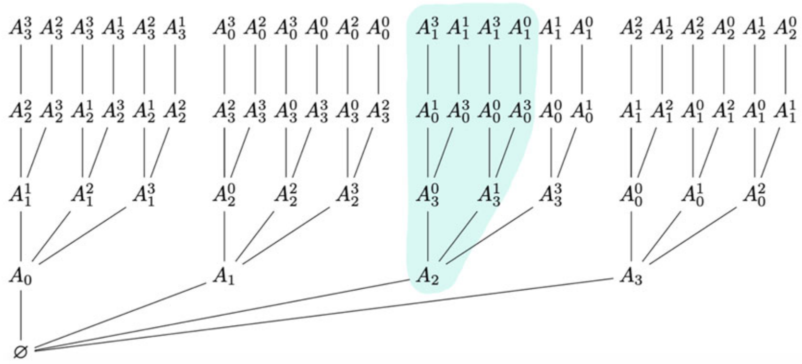

Figure 1. A diagram illustrating how the A_i^j's are being assigned to the elements of the $\{y_0, y_1, y_2, y_3\}$-shrub constructed in Proposition 11.

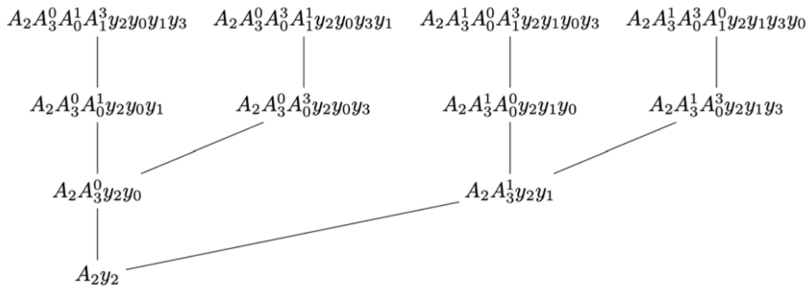

Figure 2. Segment of a shrub highlighted in Figure 1. Here the union signs are omitted because of the spacing, for example $A_2A_3^1A_0^0y_2y_1y_0$ corresponds to the shrub vertex $A_2 \cup A_3^1 \cup A_0^0 \cup \{y_2, y_1, y_0\}$.

Note that the image of $\mathcal{O}(\mathcal{Y})$ under τ is an up-tree, $\mathcal{T}$, whose minimum vertex is $\varnothing$, see Figures 1 and 2. We see that each maximum vertex of $\mathcal{T}$ is joined to $\varnothing$ by a unique chain, a maximal chain. Furthermore, non-zero vertices that belong to distinct maximal chains are incomparable. Observe that τ is a $\mathcal{Y}$-good embedding of $\mathcal{O}(\mathcal{Y})$ into Q. Indeed, for any ordered sequence of distinct vertices $(y_{i_1}, \ldots, y_{i_j})$, we have $\tau((y_{i_1}, \ldots, y_{i_j})) \cap \mathcal{Y} = \{y_{i_1}, \ldots, y_{i_j}\}$. In addition $(y_{i_1}, \ldots, y_{i_q}) <_{\mathcal{O}} (y_{i_1}, \ldots, y_{i_p})$ if and only if $\tau((y_{i_1}, \ldots, y_{i_q}))$ and $\tau((y_{i_1}, \ldots, y_{i_p}))$ are in the same maximal chain of T in the corresponding order. □

3. Duality theorem

In this section, we show a duality statement which is the key argument for the proof of Theorem 3. Recall the following definitions. We call an embedding ϕ of $\mathcal{Q}(\mathcal{X})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ for disjoint $\mathcal{X}$ and $\mathcal{Y}$, *$\mathcal{X}$-good* if $\phi(X) \cap \mathcal{X} = X$ for all $X \subseteq \mathcal{X}$. We also call a copy of $\mathcal{Q}(\mathcal{X})$ in $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ *$\mathcal{X}$-good* if there is an $\mathcal{X}$-good embedding of $\mathcal{Q}(\mathcal{X})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$. An embedding τ of $\mathcal{O}(\mathcal{Y})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ is *$\mathcal{Y}$-good* if $\tau(S) \cap \mathcal{Y} = \underline{S}$ for all $S \in \mathcal{O}(\mathcal{Y})$. We say that a copy of $\mathcal{O}(\mathcal{Y})$ is a *$\mathcal{Y}$-shrub* if there exists a $\mathcal{Y}$-good embedding of $\mathcal{O}(\mathcal{Y})$ into $\mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$.

Theorem 12. *(Duality Theorem) For two disjoint sets $\mathcal{X}$ and $\mathcal{Y}$, let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ be a blue/red coloured Boolean lattice which contains no blue copy of Λ. Then there is exactly one of the following in Q:*

- *a red $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$, or*
- *a blue $\mathcal{Y}$-good copy of $\mathcal{O}(\mathcal{Y})$, that is a blue $\mathcal{Y}$-shrub.*

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Informally speaking, this duality statement claims that for any bipartition $\mathcal{X} \cup \mathcal{Y}$ of the ground set of a Boolean lattice there exists either a red copy of $\mathcal{Q}(\mathcal{X})$ that is restricted to $\mathcal{X}$ or a blue copy of the factorial tree $\mathcal{O}(\mathcal{Y})$ restricted to $\mathcal{Y}$. This result can be seen as a strengthening of Lemma 8 in the special case when we forbid a blue copy of Λ. The Duality Theorem implies a criterion for blue/red coloured Boolean lattices Q to have neither a blue copy of Λ nor a red copy of Q_n.

Corollary 13. *Let $n, k \in \mathbb{N}$ and $N = n + k$. Let $Q = \mathcal{Q}([N])$ be a blue/red coloured Boolean lattice with no blue copy of Λ. There is no red copy of Q_n in Q if and only if for every $\mathcal{Y} \in \binom{[N]}{k}$ there exists a blue $\mathcal{Y}$-shrub in Q.*

Proof. Lemma 7 provides that there is a red copy of Q_n in Q if and only if there exists a partition $[N] = \mathcal{X} \cup \mathcal{Y}$ of the ground set of Q with $|\mathcal{X}| = n$ and $|\mathcal{Y}| = k$ as well as an $\mathcal{X}$-good embedding ϕ of $Q(\mathcal{X})$ into Q with a monochromatic red image.

If there is a red copy of Q_n in Q, then for $\mathcal{X}, \mathcal{Y}$ from Lemma 7 there is also an $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$. Thus by Theorem 12 there is no blue $\mathcal{Y}$-shrub.

On the other hand, if there is no red copy of Q_n in Q, there is no red $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$ for any $\mathcal{X} \in \binom{[N]}{n}$. Then for an arbitrary n-element subset $\mathcal{X}$ of $[N]$, let $\mathcal{Y} = [N] \setminus \mathcal{X}$. Now Theorem 12 implies that there exists a blue $\mathcal{Y}$-shrub. In particular, there is a blue $\mathcal{Y}$-shrub for any k-element subset $\mathcal{Y}$ of $[N]$. □

Throughout the section, let $\mathcal{X}$ and $\mathcal{Y}$ be fixed disjoint sets. Let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ be a Boolean lattice on ground set $\mathcal{X} \cup \mathcal{Y}$. We fix an arbitrary blue/red colouring of Q with no blue copy of Λ. We always let $n, k \in \mathbb{N}$ such that $|\mathcal{X}| = n$, $|\mathcal{Y}| = k$ and let $N = n + k$. For $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$, we usually denote the vertex $X \cup Y$ by (X, Y).

In order to characterise colourings of Q which do not contain an embedding ϕ of $\mathcal{Q}(\mathcal{X})$ into Q such that for every $X \in \mathcal{Q}(\mathcal{X})$, $\phi(X)$ is red and $\phi(X) \cap \mathcal{X} = X$, we introduce the following notation.

For $X \subseteq \mathcal{X}$ and $Y \subseteq \mathcal{Y}$, we say that the vertex $(X, Y) \in Q$ is *embeddable* if there is an embedding ϕ:$\mathcal{Q}(\mathcal{X}) \cap \{X' \subseteq \mathcal{X} : X' \supseteq X\} \to Q$ with a monochromatic red image, such that $\phi(X') \cap \mathcal{X} = X'$ for all X' and $\phi(X) \supseteq (X, Y)$. We say that ϕ *witnesses* that (X, Y) is embeddable.

This definition immediately implies:

Observation 14. *$(\varnothing, \varnothing)$ is not embeddable if and only if there is no embedding $\phi : \mathcal{Q}(\mathcal{X}) \to Q$ such that for every $X' \subseteq \mathcal{X}$, $\phi(X')$ red and $\phi(X') \cap \mathcal{X} = X'$.* □

The key ingredient for the proof of the Duality Theorem, Theorem 12, is the following characterisation of embeddable vertices.

Lemma 15. *Let $X \subseteq \mathcal{X}$, $Y \subseteq \mathcal{Y}$. Let $Q = \mathcal{Q}(\mathcal{X} \cup \mathcal{Y})$ be a blue/red coloured Boolean lattice with no blue copy of Λ. Then (X, Y) is embeddable if and only if either*

(i) (X, Y) is blue and there is a $Y' \subseteq \mathcal{Y}$ with $Y' \supset Y$ such that (X, Y') is embeddable, or

(ii) (X, Y) is red and for all $X' \subseteq \mathcal{X}$ with $X' \supset X$, (X', Y) is embeddable.

Note that if $X \subseteq \mathcal{X}$ and $(X, \mathcal{Y})$ is blue, then $(X, \mathcal{Y})$ is not embeddable.

Proof. First suppose that (X, Y) is embeddable. Let ϕ be an embedding of $\mathcal{Q}(\mathcal{X}) \cap \{X' \subseteq \mathcal{X} : X' \supseteq X\}$ into Q witnessing that (X, Y) is embeddable.

If (X, Y) is blue, then $\phi(X) \supset (X, Y)$ because ϕ has a monochromatic red image. Thus there exists $Y' \subseteq \mathcal{Y}$ with $Y' \supset Y$ such that $\phi(X) = (X, Y')$. But then ϕ also witnesses that (X, Y') is embeddable, so Condition (i) is fulfilled.

If (X, Y) is red, pick some arbitrary $X^* \subseteq \mathcal{X}$ such that $X^* \supset X$. Then the function $\phi^* : \mathcal{Q}(\mathcal{X}) \cap \{X' \subseteq \mathcal{X} : X' \supseteq X^*\} \to Q$, $\phi^*(X') = \phi(X')$ is a restriction of ϕ and therefore an embedding with a monochromatic red image such that $\phi^*(X') \cap \mathcal{X} = X'$ for all X' and $\phi^*(X^*) \supseteq (X^*, Y)$. Thus

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for every $X^* \subseteq \mathcal{X}$ with $X^* \supset X$, the vertex (X^*, Y) is embeddable, that is Condition (ii) is fulfilled.

Now, suppose that Condition (i) or Condition (ii) hold. If (i) holds, then (X, Y) is blue and there is some $Y' \supset Y$ such that (X, Y') is embeddable. Then the embedding witnessing that also verifies that (X, Y) is embeddable.

For the rest of the proof we assume that (ii) holds, that is that (X, Y) is red and for any $X' \subseteq \mathcal{X}$ with $X' \supset X$ the vertex (X', Y) is embeddable. We define the required embedding $\phi : \mathcal{Q}(\mathcal{X}) \cap \{X' \subseteq \mathcal{X} : X' \supseteq X\} \to Q$ for every X' with $X \subseteq X' \subseteq \mathcal{X}$ depending on the number of minimal X^*'s, $X \subseteq X^* \subseteq X'$ such that (X^*, Y) is blue as follows. Let X' with $X \subseteq X' \subseteq \mathcal{X}$ be arbitrary.

(1) If for all X^* with $X \subseteq X^* \subseteq X'$, the vertex (X^*, Y) is red, let $\phi(X') = (X', Y)$. Note that this case includes $X' = X$.

(2) If there is a unique minimal X^* such that $X \subseteq X^* \subseteq X'$ and (X^*, Y) is blue, then (X^*, Y) is embeddable by Condition (ii). Let ϕ_{X^*} be an embedding witnessing that. Then set $\phi(X') = \phi_{X^*}(X')$.

(3) Otherwise, let $\phi(X') = (X', \mathcal{Y})$.

Cases (1)-(3) determine a partition of the set $\{X' \subseteq \mathcal{X} : X' \supseteq X\}$ into three pairwise disjoint parts. Let $\mathcal{M}_j, j \in [3]$, be the set of those vertices X' for which ϕ was assigned in Case (j). Note that $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 = \{X' \subseteq \mathcal{X} : X' \supseteq X\}$.

Claim 1. The function ϕ witnesses that (X, Y) is embeddable.

- Clearly, for every $X' \subseteq \mathcal{X}$ with $X' \supseteq X$, $\phi(X') \cap \mathcal{X} = X'$.
- (X, Y) is red, so $X \in \mathcal{M}_1$. Thus $\phi(X) = (X, Y)$.
- The argument verifying that $\phi(X')$ is red for every $X' \subseteq \mathcal{X}$ with $X' \supseteq X$ depends on i such that $X' \in \mathcal{M}_i$. If $X' \in \mathcal{M}_1$, it is immediate that $\phi(X')$ is red. If $X' \in \mathcal{M}_2$, ϕ_{X^*} has a monochromatic red image, thus $\phi(X') = \phi_{X^*}(X')$ is also red. Now consider the case that $X' \in \mathcal{M}_3$, that is there are $X_1, X_2 \subseteq \mathcal{X}$ with $X_1 \neq X_2$ and $X \subseteq X_i \subseteq X'$, $i \in [2]$, such that (X_i, Y) are blue and X_i are both minimal with this property. The latter condition implies that X_1 and X_2 are incomparable, in particular (X_1, Y) and (X_2, Y) are incomparable as well. Moreover, observe that $X_i \neq X'$, $i \in [2]$, because X' is by definition comparable to both X_1 and X_2. Now assume for a contradiction that $\phi(X') = (X', \mathcal{Y})$ is blue. Recall that (X_1, Y) and (X_2, Y) are blue. We know that $X_i \subset X'$ and $Y \subseteq \mathcal{Y}$, thus $(X_i, Y) \subset (X', \mathcal{Y})$. As a consequence, (X_1, Y), (X_2, Y), and $(X', \mathcal{Y})$ induce a blue copy of Λ in Q, which is a contradiction. Thus ϕ has a monochromatic red image.
- It remains to show that ϕ is an embedding. Let $X_1, X_2 \subseteq \mathcal{X}$ be arbitrary with $X \subseteq X_i \subseteq X'$, $i \in [2]$. We shall show that $X_1 \subseteq X_2$ if and only if $\phi(X_1) \subseteq \phi(X_2)$. One direction is easy to prove: If $\phi(X_1) \subseteq \phi(X_2)$, then $X_1 = \phi(X_1) \cap \mathcal{X} \subseteq \phi(X_2) \cap \mathcal{X} = X_2$. Now suppose that $X_1 \subseteq X_2$. Let $Y_1, Y_2 \subseteq \mathcal{Y}$ such that $\phi(X_1) = (X_1, Y_1)$ and $\phi(X_2) = (X_2, Y_2)$. Then we shall show that $Y_1 \subseteq Y_2$. Note that $Y \subseteq Y_i \subseteq \mathcal{Y}$ for $i \in [2]$. Assume that at least one of X_1 or X_2 is in $\mathcal{M}_1 \cup \mathcal{M}_3$. If $X_1 \in \mathcal{M}_1$, then $Y_1 = Y$ and we are done as $Y \subseteq Y_2$. Furthermore, if $X_2 \in \mathcal{M}_3$, then $Y_2 = \mathcal{Y}$ and we are done as well since $Y_1 \subseteq \mathcal{Y}$. If $X_1 \in \mathcal{M}_3$, then $X_1 \subseteq X_2$ implies that X_2 is also in $\mathcal{M}_3$. Conversely, if $X_2 \in \mathcal{M}_1$, the fact that $X_2 \supseteq X_1$ yields that $X_1 \in \mathcal{M}_1$ and we are done as before.

  As a final step, suppose that $X_1, X_2 \in \mathcal{M}_2$. This implies that for each $i \in [2]$, there is a unique minimal vertex X_i^* such that $X \subseteq X_i^* \subseteq X_i$ and (X_i^*, Y) is blue. Applying the initial assumption, $X_1^* \subseteq X_1 \subseteq X_2$. By minimality of X_2^*, we obtain that $X_2^* \subseteq X_1^*$. Now this provides that $X_2^* \subseteq X_1^* \subseteq X_1$. Using the minimality of X_1^*, we see that $X_1^* \subseteq X_2^*$, so $X_1^* = X_2^*$.

  Recall that $(X_1^*, Y) = (X_2^*, Y)$ is embeddable since $X_1, X_2 \in \mathcal{M}_2$. Consider the function

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$\phi_{X_1^*} = \phi_{X_2^*}$ witnessing that. Because $\phi_{X_1^*}$ is an embedding and $X_1^* \subseteq X_1 \subseteq X_2$, we obtain $\phi_{X_1^*}(X_1) \subseteq \phi_{X_1^*}(X_2)$. Combining the given conditions,

$$\phi(X_1) = \phi_{X_1^*}(X_1) \subseteq \phi_{X_1^*}(X_2) = \phi_{X_2^*}(X_2) = \phi(X_2),$$

that implies that $Y_1 \subseteq Y_2$.

This concludes the proof of the Claim and the Lemma.

Corollary 16. *Let $X \subseteq \mathcal{X}$ and $S \in \mathcal{O}(\mathcal{Y})$ such that $(X, \underline{S})$ is not embeddable. Then there exists some $X' \subseteq \mathcal{X}$, $X' \supseteq X$, such that $(X', \underline{S})$ is blue and not embeddable.*

Proof. If $(X, \underline{S})$ is blue, we are done. Otherwise Lemma 15 yields an $X_1 \subseteq \mathcal{X}$, $X_1 \supset X$ such that $(X_1, \underline{S})$ is not embeddable. By repeating this argument, we find an $X' \subseteq \mathcal{X}$, $X' \supseteq X$, with $(X', \underline{S})$ is blue and not embeddable. □

Next we show a connection between embeddable vertices and the existence of a weak $\mathcal{Y}$-shrub. Recall that a weak $\mathcal{Y}$-shrub is a subposet $\mathcal{P}$ of Q such that there is a function $\tau{:}\mathcal{O}(\mathcal{Y}) \to Q$ with image $\mathcal{P}$ such that for every $S \in \mathcal{O}(\mathcal{Y})$, $\tau(S) \cap \mathcal{Y} = \underline{S}$, and for every $S, T \in \mathcal{O}(\mathcal{Y})$ with $S <_{\mathcal{O}} T$, $\tau(S) \subset \tau(T)$.

Lemma 17. *If $(\varnothing, \varnothing)$ is not embeddable, then there is a monochromatic blue weak $\mathcal{Y}$-shrub.*

Proof. We construct $\tau : \mathcal{O}(\mathcal{Y}) \to Q$ iteratively and increasingly with respect to the order of $\mathcal{O}(\mathcal{Y})$. Suppose that $(\varnothing, \varnothing)$ is not embeddable. By Corollary 16 there is some $X_\varnothing \subseteq \mathcal{X}$ such that $(X_\varnothing, \varnothing)$ is blue and not embeddable. Let $\tau(\varnothing) = (X_\varnothing, \varnothing)$. From here, we continue iteratively. Suppose that for $S \in \mathcal{O}(\mathcal{Y})$, $\underline{S} \neq \mathcal{Y}$, we have defined $X_S \subseteq \mathcal{X}$ such that

(1) $X_S \supseteq X_T$ for every $T \leq_{\mathcal{O}} S$ and
(2) $\tau(S) = (X_S, \underline{S})$ is blue and not embeddable.

Consider an arbitrary $S' \in \mathcal{O}(\mathcal{Y})$ such that $S <_{\mathcal{O}} S'$ and $|S'| = |S| + 1$. By Lemma 15 applied for X_S and $\underline{S}$, we obtain that $(X_S, \underline{S'})$ is not embeddable. Then Corollary 16 yields that there is some $X_{S'} \subseteq \mathcal{X}$, $X_{S'} \supseteq X_S$, such that $(X_{S'}, \underline{S'})$ is blue and not embeddable. Observe that for $T \in \mathcal{O}(\mathcal{Y})$ with $T \leq_{\mathcal{O}} S'$, either $T = S'$ and so $X_T = X_{S'}$, or $T \leq_{\mathcal{O}} S$ and so by (1) $X_T \subseteq X_S \subseteq X_{S'}$. Let $\tau(S') = (X_{S'}, \underline{S'})$.

Using this procedure, we define τ for all $S \in \mathcal{O}(\mathcal{Y})$. Let $\mathcal{P}$ be the subposet of Q induced by the image of τ. We shall show that $\mathcal{P}$ is a weak $\mathcal{Y}$-shrub witnessed by the function τ. By (2), for every $S \in \mathcal{O}(\mathcal{Y})$, $\tau(S)$ is blue and $\tau(S) \cap \mathcal{Y} = \underline{S}$.

Let $S, T \in \mathcal{O}(\mathcal{Y})$ with $S <_{\mathcal{O}} T$. Let $X_S, X_T \subseteq \mathcal{X}$ such that $\tau(S) = (X_S, \underline{S})$ and $\tau(T) = (X_T, \underline{T})$. Clearly, $\underline{S} \subset \underline{T}$. Moreover, item (1) implies that $X_S \subseteq X_T$. Consequently, $\tau(S) \subset \tau(T)$. □

Combining the previously presented Lemmas, we can now prove the Duality Theorem.

Proof of Theorem 12. Let $\mathcal{X}$ and $\mathcal{Y}$ be disjoint sets. Let $Q = Q(\mathcal{X} \cup \mathcal{Y})$ be a blue/red coloured Boolean lattice which contains no blue copy of Λ.

First suppose that there is no red $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$. By Observation 14, $(\varnothing, \varnothing)$ is not embeddable and Lemma 17 provides that there is a blue weak $\mathcal{Y}$-shrub in Q. Using Proposition 10 we obtain a blue $\mathcal{Y}$-shrub in Q. This shows that there is either a red $\mathcal{X}$-good copy of $\mathcal{Q}(\mathcal{X})$ or a blue $\mathcal{Y}$-shrub.

Next we show that both events could not happen simultaneously. Let $n = |\mathcal{X}|$, $k = |\mathcal{Y}|$ and $N = n + k$. Assume that there exist both an $\mathcal{X}$-good embedding $\phi : \mathcal{Q}(\mathcal{X}) \to Q$ with monochromatic red image as well as a $\mathcal{Y}$-good embedding $\tau : \mathcal{O}(\mathcal{Y}) \to Q$ with a monochromatic blue image.

We apply an iterative argument in order to find a contradiction. Let $Y_0 = \varnothing$ and let $S_0 = (Y_0, \leq)$ be the empty ordered set. Now let $X_1 \subseteq \mathcal{X}$ such that $\tau(S_0) = (X_1, \underline{S_0})$ and let $Y_1 \in \mathcal{Y}$ such that $\phi(X_1) = (X_1, Y_1)$. Since $\phi(X_1)$ is red but $\tau(S_0)$ is blue, we know that $\phi(X_1) \neq \tau(S_0)$ and thus $Y_1 \neq \underline{S_0} = \varnothing$, so $|Y_1| \geq 1$.

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Now say that we already defined $X_1, \ldots, X_i, Y_0, \ldots, Y_i, S_0, \ldots, S_{i-1}$ for some $i \in [k]$ such that

- $S_{i-1} \in \mathcal{O}$ and $\underline{S_{i-1}} = Y_{i-1}$,
- $\tau(S_{i-1}) = (X_i, \underline{S_{i-1}})$,
- $\phi(X_i) = (X_i, Y_i)$, and
- $Y_{i-1} \subset Y_i \subseteq \mathcal{Y}$ and $|Y_i| \geq i$.

Fix any ordering S_i of Y_i such that $S_{i-1} <_{\mathcal{O}} S_i$. Such S_i exists because $\underline{S_{i-1}} = Y_{i-1} \subset Y_i$.

Then let X_{i+1} be such that $\tau(S_i) = (X_{i+1}, \underline{S_i})$. Since $S_{i-1} <_{\mathcal{O}} S_i$ and τ is an embedding, $(X_i, \underline{S_{i-1}}) = \tau(S_{i-1}) \subseteq \tau(S_i) = (X_{i+1}, \underline{S_i})$, therefore $X_i \subseteq X_{i+1}$. Note that $\phi(X_i) = (X_i, \underline{S_i})$ is coloured red but $\tau(S_i) = (X_{i+1}, \underline{S_i})$ is blue. Therefore $X_i \neq X_{i+1}$, consequently $X_i \subset X_{i+1}$ and in particular $\phi(X_i) \subset \phi(X_{i+1})$ because ϕ is an embedding.

Next let $Y_{i+1} \subseteq \mathcal{Y}$ such that $\phi(X_{i+1}) = (X_{i+1}, Y_{i+1})$. Then $Y_{i+1} \supseteq Y_i$ and furthermore, because (X_{i+1}, Y_i) is blue but $\phi(X_{i+1})$ is red, $Y_{i+1} \neq Y_i$. Consequently $Y_{i+1} \supset Y_i$, and in particular $|Y_{i+1}| \geq |Y_i| + 1 \geq i + 1$.

Iteratively, we obtain $Y_{k+1} \subseteq \mathcal{Y}$ with $|Y_i| \geq k + 1$, a contradiction to $|\mathcal{Y}| = k$. □

4. Random colouring with many blue shrubs

We shall provide a colouring that will give us a lower bound on $R(\Lambda, Q_n)$. Note that we do not provide an explicit construction but only prove the existence of such a colouring.

Theorem 18. *Let $N \in \mathbb{N}$ be sufficiently large and $k = \frac{10}{216} \frac{N}{\ln(N)}$. Consider the Boolean lattice $Q = \mathcal{Q}([N])$. Then for sufficiently large N, there exists a blue/red colouring of Q which contains no blue copy of Λ and such that for each $\mathcal{Y} \in \binom{[N]}{k}$, there is a blue $\mathcal{Y}$-shrub in Q.*

Proof of Theorem 18. Let $\alpha = 21.6$ and $\beta = 0.134$. Let $N \in \mathbb{N}$ and $k = \frac{1}{\alpha} \frac{N}{\ln(N)}$, let $Q = \mathcal{Q}([N])$.

The idea of the proof is to construct a $\mathcal{Y}$-shrub, denoted $\mathcal{P}_{\mathcal{Y}}$, for every $\mathcal{Y} \in \binom{[N]}{k}$, with an additional property so that the selected shrubs are independent. Since each shrub does not contain a copy of Λ, it follows that the independent union of all the $\mathcal{P}_{\mathcal{Y}}$'s also does not contain a copy of Λ. We obtain these shrubs by randomly choosing a *$\mathcal{Y}$-framework* for every $\mathcal{Y} \in \binom{[N]}{k}$ and then constructing a $\mathcal{Y}$-shrub based on each of them. Afterwards we define a colouring where every vertex in each constructed shrub is coloured blue and the remaining vertices red.

A *$\mathcal{Y}$-framework* of $\mathcal{Y} \in \binom{[N]}{k}$ is a 4-tuple $(\mathcal{Y}, A_{\mathcal{Y}}, Z_{\mathcal{Y}}, X_{\mathcal{Y}})$ such that

- $\mathcal{Y}, A_{\mathcal{Y}}\ Z_{\mathcal{Y}}$ are pairwise disjoint and $\mathcal{Y} \cup A_{\mathcal{Y}} \cup Z_{\mathcal{Y}} = [N]$,
- $|A_{\mathcal{Y}}| = \frac{3}{2} k \ln k - k$,
- $X_{\mathcal{Y}} \subseteq Z_{\mathcal{Y}}$.

A $\mathcal{Y}$-framework is *random* if

- $A_{\mathcal{Y}} \in \binom{[N] \setminus \mathcal{Y}}{\frac{3}{2} k \ln k - k}$ is chosen uniformly at random, and
- each element of $Z_{\mathcal{Y}} = [N] \setminus (\mathcal{Y} \cup A_{\mathcal{Y}})$ is included in $X_{\mathcal{Y}}$ independently at random with probability $\frac{1}{2}$.

Now draw a random $\mathcal{Y}$-framework for every $\mathcal{Y} \in \binom{[N]}{k}$. Observe that by choice of k, $k \ln k = \frac{N}{\alpha} \cdot \frac{\ln(N) - \ln(\alpha) - \ln\ln(N)}{\ln(N)}$, so $\frac{20N}{21\alpha} \leq k \ln k \leq \frac{N}{\alpha}$. Since $|Z_{\mathcal{Y}}| = N - \frac{3}{2} k \ln k$, we have $\left(1 - \frac{3}{2\alpha}\right) N \leq |Z_{\mathcal{Y}}| \leq \left(1 - \frac{10}{7\alpha}\right) N$.

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Claim 2. W.h.p. for every $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$ with $\mathcal{Y}_1 \neq \mathcal{Y}_2$, $|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \geq \beta N$.

Consider some arbitrary $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$, $\mathcal{Y}_1 \neq \mathcal{Y}_2$. Observe that $\left(1 - \frac{3}{\alpha}\right) N \leq |Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \leq |Z_{\mathcal{Y}}| \leq \left(1 - \frac{10}{7\alpha}\right) N$. In a random $\mathcal{Y}$-framework, each element of $Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}$ is contained in $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}$ independently with probability $\frac{1}{2}$. Consequently, $|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_1}| \sim \text{Bin}\left(|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|, \frac{1}{2}\right)$ and $\mathbb{E}\left(|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_1}|\right) = \frac{1}{2}|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|$. We have $1 - \frac{3}{\alpha} \geq 2\beta$. In addition, $\frac{\left(\frac{1}{2} - \frac{3}{2\alpha} - \beta\right)^2}{1 - \frac{10}{7\alpha}} > \frac{2}{\alpha}$, thus there exist some $\epsilon > 0$ such that $\frac{\left(\frac{1}{2} - \frac{3}{2\alpha} - \beta\right)^2}{1 - \frac{10}{7\alpha}} \geq \epsilon + \frac{2}{\alpha}$. Applying Chernoff's inequality gives

$$\begin{aligned}
\mathbb{P}\left(|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \leq \beta N\right) &= \mathbb{P}\left(|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \leq \frac{|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|}{2} - \left(\frac{|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|}{2} - \beta N\right)\right) \\
&\leq \exp\left(-\frac{\left(\frac{|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|}{2} - \beta N\right)^2}{|Z_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|}\right) \\
&\leq \exp\left(-\frac{\left(\left(\frac{1}{2} - \frac{3}{2\alpha}\right) - \beta\right)^2}{\left(1 - \frac{10}{7\alpha}\right)} \cdot N\right) \\
&\leq \exp\left(-\left(\frac{2}{\alpha} + \epsilon\right) \cdot N\right).
\end{aligned}$$

Let E_1 be the event that for some distinct $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$, $|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \leq \beta N$. Then

$$\begin{aligned}
\mathbb{P}(E_1) &= \binom{N}{k}\left(\binom{N}{k} - 1\right) \mathbb{P}(|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}| \leq \beta N) \\
&\leq N^{2k} \exp\left(-\left(\frac{2}{\alpha} + \epsilon\right) \cdot N\right) \\
&\leq \exp\left(\frac{2N \ln(N)}{\alpha \ln(N)} - \left(\frac{2}{\alpha} + \epsilon\right) \cdot N\right) \\
&= \exp(-\epsilon N) \to 0, \text{ as } N \to \infty.
\end{aligned}$$

This proves Claim 1.

Claim 3. W.h.p. for every $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$ with $\mathcal{Y}_1 \neq \mathcal{Y}_2$, $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \not\subseteq X_{\mathcal{Y}_2}$.

We can suppose that the collection of random frameworks fulfils the property of Claim 1. Let $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$ be such that $\mathcal{Y}_1 \neq \mathcal{Y}_2$. Note that each element of $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}$ is contained in $X_{\mathcal{Y}_2}$ with probability $\frac{1}{2}$. Thus,

$$\mathbb{P}\left(X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \subseteq X_{\mathcal{Y}_2}\right) = \left(\frac{1}{2}\right)^{|X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}|} \leq 2^{-\beta N}.$$

Let E_2 be the event that there exist $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$ with $\mathcal{Y}_1 \neq \mathcal{Y}_2$ such that $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \not\subseteq X_{\mathcal{Y}_2}$. Since $\frac{2}{\alpha} < \ln(2)\beta$, we have

$$\mathbb{P}(E_2) \leq N^{2k}\mathbb{P}\left(X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \subseteq X_{\mathcal{Y}_2}\right) \leq N^{2k} \cdot 2^{-\beta N} = \exp\left(\frac{2}{\alpha}N - \ln(2)\beta N\right) \to 0, \text{ as } N \to \infty.$$

This proves Claim 2.

In particular, there exists a collection of $\mathcal{Y}$-frameworks $(\mathcal{Y}, A_{\mathcal{Y}}, Z_{\mathcal{Y}}, X_{\mathcal{Y}})$, $\mathcal{Y} \in \binom{[N]}{k}$, such that for every $\mathcal{Y}_1, \mathcal{Y}_2 \in \binom{[N]}{k}$ with $\mathcal{Y}_1 \neq \mathcal{Y}_2$, $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \not\subseteq X_{\mathcal{Y}_2}$.

Note that $|A_{\mathcal{Y}}| = \frac{3}{2}k\ln k - k \geq k(\log k + \log\log k)$. Let $\mathcal{P}'_{\mathcal{Y}}$ be a $\mathcal{Y}$-shrub in $\mathcal{Q}(A_{\mathcal{Y}} \cup \mathcal{Y})$ as guaranteed by Lemma 11. Note that $\mathcal{P}_{\mathcal{Y}}$'s are not necessarily independent. Let $\mathcal{P}_{\mathcal{Y}}$ be obtained from $\mathcal{P}'_{\mathcal{Y}}$ by replacing each vertex W of $\mathcal{P}'_{\mathcal{Y}}$ with $W \cup X_{\mathcal{Y}}$. Then $\mathcal{P}_{\mathcal{Y}}$ is a $\mathcal{Y}$-shrub in Q.

Claim 4. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two distinct k-element subsets of $[N]$. Then $\mathcal{P}_{\mathcal{Y}_1}$ and $\mathcal{P}_{\mathcal{Y}_2}$ are independent.

Consider arbitrary elements $U_i \in \mathcal{P}_{\mathcal{Y}_i}$, $i \in [2]$. Recall that $X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2} \not\subseteq X_{\mathcal{Y}_2}$, which implies that there exists some $z \in (X_{\mathcal{Y}_1} \cap Z_{\mathcal{Y}_2}) \backslash X_{\mathcal{Y}_2}$. Note that $z \in U_1$ since $X_{\mathcal{Y}_1} \subseteq U_1$. Moreover $z \notin U_2$ since $z \in Z_{\mathcal{Y}_2} \setminus X_{\mathcal{Y}_2}$ and $(Z_{\mathcal{Y}_2} \setminus X_{\mathcal{Y}_2}) \cap U_2 = \varnothing$. In particular, $z \in U_1 \backslash U_2$. Similarly, there is an element $w \in U_2 \backslash U_1$. Thus $U_1 \not\sim U_2$.

We consider the following colouring $c: Q \to \{\text{blue}, \text{red}\}$. For $X \subseteq [N]$, let

$$c(X) = \begin{cases} \text{blue}, & \text{if } X \in \bigcup_{\mathcal{Y} \in \binom{[N]}{k}} \mathcal{P}_{\mathcal{Y}}, \\ \text{red}, & \text{otherwise.} \end{cases}$$

Note that for every $\mathcal{Y} \in \binom{[N]}{k}$, $\mathcal{P}_{\mathcal{Y}}$ witnesses that there is a blue $\mathcal{Y}$-shrub in Q. Recall that a $\mathcal{Y}$-shrub is an up-tree. Applying Claim 3 the blue subposet of Q is a collection of independent up-trees. Then Lemma 5 provides that the colouring c does not contain a blue copy of Λ.

5. Proof of Theorems 3 and 2

Proof of Theorem 3.
Upper Bound: Let $k = (1+\epsilon)\frac{n}{\log(n)}$ and consider an arbitrary blue/red coloured Boolean lattice Q on ground set $[n+k]$ with no blue copy of Λ. Pick any $\mathcal{Y} \in \binom{[n+k]}{k}$ and assume that there is a blue $\mathcal{Y}$-shrub in Q. Recall that the maximal elements of the $\mathcal{Y}$-shrub form an antichain of size $k!$. Sperner's theorem provides that the largest antichain in Q has size $\binom{n+k}{\lfloor \frac{n+k}{2} \rfloor}$, so $k! \leq \binom{n+k}{\lfloor \frac{n+k}{2} \rfloor} \leq 2^{n+k}$.

We also have that $k! > \left(\frac{k}{e}\right)^k = 2^{k(\log k - \log e)}$. By the choice of k, we obtain for sufficiently large n,

$$k\log k \geq \frac{(1+\epsilon)n}{\log n}\big(\log(n) - \log\log(n)\big) > \left(1 + \frac{\epsilon}{2}\right)n.$$

In particular for sufficiently large n, $k\log k - k\log e > n + k$, a contradiction. Thus Q does not contain a blue $\mathcal{Y}$-shrub for this fixed $\mathcal{Y}$. Then Corollary 13 yields that there is a red copy of Q_n in Q. Consequently, each blue/red coloured Boolean lattice of dimension $n+k$ contains either a blue copy of Λ or a red copy of Q_n.

Lower Bound: Let N sufficiently large, let $k = \frac{10}{216}\frac{N}{\ln(N)}$ and $n = N - k$. Note that $k \leq \frac{N}{2}$, thus $n \leq N \leq 2n$. Let $Q = \mathcal{Q}([N])$. By Theorem 18 there exists a colouring of Q with no blue copy of Λ such that for every $\mathcal{Y} \in \binom{[N]}{k}$, there is a blue $\mathcal{Y}$-shrub. By Corollary 13, there is no red copy of Q_n in this colouring, thus $R(\Lambda, Q_n) \geq N = n + k$. It remains to bound k in terms of n. Indeed,

$$k = \frac{10}{216} \cdot \frac{N}{\ln(N)} \geq \frac{10}{216} \cdot \frac{n}{\ln(2n)} = \frac{10}{216} \cdot \frac{\log(e)n}{\log(2n)} \geq \frac{1}{15} \cdot \frac{n}{\log(n)},$$

which concludes the proof. □

Proof of Theorem 2. The lower bound on $R(P, Q_n)$ for P containing either Λ or V follows from Theorem 3.

Consider now a poset P that contains neither a copy of Λ nor a copy of V. By Corollary 6, P is a union of independent chains. Assume that P has k independent chains on at most ℓ vertices each. Let K be an even integer such that $\binom{K}{K/2} \geq k$. Let $\mathcal{Y}$ be a set of size K and let $\mathcal{X}$ be a set, disjoint

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from $\mathcal{Y}$ of size $n+\ell$. Consider an arbitrary colouring of $\mathcal{Q}(\mathcal{X}\cup\mathcal{Y})$. Assume that there is no red copy of Q_n. We shall show that there is a blue copy of P.

Let $Y_1,\dots,Y_k$ form an antichain in $\mathcal{Q}(\mathcal{Y})$, its existence is guaranteed by Sperner's theorem. Let Q^i be a copy of $\mathcal{Q}(\mathcal{X})$ obtained as an image of an embedding $\phi_i:\mathcal{Q}(\mathcal{X})\to\mathcal{Q}(\mathcal{X}\cup Y_i)$, $\phi_i(X)=X\cup Y_i$ for any $X\subseteq\mathcal{X}$. Consider the blue vertices in Q^i. If there is no blue chain on ℓ vertices in Q^i, Corollary 9 implies the existence of a red copy of Q_n in Q^i, a contradiction. Thus for every $i\in[k]$, there is a blue copy P_i of a chain on ℓ vertices in Q^i. Note that for any $A\in Q^i$, $B\in Q^j$, $i\neq j$, $A\not\sim B$, since $A\cap\mathcal{Y}=Y_i\not\sim Y_j=B\cap\mathcal{Y}$. Thus the P_i's are independent chains on ℓ vertices each. Their union contains a copy of P. This shows that $R(P,Q_n)\le n+K+\ell=n+f(P)$. □

6. Conclusion

In this paper we considered the poset Ramsey number $R(P,Q_n)$ for a fixed poset P and large n. We showed a sharp jump in asymptotic behaviour of this Ramsey number depending on P. For 'simple' posets P, those containing neither a copy of Λ nor a copy of V, the poset Ramsey number of P versus Q_n deviates from the trivial lower bound n by at most an additive constant. As pointed out in the proof of Theorem 2 these 'simple' posets are given by the unions of independent chains. 'Complicated' posets, those that contain a copy of Λ or V, behave differently. In this case, $R(P,Q_n)$ is always notably larger than the trivial lower bound n by at least an additive term $\Omega(n/\log n)$.

The best known upper bound for a fixed poset P and large n is given by $R(P,Q_n)\le C_P\cdot n$, for a constant C_P, as was shown by Lu and Thompson [20]. Here C_P is close to the two-dimension of P, that is the dimension of the smallest Boolean lattice containing a copy of P. However, we believe that the true value of $R(P,Q_n)$ is significantly closer to our lower bound, namely that the difference $R(P,Q_n)-n$ is sublinear in terms of n.

Conjecture 19. *Let $n\in\mathbb{N}$ and P be a fixed poset independent from n. Then*

$$R(P,Q_n)=n+o(n).$$

Note that this conjecture is equivalent to verifying in the classical off-diagonal setting of poset Ramsey numbers whether for all m fixed and n large, it holds that $R(Q_m,Q_n)=n+o(n)$. This is related to a conjecture raised by Lu and Thompson [20] claiming that $R(Q_m,Q_n)=o(n^2)$ for $n\ge m$ with n and m sufficiently large.

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