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METRIZABLE (LF)-SPACES, (DB)-SPACES, AND THE SEPARABLE QUOTIENT PROBLEM

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The existence of metrizable (LF)-spaces was announced by Stephen A. Saxon ("Metrizable generalized (LF)-spaces", 701-46-14), in *Notices Amer. Math. Soc.* 20 (1973), A-143. Elsewhere, the authors have discovered an abundant existence of metrizable and normable (generalized) (LF)-spaces, while observing that an (LF)-space is metrizable if and only if it is Baire-like. Recently, W. Robertson, I. Tweddle and F.E. Yeomans introduced the class of locally convex spaces E having the property

(db) if E is the union of an increasing sequence (E_n) of vector subspaces, then some E_n is dense and barrelled.

They noted that unordered Baire-like implies (db) which in turn implies Baire-like. No distinguishing examples were given. It is noted that the metrizable (LF)-spaces are precisely those (LF)-spaces which distinguish between Baire-like spaces and (db)-spaces, since no (LF)-space is a (db)-space. An easy example of a metrizable (LF)-space is provided. We also show, by means of a simple construction, that every infinite-dimensional Fréchet space has a dense subspace which is (db) but not unordered Baire-like. The interaction between metrizable (LF)-spaces, (db)-spaces and the classical separable quotient problem is discussed.

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In particular, it is proved that every (LF)-space "has a separable quotient", and a Fréchet space "has a separable quotient" if and only if it has a dense non-(db)-subspace.

1. Preliminary considerations

We recall from [8], [13] the following definitions. A locally convex space E is

(1) *Baire* if E is not the union of a sequence of nowhere dense sets;

(2) *unordered Baire-like* if E is not the union of a sequence of nowhere dense, absolutely convex sets;

(3) *Baire-like* if E is not the union of an increasing sequence of nowhere dense, absolutely convex sets;

(4) *quasi-Baire* if E is barrelled, and is not the union of an increasing sequence of nowhere dense subspaces.

The diagram

Baire $\Rightarrow$ unordered Baire-like $\Rightarrow$ Baire-like $\Rightarrow$ quasi-Baire $\Rightarrow$ barrelled

indicates the obvious inclusion relationships among the five classes. Examples exist [7], [8], [13] to show that these classes are distinct. However, if E does not contain (an isomorphic copy of) φ , an $\aleph_0$-dimensional vector space equipped with the strongest locally convex topology, in particular if E is metrizable, then the notions of being barrelled, quasi-Baire and Baire-like are equivalent for E . In [13] it has been proved that E is unordered Baire-like if and only if it has

property (R-R): if E is covered by a sequence of subspaces, at least one of the subspaces is both dense and barrelled.

We will call E a (db)-*space* if it satisfies

property (R-T-Y): if E is covered by an increasing sequence of subspaces, at least one of the subspaces is (and hence almost all of them are) both dense and barrelled.

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These spaces were introduced by Robertson, Tweddle and Yeomans in [6] with the observation that

$$\text{unordered Baire-like} \Rightarrow \text{(db)} \Rightarrow \text{Baire-like.}$$

Techniques of [13] show that (db)-spaces enjoy the usual permanence properties; that is, arbitrary products, quotients and countable-codimensional subspaces of (db)-spaces are (db)-spaces. (It has recently been shown by Arias de Reyna [1] that, assuming Martin's Axiom, every separable Banach space has a dense 1-codimensional subspace which is not Baire.)

The definitions of quasi-Baire and (db)-spaces naturally bring to mind that of (LF)-spaces: a locally convex space (E, τ) is an (LF)-*space* [(LB)-*space*] if there is a strictly increasing sequence (E_n, τ_n) of Fréchet [Banach] spaces (called a *defining sequence*) such that

$$E = \bigcup_{n=1}^{\infty} E_n , \quad \tau_{n+1}|_{E_n} \lesssim \tau_n$$

and τ is the strongest locally convex Hausdorff topology such that $\tau|_{E_n} \lesssim \tau_n$ for each n.

NOTE. "$\lesssim$" means "weaker (coarser) than" and $\tau|_G$ denotes the relativization of the topology τ to the subspace G. If $\tau_{n+1}|_{E_n} = \tau_n$ for each n, then $\tau|_{E_n} = \tau_n$ for each n, and (E, τ) is then called a *strict* (LF)-*space*.

2. Distinguishing between Baire-like and (db)-spaces

It is well-known that all (LF)-spaces are barrelled, and we make the observation that no (LF)-space is a (db)-space. For no E_n can be both dense and barrelled in (E, τ) by Pták's open mapping theorem ([4], Proposition 2, p. 299) applied to the identity map from (E_n, τ_n) onto $(E_n, \tau|_{E_n})$, since $E_n \subsetneqq E_{n+1}$ for each n. By a similar argument, no (LB)-space is Baire-like: consider an increasing sequence of multiples of the unit balls of $\{E_n\}_{n=1}^{\infty}$. No strict (LF)-space is quasi-Baire, since in

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the above definition, each E_n is a closed proper subspace of E .

In [11] we partitioned the class of (LF)-spaces into three mutually disjoint, non-empty classes as follows:

DEFINITION. An (LF)-space (E, τ) is said to be of *type* (i) , or simply an $(\mathrm{LF})_i$*-space*, if it satisfies condition (i) below $(i = 1, 2, 3)$:

(1) (E, τ) has a defining sequence none of whose members is dense in (E, τ) ;

(2) (E, τ) is non-metrizable and has a defining sequence each of whose members is dense in (E, τ) [equivalently, (E, τ) is non-metrizable and has a defining sequence at least one of whose members is dense in (E, τ)];

(3) (E, τ) is metrizable.

In [11] we have shown that the (LF)-space (E, τ) is of:

type (1) if and only if it contains a complemented copy of φ , if and only if it contains a closed $\aleph_0$-codimensional subspace, if and only if it is not quasi-Baire;

type (2) if and only if it contains φ but not φ complemented, if and only if it is quasi-Baire but not Baire-like;

type (3) if and only if it does not contain φ , if and only if it is Baire-like.

Hence we see that:

$(\mathrm{LF})_1$-spaces are precisely those (LF)-spaces which distinguish between barrelled and quasi-Baire spaces;

$(\mathrm{LF})_2$-spaces are precisely those (LF)-spaces which distinguish between quasi-Baire and Baire-like spaces;

$(\mathrm{LF})_3$-spaces are precisely those (LF)-spaces which distinguish between Baire-like and (db)-spaces.

We note that each of these distinguishing classes of (LF)-spaces is indeed rich: every strict (LF)-space is of type (1); every (LB)-space with a defining sequence of dense subspaces, for example l_q- $(q > 1)$ of [11] is of type (2); metrizable (and normable) (LF)-spaces exist in abundance [10], for example, the familiar sequence and function spaces all contain dense (LF)-subspaces. We give here a remarkably simple, concrete

EXAMPLE 1. Let ω denote the Fréchet space of all scalar sequences with the product topology. The Banach space l_1 is a dense vector subspace of ω. For $n = 1, 2, \ldots$, set

$$E_n = \underbrace{\omega \times \omega \times \ldots \times \omega}_{n \text{ factors}} \times l_1 \times l_1 \times \ldots .$$

Then $\{E_n\}$ is a strictly increasing sequence of Fréchet spaces, and E_{n+1} induces a topology on E_n coarser than its product topology. One easily sees that $E = \bigcup_{n=1}^{\infty} E_n$ is a dense subspace of the Fréchet space $F = \omega \times \omega \times \ldots$ which, with the relative topology, is a (metrizable) (LF)-space. Thus ω contains a dense (LF)-subspace, since ω is isomorphic to F.

Thus many (perhaps all) infinite-dimensional Fréchet spaces contain dense subspaces which are Baire-like but not (db) (*cf.* [10]). We characterize such subspaces in the following

THEOREM 1. *Let N be a dense, Baire-like (equivalently, barrelled) subspace of a Fréchet space (E, τ). N is not a* (db)*-space if and only if there exists a subspace M of F such that $M \supset N$ and M with a topology stronger than the relative topology is an* (LF)*-space.*

Proof. Suppose M exists as above. Then there exists an increasing sequence $\{(F_n, \tau_n)\}$ of Fréchet spaces whose union is M, and such that $\tau|_{F_n} \lesssim \tau_n$, for each n. Suppose that for some n, F_n is both dense and barrelled in $(M, \tau|_M)$. Then Pták's open mapping theorem applies to the identity map from (F_n, τ_n) onto $(F_n, \tau|_{F_n})$, forcing the conclusion that F_n is a dense, complete subspace of $(M, \tau|_M)$, contradicting the fact that $F_n \subsetneq M$, since $F_n \subsetneq F_{n+1} \subset M$. Thus $(M, \tau|_M)$ is not a (db)-space. Hence M cannot contain a dense subspace which is a (db)-space.

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Specifically, N is not a (db)-space.

Conversely, suppose N is a dense, barrelled, non-(db)-subspace of F. Let $\{G_n\}$ be an increasing sequence of subspaces of N such that $\bigcup_{n=1}^{\infty} G_n = N$, and such that no G_n is both dense and barrelled in N. Thus for each n, we can choose a barrel B_n in G_n whose closure $\overline{B}_n$ in F is not a neighbourhood of 0. Let H_n be the linear span $\mathrm{sp}(\overline{B}_n)$ of $\overline{B}_n$ and let η_n be the topology on H_n which has as a base of neighbourhoods of 0 the set $\left\{k^{-1}\overline{B}_n \cap V_k\right\}_{k=1}^{\infty}$, where $\{V_k\}_{k=1}^{\infty}$ is a countable base of closed neighbourhoods of 0 in (F, τ). Thus for each n, (H_n, η_n) is a Fréchet space (by Proposition 5, [4], p. 207). Let $F_k = \bigcap_{n=k}^{\infty} H_n$, and endow F_k with the locally convex topology τ_k which is the supremum of the relativizations of the η_n's $(n \geq k)$, so that (by Lemma 2 of [10]) (F_k, τ_k) is a Fréchet space for $k = 1, 2, \ldots$. Clearly, $F_k \subset F_{k+1}$ and $\tau_{k+1}|_{F_k} \leq \tau_k$ for each k. If $H_n \supset N$, then $\overline{B}_n \cap N$ would be a neighbourhood of 0 in the barrelled space N, and hence $\overline{B}_n$ would be a neighbourhood of 0 in F, since N is dense in F, a contradiction. Thus $H_n \not\supset N$ for all n, so that

(*) $$F_k \not\supset N \text{ for all } k .$$

Now, $F_k = \bigcap_{n \geq k} H_n \supset \bigcap_{n \geq k} \mathrm{sp}(B_n) = \bigcap_{n \geq k} G_n = G_k$, so that

(**) $$\bigcup_{k=1}^{\infty} F_k \supset N .$$

By (*) and (**), there exists a subsequence $\{F_{k_j}\}_{j=1}^{\infty}$ of $\{F_k\}_{k=1}^{\infty}$ which is strictly increasing so that $M = \bigcup_{j=1}^{\infty} F_{k_j} = \bigcup_{k=1}^{\infty} F_k \supset N$, and M with a topology stronger than $\tau|_M$ is an (LF)-space. $[\tau|_{F_{k_j}} \leq \tau_{k_j}$ for all j .$]$

COROLLARY. *A Fréchet space F contains a dense subspace which is Baire-like (equivalently, barrelled) but not* (db) *if and only if it contains a dense barrelled subspace which, with a topology stronger than the relative topology, is an* (LF)*-space.*

Proof. If $N \subset M \subset F$ and N is dense and barrelled in F , so is M .

EXAMPLE 2. Let E be any Banach space with a dense subspace M which is an (LF)-space (*cf.* [*10*]). Let x be in E , not in M , and choose a linear functional f such that $f(M) = \{0\}$ and $f(x) = 1$. Choose g in E' such that $g(x) = 1$, and set $S = f^{\perp}$, $H = g^{\perp}$. By Example 3.4 of [*12*], the norm $\|\cdot\|_B$ defined on S in terms of the original norm $\|\cdot\|$ by

$$\|y\|_B = \|y-g(y)\cdot x\| \quad (y \text{ in } S)$$

generates a complete topology (strictly) weaker than the (incomplete) relative topology. Note that $\|y\|_B = \|y\|$ for all y in $S \cap H$. Thus $M \cap H$ is a 1-codimensional subspace of M on which both topologies coincide. Hence $M \cap H$ is barrelled [*9*], and since any 1-dimensional extension of a barrelled space is barrelled, we have that M , under both topologies, is barrelled. Note that g is not $\|\cdot\|_B$-continuous on M . Hence we see that M *is a dense, barrelled subspace of the Banach space* $(S, \|\cdot\|_B)$ *which, with a topology strictly stronger than the relative topology, is an* (LF)*-space.* [This example shows that a continuous linear map from a metrizable (LF)-space onto a metrizable, barrelled (and hence Baire-like) space need not be open.]

3. The separable quotient problem

We see from the above example that in Theorem 1 and its corollary, the phrase "with a topology stronger than the relative topology" cannot, *a priori*, be omitted. Nevertheless, it may be that every infinite-dimensional Fréchet space has a dense (LF)-subspace. If this should be the case, the phrase *can* be omitted in the corollary; in fact, every infinite-

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dimensional Fréchet space would then contain a dense subspace which is Baire-like but not (db), yielding, *via* the following Theorem 2, an affirmative solution to the classical separable quotient problem: *Does every infinite-dimensional Fréchet space* F *have a closed subspace* M *such that the Fréchet space* F/M *is infinite-dimensional and separable*? Conversely, in [*10*] it is shown that if the splitting and separable quotient problems have affirmative solutions, then every infinite-dimensional Fréchet space does indeed have a dense (LF)-subspace.

We raise the open

QUESTION. *For each Fréchet space* F, *is it true that* F *has a dense, Baire-like (equivalently, barrelled), non-*(db)*-subspace if and only if* F *has a dense* (LF)*-subspace?*

Further discoveries of the intimate interaction among (db)-spaces, metrizable (LF)-spaces and the separable quotient problem are found below.

THEOREM 2. *Let* F *be a Fréchet space. The following statements are equivalent:*

(i) F *has a separable (infinite-dimensional) quotient (by a closed subspace);*

(ii) F *has a dense, non-barrelled subspace;*

(iii) F *has a dense, non-*(db)*-subspace;*

(iv) (Bennett and Kalton [*2*]) F *has a dense, proper subspace which, with a topology stronger than the relative topology, is a Fréchet space;*

(v) F *has a dense subspace which, with a topology stronger than the relative topology, is a metrizable* (LF)*-space.*

Proof. The equivalence of statements *(i)*, *(ii)*, *(iv)*, *(v)* and several others has been established in [*2*], [*10*] and [*12*]. We prove that *(ii)* is equivalent to *(iii)*. Trivialy, *(ii)* implies *(iii)*.

Conversely, suppose every dense subspace of F is barrelled. Let M be a dense subspace which is the union of an increasing sequence of subspaces $\{M_n\}_{n=1}^{\infty}$. Since M is metrizable and barrelled, hence quasi-Baire, some M_n is dense in M, thus dense in F, thus barrelled as well. That is, M is a (db)-space. Hence the contrapositive of [*(iii)* implies *(ii)*] is valid, and *(i)*-*(v)* are equivalent.

Let us write $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ to denote that (E, τ) is an (LF)-space with defining sequence $(E_n, \tau_n)_{n=1}^{\infty}$. Using §19, 4.(1), p. 222 of [5], the Open Mapping Theorem, and the facts that $\tau|_{E_n} \leq \tau_n$ for each n and τ is Hausdorff, one easily proves that *the following statements are equivalent*:

(1) $\tau_{n+1}|_{E_n} = \tau_n$ *for each* n ;

(2) $\tau|_{E_n} = \tau_n$ *for each* n ;

(3) E_n *is* τ_{n+1}*-closed in* E_{n+1} *for each* n ;

(4) E_n *is* τ*-closed in* E *for each* n ;

(5) [H. Jarchow] $(E_n, \tau|_{E_n})$ *is sequentially complete for each* n .

Modulo isomorphisms, φ is the only strict (LF)-space for which every defining sequence is strict; that is, satisfies the equivalent conditions (1)-(5). And only a space of the form $F \times \varphi$ has all of its defining sequences "almost strict", for we prove in [11] that, *for an* (LF)*-space* (E, τ) , *the following statements are equivalent*:

(a) *given any defining sequence* $(F_n, \eta_n)_{n=1}^{\infty}$ *for* (E, τ) , $\eta_{n+1}|_{F_n} = \eta_n$ *for almost all* n ;

(b) *there exists a defining sequence* $(E_n, \tau_n)_{n=1}^{\infty}$ *for* (E, τ) *such that* E_{n+1}/E_n *is finite-dimensional for each* n ;

(c) (E, τ) *is isomorphic to* $F \times \varphi$ *for some Fréchet space* F .

Now suppose that $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ is an arbitrary (LF)-space. For $n = 1, 2, \ldots$, choose x_n in $E_{n+1} \backslash E_n$, and let F_n be the Fréchet space $E_n \oplus \operatorname{sp}(x_n)$. Clearly, F_{n-1} is a vector subspace of E_n , thus non-dense in the Fréchet space F_n , and $(E, \tau) = \varinjlim_n F_n$. That is, *every* (LF)*-space, even one of type* (2) *or* (3)*, has a defining sequence of Fréchet spaces none of which is dense in the succeeding one.* On the other hand, suppose $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ and E_{n-1} is dense in (E_n, τ_n) for each $n \geq 2$, as in the case of our Example 1 and the (LB)-spaces $l_{q^-} = \bigcup_{1 \leq p < q} l_p$ $(q > 1)$ [*10*]. Then every τ-neighborhood of a point in E_n is a τ-neighborhood of a point in E_{n-1} $\left(\tau|_{E_n} \leq \tau_n\right)$, hence a neighborhood of a point in $E_{n-2}, \ldots, E_1$, so that E_1 is dense in $E = \bigcup_{n=1}^{\infty} E_n$, and E is an (LF)-space of type (2) or (3). By the equivalence of *(i)* and *(iv)*, Theorem 2, we have

COROLLARY 1. *Let* $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ *, where* E_{n-1} *is dense in* (E_n, τ_n) *for* $n = 2, 3, \ldots$. *Then* (E, τ) *is of type* (2) *or* (3)*, and each* (E_n, τ_n) *has a separable quotient* $(n \geq 2)$.

COROLLARY 2. *There exists a defining sequence for the* (LF)*-space* (E, τ) *each of whose members has a separable quotient if and only if* (E, τ) *is not isomorphic to* $F \times \varphi$ *, where* F *is any Fréchet space not having a separable quotient.*

NOTE. Since "separable quotients" must be infinite-dimensional, no finite-dimensional spaces can have them.

Proof. If (E, τ) is isomorphic to $F \times \varphi$, where F is a Fréchet space, then by the Equivalence Theorem [*10*], almost all of the members of any defining sequence must be isomorphic to a Fréchet space of the form $F \times L$, where L is finite-dimensional, so that $F \times L$ has a separable quotient if and only if F does. Any subsequence of a defining sequence is one also, and the result follows for E isomorphic to $F \times \varphi$.

If *(c)* of the preceding discussion fails, then by the negation of *(a)*, there exists a defining sequence $(F_n, \eta_n)_{n=1}^{\infty}$ such that $\eta_{n+1}|_{F_n} \neq \eta_n$ for

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infinitely many n. Taking the closure of F_n in (F_{n+1}, η_{n+1}) for all such n yields a defining sequence of Fréchet spaces each of which has a separable quotient, by the Open Mapping Theorem and the equivalence of *(i)* and *(iv)*.

COROLLARY 3. *For every non-strict* (LF)*-space* (*in particular, for every* $(\mathrm{LF})_2$ *or* $(\mathrm{LF})_3$*-space*) *there exists a defining sequence each of whose members has a separable quotient.*

Proof. $F \times \varphi$ is a strict (LF)-space for F a Fréchet space.

While we have no proof that every infinite-dimensional Fréchet space has a separable quotient (unknown even for Banach spaces), we conclude this section by showing that every (LF)-space has a separable quotient.

THEOREM 3. *Let* (E, η) *be a Hausdorff barrelled space, and suppose* $(E, \tau) = \varinjlim_n (E_n, \tau_n)$ *for some topology* τ *with* $\eta \leq \tau$. *Then* (E, η) *has a separable quotient.*

NOTE. Compare with the corollary to Theorem 1, Example 2, and *(v)* of Theorem 2.

Proof. By [11], if (E, η) is not quasi-Baire, it contains a complemented copy of φ, and φ is infinite-dimensional and separable. Therefore let us assume, without loss of generality, that (E, η) is quasi-Baire, so that E_k is dense in (E, η) for some k. By Pták's Open Mapping Theorem, the identity map from (E_k, τ_k) into (E, η) cannot be almost open $(E_k \subsetneq E_{k+1} \subset E)$; that is, there exists an absolutely convex τ_k-neighborhood V of 0 whose closure $\overline{V}$ in (E, η) is not an η-neighborhood of 0. Now $F = \mathrm{sp}(\overline{V})$ is dense in (E, η), so $\overline{V}$ is not an η-relative neighborhood of 0 in F, although it is a barrel in F. By [9], then, F is of uncountable codimension in the barrelled space (E, η). Thus for some j, $F \cap E_j$ is infinite-codimensional in E_j. Let $\{U_n\}_{n=1}^{\infty}$ be a basic sequence of absolutely convex neighborhoods of 0 in (E_j, τ_j) such that $U_n + U_n \subset U_{n-1}$ for $n = 2, 3, \ldots$. Choose x_1 in $U_1 \backslash F$ and f_1 in E', where E' denotes the dual of (E, η), such

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that $f_1(x_1) = 1$ and f_1 is in

$$\overline{V}^0 = \{f \in E' : |f(x)| \leq 1 \text{ for all } x \text{ in } \overline{V}\} .$$

Let $V_1 = \overline{V}$, $V_2 = V_1 + \{ax_1 : |a| \leq 1\}$, and choose x_2 in $U_2 \backslash \mathrm{sp}(V_2)$ such that $f_1(x_2) = 0$, and choose f_2 in V_2^0 such that $f_2(x_2) = 1$. (V_m is absolutely convex and closed, $f_1^{-1}(0) \cap E_j$ is finite-codimensional and $\mathrm{sp}(V_m) \cap E_j$ is infinite-codimensional in E_j , for $m = 1, 2$.) Continue in this fashion to obtain sequences $\{V_n\}$, $\{x_n\}$ and $\{f_n\}$ with $V_1 = \overline{V}$, $V_{n+1} = V_n + \{ax_n : |a| \leq 1\}$, x_n in $U_n \backslash \mathrm{sp}(V_n)$, f_n in V_n^0 , $f_n(x_i) = 1$ if $i = n$, and 0 if $i > n$, for $n = 1, 2, \ldots$. Given y in V_1 , inductively define

$$a_1 = -f_1(y) , \quad a_n = -f_n\left(y + \sum_{i=1}^{n-1} a_i x_i\right) \text{ for } n \geq 2 .$$

Note that $|a_1| \leq 1$, and if $|a_i| \leq 1$ for $1 \leq i \leq p$, then $y + \sum_{i=1}^{p} a_i x_i$ is in V_{p+1} , so that $|a_{p+1}| \leq 1$. Thus $a_i x_i$ is in U_i for each i , and $\sum_{i=1}^{\infty} a_i x_i$ converges (absolutely) to some z in (E_j, τ_j) . Since $\eta|_{E_j} \lesssim \tau|_{E_j} \lesssim \tau_j$, the series is η-convergent to z , and for each n ,

$$f_n(y+z) = f_n\left(y + \sum_{i=1}^{n-1} a_i\right) + a_n f_n(x_n) + \sum_{i=n+1}^{\infty} a_i f_n(x_i) = -a_n + a_n + 0 = 0 .$$

It follows that $y + z$ is in the η-closed subspace $N = \bigcap_{n=1}^{\infty} f_n^{-1}(0)$, and

$$\left\{y + z - \sum_{i=1}^{n} a_i x_i\right\}_{n=1}^{\infty}$$

is a sequence in $N + \mathrm{sp}\left(\{x_i\}_{i=1}^{\infty}\right)$ which is η-convergent to y . Therefore, the η-closure of the subspace $N + \mathrm{sp}\left(\{x_i\}_{i=1}^{\infty}\right)$ contains V_1 , hence F , and hence E ; that is,

$N + \mathrm{sp}\left(\{x_i\}_{i=1}^{\infty}\right)$ is dense in (E, η). We conclude that $\mathrm{sp}\left(\{x_i+N\}_{i=1}^{\infty}\right)$ is dense in the quotient space E/N of (E, η), and clearly, E/N is infinite-dimensional and separable.

COROLLARY. *Every* (LF)*-space has a separable quotient.*

4. Distinguishing between unordered Baire-like and (db)-spaces

We now provide a wide class of metrizable (db)-spaces which are not unordered Baire-like by exploiting Example 1.2 of [7] to obtain

THEOREM 4. *Every infinite-dimensional Fréchet space has a dense subspace which is a* (db)*-space but not an unordered Baire-like space.*

Proof. Let F be an infinite-dimensional Fréchet space, let $\{(x_i, f_i)\}$ be a biorthogonal sequence in $F \times F'$, and let $E = \mathrm{sp}(A)$, where

$A = \{x \in F$: the real and imaginary parts of $f_i(x)$ are rational for each $i\}$.

One easily sees that E is dense in F, and by Theorem 1.1 of [7], E is not unordered Baire-like (nor even "unordered" quasi-Baire).

Let $\{U_n\}$ be a decreasing sequence of absolutely convex basic neighbourhoods of 0 in F such that $U_{n+1} + U_{n+1} \subset U_n$ for each n. Suppose $\{E_n\}$ is an increasing sequence of dense, non-barrelled subspaces whose union is E. Then for $n = 1, 2, \ldots$ there exists $B_n \subset F'$ such that B_n is pointwise bounded on E_n but not on F. Since the Fréchet space F is unordered Baire-like, there exists x in F such that B_n is unbounded on x for all $n = 1, 2, \ldots$. $\left(\{kB_n^0 : k, n = 1, 2, \ldots\}\right.$ is a countable collection of nowhere dense, absolutely convex sets in F.$\left.\right)$ Now, if B_1 is bounded on $x + ax_1$ for some $a \neq 0$, then B_1 is unbounded on x_1 so that B_1 is unbounded on $x + bx_1 = (x+ax_1) + (b-a)x_1$ for all $b \neq a$. Hence there exists a_1 such that $a_1x_1 \in U_1$, $f_1(x+a_1x_1)$ is "rational", and B_1 is unbounded on $x + a_1x_1$. Now there

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exists $g_{1,1}$ in B_1 such that $|g_{1,1}(x+a_1x_1)| > 1$, and there exists $n_2 > n_1 = 1$ such that x_1 is in E_{n_2}. Therefore, B_{n_2} is bounded on x_1, hence unbounded on $x + a_1x_1$: B_{n_j} is unbounded on $x + a_1x_1$ for $j = 1, 2$. Choose $g_{1,2} \in B_1$ and $g_{2,j} \in B_{n_2}$ $(j = 1, 2)$ such that $|g_{i,j}(x+a_1x_1)| > j$ $(1 \leq i, j \leq 2)$ and choose a_2 such that $a_2x_2 \in U_2$, $f_2(x+a_1x_1+a_2x_2) = f_2(x) + a_2$ is "rational" and $|g_{i,j}(x+a_1x_1+a_2x_2)| > j$ $(1 \leq i, j \leq 2)$.

Continuing in this fashion, we obtain a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers, a scalar sequence $\{a_i\}_{i=1}^{\infty}$ and sequences $\{g_{i,j}\}_{j=1}^{\infty} \subset B_{n_i}$ $(i = 1, 2, \ldots)$ such that $\sum_{i=1}^{\infty} a_ix_i$ converges (absolutely) to some y in F, with $x + y \in A$ and $|g_{i,j}(x+y)| \geq j$ for all $i, j = 1, 2, \ldots$. But this implies that $x + y$ is not in $\bigcup_{i=1}^{\infty} E_{n_i} = \bigcup_{n=1}^{\infty} E_n = E \supset A$, a contradiction. We must conclude that no such sequence $\{E_n\}$ exists. Thus E is barrelled (take $E_n = E$ for each n) and therefore, by metrizability, is quasi-Baire, so that if E is the union of an increasing sequence of subspaces, almost all of them must be dense, and, by the above argument, at least one of them must be both dense and barrelled. That is, E is a (db)-space.

REMARK. Theorem 4 and Example 1 not only distinguish between unordered Baire-like, (db) and Baire-like spaces in the class of metrizable spaces, but also in the smallest (non-trivial) variety, namely the variety of all locally convex spaces with their weak topology [3]. Apart from providing a class of Baire-like, non-(db)-spaces, the metrizable (LF)-spaces also constitute a class of incomplete quotients of complete spaces (*cf.* [5], page 225).

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