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# Left Invariant Einstein–Randers Metrics on Compact Lie Groups

Hui Wang and Shaoqiang Deng

*Abstract.* In this paper we study left invariant Einstein–Randers metrics on compact Lie groups. First, we give a method to construct left invariant non-Riemannian Einstein–Randers metrics on a compact Lie group, using the Zermelo navigation data. Then we prove that this gives a complete classification of left invariant Einstein–Randers metrics on compact simple Lie groups with the underlying Riemannian metric naturally reductive. Further, we completely determine the identity component of the group of isometries for this type of metrics on simple groups. Finally, we study some geometric properties of such metrics. In particular, we give the formulae of geodesics and flag curvature of such metrics.

## 1 Introduction

The purpose of this paper is to study left invariant Einstein–Randers metrics on compact Lie groups. Homogeneous Einstein Riemannian manifolds have been studied extensively by many researchers, see [1] for an excellent exposition. However, only very little work has been done on homogeneous Einstein–Finsler metrics. This is mainly due to the fact that Einsetin–Finsler metrics are much more complicated than Einstein Riemannian metrics. For example, up to now, we do not have any useful existence (or non-existence) results on invariant Einstein–Finsler metrics on homogeneous manifolds. In comparison, Einstein–Randers metrics are relatively easier to handle than general Einstein–Finsler metrics. In fact, we have a very convenient criterion for a Randers metric to be an Einstein metric, see Lemma 1.1 and Proposition 1.2 below.

We first recall the definition of Einstein–Finsler metrics; for details we refer to [13]. Let $(M,F)$ be a connected Finsler space, $x \in M$, $y \in T_x(M)\backslash\{0\}$. The Ricci scalar $\mathrm{Ric}(x,y)$ is defined to be the sum of those $n-1$ flag curvatures $K(x,y,e_\nu)$, where $\{e_\nu : 1 \le \nu \le n-1\}$ is any collection of $n-1$ orthonormal transverse edges perpendicular to the flagpole, *i.e.*,

$$\mathrm{Ric}(x,y) := \sum_{\nu=1}^{n-1} R_{\nu\nu}.$$

Its easily seen that this sum is independent of the choice of the specific $n-1$ flags with transverse edges orthogonal to $y$. Note that the Ricci scalar depends on both $x$

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and $y$. A Finsler manifold whose Ricci scalar depends only on $x$ is called an Einstein–Finsler manifold, *i.e.*, $\mathrm{Ric}(x, y) = (n-1)K(x)$ for some function $K(x)$ on M. When $\dim M \geq 3$ and the metric is of the Randers type on $M$ (see [13]), the function $K(x)$ is necessarily a constant.

A Randers metric is built from a Riemannian metric and a 1-form: $F = \alpha + \beta$, where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form whose length with respect to $\alpha$ is everywhere less than 1. There is another presentation of such metrics by the so-called navigation data (see [3]):

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda}, \tag{1.1}$$

where $h$ is a Riemannian metric, $W$ is a vector field on $M$ with $h(W, W) < 1$ and $\lambda = 1 - h(W, W)$. The pair $(h, W)$ is called the navigation data of the Randers metric $F$. This version of Randers metric is convenient when handling Einstein–Randers metrics and constant flag curvature metrics. In fact, we have the following lemma (see [3, 13]).

***Lemma 1.1*** *Suppose $(M, F)$ is a Randers space with the navigation data $(h, W)$. Then $(M, F)$ is Einstein with Ricci scalar $\mathrm{Ric}(x) = (n-1)K(x)$ if and only if there exists a constant $\sigma$ such that*

- *$h$ is Einstein with Ricci scalar $(n-1)(K(x) + \frac{1}{16}\sigma^2)$,*
- *$W$ is an infinitesimal homothety of $h$, namely,*

$$\mathcal{L}_W h = -\sigma h.$$

*Furthermore, $F$ is Riemannian if and only if $W = 0$, and $\sigma$ must be zero whenever $h$ is not flat.*

Based on this result, Deng and Hou obtained a characterization of homogeneous Einstein–Randers metrics (see [6]):

***Proposition 1.2*** *Let $G$ be a connected Lie group and $H$ be a closed subgroup of $G$ such that $G/H$ is a reductive homogeneous space with a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{h} = \mathrm{Lie}\, H$. Suppose $h$ is a $G$-invariant Riemannian metric on $G/H$ and let $W$ be an invariant vector field on $G/H$ generated by an $H$-invariant vector $X$ in $\mathfrak{m}$. Then the Randers metric $F$ with navigation data $(h, W)$ is Einstein with Ricci constant $K$ if and only if $h$ is Einstein with Ricci constant $K$ and $X$ satisfies*

$$h([X, Z_1]_{\mathfrak{m}}, Z_2) + h(Z_1, [X, Z_2]_{\mathfrak{m}}) = 0 \text{ for all } Z_1, Z_2 \in \mathfrak{m}.$$

*In this case, $W$ is necessarily a Killing vector field.*

From the above results, it is easily seen that the Randers metric $F$ described by (1.1) on a Lie group $G$ is a left invariant Einstein metric if and only if $h$ is a left invariant Einstein–Riemannian metric on $M$ and $W$ is a left invariant Killing vector field. Left invariant Einstein Riemannian metrics on Lie groups have been studied extensively by D'Atri and Ziller in [7]. They showed that on any compact connected simple Lie group $G$ there exist two left invariant Einstein–Riemannian metrics, one of them bi-invariant and the other not. Furthermore, both of the metrics are naturally reductive. Much work also has been done on other types of Lie groups. For example, it was shown that on a noncommutative nilpotent Lie group there are no left invariant Einstein–Riemannian metrics [11], and on some solvable Lie groups there exist many Einstein metrics, *e.g.,* the Bergman metrics on bounded homogeneous domains, all of them symmetric naturally reductive [8]. However, on noncompact semisimple groups there are no known left invariant Einstein–Riemannian metrics.

Einstein–Finsler metrics are very important in Finsler geometry. On several occasions, the late Professor S. S. Chern openly asked whether every smooth manifold admits an Einstein–Finsler metric. However, little about this question is known so far. Moreover, Einstein–Finsler manifolds have some remarkable properties, such as the constancy of $S$-curvature (see [2]).

In this paper, we will explicitly describe left invariant Einstein–Randers metrics on compact Lie groups. We first present a method to construct a special class of left invariant Einstein–Randers metrics on a compact Lie group. In the case of compact simple Lie groups, we show that these are all the left invariant Einstein–Randers metrics with the navigation data $(h, W)$ such that $h$ is naturally reductive. In this case, we will also determine the identity component of the group of isometries of the left invariant Einstein-Riemannian metrics and present an isometric classification of such metrics. Finally, we obtain the formulae of geodesics and flag curvature of such metrics.

The arrangement of this paper is as follows. In Section 2, we explicitly describe left invariant Einstein–Randers metrics on a compact Lie group. In Section 3, we present the classification of such metrics on simple compact Lie groups under isometries. Finally, we study some properties about geodesics and curvature in Section 4.

## 2 Left Invariant Einstein–Randers Metrics

In this section we will give a description of a special class of left invariant Einstein–Randers metrics on a compact connected Lie group. We first recall briefly some results about left invariant Einstein-Riemannian metrics on compact Lie groups (for details the reader is referred to [7]). If $G$ is abelian, then any left invariant metric is Einstein and Ricci-flat. Therefore we assume that $G$ is non-abelian. Suppose $\mathfrak{g} = \operatorname{Lie} G$, and $K$ is a connected subgroup of $G$ with $\mathfrak{k} = \operatorname{Lie} K$. Then $\mathfrak{g}$ and $\mathfrak{k}$ are compact Lie algebras. $\mathfrak{k}$ splits into the direct sum of its center and simple ideals: $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, where $\mathfrak{k}_0 = \mathfrak{z}(\mathfrak{k})$ is the center of $\mathfrak{k}$. Suppose $g$ is a bi-invariant Riemannian metric on the compact Lie group $G$ (such a metric does exist, see [11]), and $\mathfrak{a} = \mathfrak{k}^{\perp}$ with respect to $g$. Then the metric

$$h(\cdot\,,\,\cdot) = \alpha g|_{\mathfrak{a}} + g(A\,\cdot\,,\,\cdot)|_{\mathfrak{k}_0} + \alpha_1 g|_{\mathfrak{k}_1} + \cdots + \alpha_r g|_{\mathfrak{k}}$$

is a left invariant naturally reductive metric, where $\alpha, \alpha_1, \dots, \alpha_r \in \mathbb{R}^{+}$, $\alpha_i \neq \alpha$, and $A$ is a symmetric endomorphism on $\mathfrak{k}_0$ with respect to the metric $g$. Note that $h$ can be realized as an invariant Riemannian metric on $G = G \times K/H$, where $H =$

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$\Delta(K) = \{(k,k) \mid k \in K\}$. Moreover, there is an $\mathrm{Ad}(G\times K)$-invariant symmetric non-degenerate bilinear form $Q$ on $\mathfrak{g} \oplus \mathfrak{k}$ whose restriction to $\mathfrak{p} = \mathfrak{h}^{\perp}$ is the given metric $h$ via the isomorphism $\mathfrak{p} \cong T_o(G \times K)/H = T_eG = \mathfrak{g}$, where $\mathfrak{h} = \mathrm{Lie}\, H = \mathrm{Lie}\, \Delta(K)$. The endomorphism $A$ in the above metric can be diagonalized: $AZ_i = \lambda_i Z_i, \lambda_i > 0, i = 1, \dots, s$. Define $\mathfrak{k}_{r+i} = \mathrm{span}\{Z_i\}, \alpha_{r+i} = \lambda_i$. Then the metric above can be written as

$$h(\cdot,\cdot) = \alpha g|_{\mathfrak{a}} + \alpha_1 g|_{\mathfrak{k}_1} + \cdots + \alpha_{r+s} g|_{\mathfrak{k}_{r+s}} \tag{2.1}$$

and $\mathfrak{z}(\mathfrak{k}) = \mathfrak{k}_0 = \mathfrak{k}_{r+1} \oplus \cdots \oplus \mathfrak{k}_{r+s}$. Suppose also that none of the $\alpha_{r+1}, \dots, \alpha_{r+s}$ is equal to $\alpha$. It is known that the metric (2.1) can be Einstein only if $\mathfrak{z}(\mathfrak{g}) = 0$ (see [7]), where $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Without loss of generality, we can assume $G$ is semisimple, since $\mathfrak{g} \neq \mathfrak{z}(\mathfrak{g})$. For simplicity, we let $g = -B$, where $B$ is the Killing form of $\mathfrak{g}$. Define symmetric bilinear forms $A_i$ on $\mathfrak{a}$ as follows:

$$A_i(X,Y) = \mathrm{Tr}_{\mathfrak{a}}(\pi_i\, \mathrm{ad}\, X)(\pi_i\, \mathrm{ad}\, Y),$$

where $\pi_i$ denotes the projections of $\mathfrak{g}$ onto $\mathfrak{k}_i$, $i = 1, \dots, r$. Let $B_i$ denote the Killing form of $\mathfrak{k}_i$. Since $\mathfrak{k}_i$, $i = 1, \dots r$, is simple, we have $B_i = c_i B$ for some $c_i > 0$.

***Lemma 2.1*** ([7]) *Let $G$ be a compact semisimple non-abelian Lie group and let $h$ be a naturally reductive left invariant Riemannian metric defined as in* (2.1)*. Then a necessary condition for $h$ to be Einstein with Einstein constant $\rho$ is that $h|_{\mathfrak{k}_0} = -\alpha_0 B|_{\mathfrak{k}_0}$ for some $\alpha_0 > 0$. In addition, if the metric is normalized such that $\alpha = 1$, then $h$ is Einstein if and only if $h$ satisfies the following conditions:*

$$\alpha_0 = \alpha_i = 4\rho, \quad i = r+1, \dots, r+s,$$

$$(1 - 4\alpha_i^2)c_i + \alpha_i^2 = 4\rho\alpha_i, \quad i = 1, \dots, r,$$

$$\sum_{i=0}^{r} (\alpha_i - 1)A_i|_{\mathfrak{a}} = \frac{1}{2}(1 - 4\rho)B|_{\mathfrak{a}}.$$

*If $h$ is an Einstein metric, then the Einstein constant $\rho$ is greater than* 0*. Moreover, if $G$ is simple, then any left invariant naturally reductive Einstein metric is of the type described above.*

***Remark 1*** For almost all the compact Lie groups, the metric described above exists. Indeed, on every compact simple Lie group except $SO(3)$ or $SU(2)$, there exists at least one such metric that is not bi-invariant (see [7]).

In order to describe Einstein–Randers metrics on the Lie group $G$, we need to study the Killing vector fields on it. Let $V = \{X \in \mathfrak{a} | [X, \mathfrak{k}] = 0\}$.

***Lemma 2.2*** *Suppose $G$ is a compact connected Lie group and $h$ is a left invariant Riemannian metric as above, $\mathfrak{g} = \mathrm{Lie}\, G$. Then the space of left invariant Killing vector fields on $(G, h)$ is $\mathfrak{k}' = V \oplus \mathfrak{k}$.*

**Proof** For $g \in G$, denote $R(g)$ the right translation: $x \mapsto xg$ of $G$. Then $R(G)$ is a Lie group, whose Lie algebra is identified with the Lie algebra of all left invariant vector fields on $G$. Thus $W \in \mathfrak{g}$ is a Killing vector field if and only if $R(\exp(tW))$ are isometries of $(G, h)$ for all $t \in \mathbb{R}$. Or equivalently, $\operatorname{ad} W$ on $\mathfrak{g}$ is skew symmetric with respect to $h$, since $h$ is left invariant.

Recall that $\mathfrak{h} = \operatorname{Lie} \Delta(K)$ is isomorphic to the Lie algebra $\mathfrak{k} = \operatorname{Lie} K$ via the mapping

$$\varphi \colon \mathfrak{h} \longrightarrow \mathfrak{k},$$
$$(X, X) \longmapsto X, \quad X \in \mathfrak{k},$$

and satisfies $[V, \varphi(\mathfrak{h})] = 0$. Since the symmetric non-degenerate bilinear form $Q$ above on $\mathfrak{g} \oplus \mathfrak{k}$ is $\operatorname{Ad}(G \times K)$-invariant, the right translation by $\exp(tW)(W \in V)$ on $G \times K$ is an isometry with respect to $Q$. Since $\exp tW$ commutes with $\Delta(K)$, the above right translation induces a map on $G \times K/\Delta(K)$ . But $h = Q|_{\mathfrak{p}}$, thus the induced map is an isometry on $G \times K/\Delta(K)$ with respect to $h$. By the isomorphism $G \times K/\Delta(K) \cong G$, the induced map is carried into the right translation on $G$ by $\exp(tW)$, which is also an isometry with respect to $h$. This implies that all the elements of $V$ are Killing vector fields with respect to $h$. Since the right translation by elements of $K$ are isometries, all the elements of $\mathfrak{k}$ are left invariant Killing vector fields on $(G, h)$. Therefore, $V \oplus \mathfrak{k} \subseteq \mathfrak{k}'$.

On the other hand, let $W$ be a Killing vector field. Then $\operatorname{ad} W$ is skew symmetric with respect to $h$. Now, $W$ can be written as $W = W_{\mathfrak{a}} + W_{\mathfrak{k}}$, $(W_{\mathfrak{a}} \in \mathfrak{a}, W_{\mathfrak{k}} \in \mathfrak{k})$. Since each element of $\mathfrak{k}$ is a Killing vector filed, we can assume $W = W_{\mathfrak{a}} \in \mathfrak{a}$. Since $W$ is a left invariant Killing vector field, the right translation by $R(\exp(tW))$ on $(G, h)$ is an isometry. Thus $\operatorname{ad} W$ is skew symmetric on $\mathfrak{g}$ with respect to $h$. Now we show that $W \in V$, or equivalently, $[W, \mathfrak{k}] = 0$. Suppose $[W, \mathfrak{k}_i] \neq 0$ for some $i$. Then there exists a $W' \in \mathfrak{a}$, such that $h([W, \mathfrak{k}_i], W') \neq 0$. But $\operatorname{ad} W$ is skew symmetric, hence $h([W, \mathfrak{k}_i], W') = -h([W, W'], \mathfrak{k}_i)$. Then by (2.1),

$$\alpha g([W, \mathfrak{k}_i], W') = -\alpha g([W, W'], \mathfrak{k}_i),$$

since $\operatorname{ad}(\mathfrak{g})$ is skew symmetric with respect to the bi-invariant metric $g$ (see [11]). But the summands $\mathfrak{a}, \mathfrak{k}_0, \mathfrak{k}_1, \dots, \mathfrak{k}_r$ are mutually orthogonal with respect to the bi-invariant metric $g$; again by (2.1) we have

$$\alpha g([W, \mathfrak{k}_i], W') = -\alpha_i g([W, W'], \mathfrak{k}_i),$$

or equivalently,

$$\alpha g([W, W'], \mathfrak{k}_i) = \alpha_i g([W, W'], \mathfrak{k}_i).$$

This implies $\alpha = \alpha_i$. But this contradicts the assumption that $h$ is a metric with $\alpha \neq \alpha_i$. Therefore, $[W, \mathfrak{k}_i] = 0$ for all $i$. Since $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$, $[W, \mathfrak{k}] = 0$, we have $W \in V$. This implies that $\mathfrak{k}' \subseteq V \oplus \mathfrak{k}$. Therefore, we have $\mathfrak{k}' = V \oplus \mathfrak{k}$. ■

Now we consider left invariant Einstein–Randers metrics on the compact Lie group $G$.

***Theorem 2.3*** *Let G be a compact semisimple Lie group. If there exists a left invariant Einstein-Riemanian metric h given by* (2.1) *satisfying the conditions in Lemma* 2.1*,* $W \in \mathfrak{k}'$, $h(W, W) < 1$, *then*

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda}, \tag{2.2}$$

*where* $\lambda = 1 - h(W, W) > 0$ *is a left invariant Einstein–Randers metric on G. Furthermore, all the left invariant Einstein–Randers metrics on the Lie group G with the underlying Riemannian-Einstein metric h in the navigation data can be obtained in this way.*

**Proof** By Proposition 1.2(see also the remarks thereafter) and Lemma 1.1, the metrics given by (2.2) are all Einstein–Randers metrics, with the constant $\sigma = 0$. Now suppose (2.2) is an Einstein–Randers metric on $G$, with navigation data $(h, W)$. Since $(G, h)$ can be viewed as a homogeneous Riemannian manifold $G = G \times K/\Delta K$, where $K$ is a connected subgroup of $G$ as above, by Proposition 1.2, the left invariant vector field $W$ must be a Killing field with respect to $h$. Or equivalently, $W \in \mathfrak{k}'$ by Lemma 2.2. Therefore, all the left invariant Einstein–Randers metrics $F(x, y)$ with the underlying Riemannian-Einstein metric $h$ are given by (2.2). ∎

***Proposition 2.4*** *The metrics described in Theorem* 2.3 *are not Berwald unless* $W = 0$.

**Proof** Recall that if the non-Riemannian Einstein–Randers metric on a compact manifold without boundary is Berwald, then $\mathrm{Ric} = 0$ (see [2]). By Lemma 1.1, the metric $h$ must be Ricci-flat. But the Einstein metric $h$ has Einstein constant $\rho > 0$ by Lemma 2.1. The contradiction implies that $(G, F)$ cannot be a Berwald manifold if $W \neq 0$. ∎

***Remark 2*** If the compact group $G$ is simple, then all naturally reductive metrics are of the form (2.1) (see [7]). Up to now, all the known left invariant Einstein-Riemannian metrics on compact Lie groups are naturally reductive.

***Remark 3*** If the connected group $G$ is abelian, then every left invariant Finsler metric on $G$ is also bi-invariant. Now suppose $h$ is a left invariant Riemannian metric on $G$, then $h$ is Einstein with $\mathrm{Ric} = 0$. Since $\mathfrak{g}$ is abelian, all the left invariant vector fields are Killing fields. Then by Lemma 1.1,

$$F(x, y) = \frac{\sqrt{h(y, W)^2 + \lambda h(y, y)}}{\lambda} - \frac{h(y, W)}{\lambda}$$

is also a left invariant Einstein–Randers metric on $G$ with $h(W, W) < 1$. In this case, the metric must be bi-invariant.

## 3 The Isometries

In Section 2, we described some left invariant Einstein–Randers metrics on the compact Lie group $G$. Two Finsler spaces $(M_1, F_1)$ and $(M_2, F_2)$ are said to be isometric if there exists a diffeomorphism $\varphi\colon M_1 \to M_2$ such that $\varphi^* F_2 = F_1$. There is a characterization of isometry between Randers spaces (see [3]).

***Lemma 3.1*** *Suppose $(M_1, F_1)$ and $(M_2, F_2)$ are two Randers spaces with navigation datas $(h_1, W_1)$ and $(h_2, W_2)$, respectively. Let $\varphi$ be a diffeomorphism from $M_1$ onto $M_2$. Then $\varphi$ is an isometry if and only if $\varphi$ satisfies $\varphi^* h_2 = h_1$ and $\varphi_* W_1 = W_2$.*

Now we consider two metrics $F_1(x, y)$ and $F_2(x, y)$ described in Theorem 2.3 with the same underlying Einstein-Riemannian metric $h$ on a connected compact simple Lie group $G$. Suppose the navigation data of $F_i(x, y)$ is $(h, W_i)(i = 1, 2)$, respectively. It is known that if the group $G$ is simple, then the identity component of the group of isometries of $h$, $I_0(G, h)$, is contained in $L(G)R(G)$, where $L(G)$ and $R(G)$ are the groups of left translations and right translations, respectively (see [12]). Moreover, $I_0(G, h)$ is of the form $G \times K'$, where $K'$ is the connected group generated by $\exp(\mathfrak{k}')$. Here $G$ acts on $G$ by left translations and $K'$ acts on $G$ by right translations (see [7]).

By Lemma 3.1, $F_1(x, y)$ and $F_2(x, y)$ are isometric if and only if there is a $\varphi \in G \times K'$ such that $\varphi_* W_1 = W_2$. If $\varphi \in G$, then $\varphi_* W_1 = W_2$ if and only if $W_1 = W_2$. Thus we can suppose $\varphi \in K'$, or equivalently, $\varphi = R(g^{-1})(g \in K')$. Then $R(g^{-1})$ is an isometry if and only if $R(g^{-1})_* W_1 = W_2$. Since $R(g^{-1})_*$ commutes with $L(g)_*$, $R(g^{-1})_* W_1$ is also a left invariant vector field. This implies that $R(g^{-1})$ is an isometry if and only if $L(g)_* R(g^{-1})_* W_1 = W_2$, since $L(g)_* W_1 = W_1$. In other words, $R(g^{-1})$ is an isometry if and only if $\mathrm{Ad}(g) W_1 = W_2$. Therefore, we have proved the following theorem.

***Theorem 3.2*** *Suppose $G$ is a compact simple Lie group and $h$ is a left invariant Einstein-Riemannian metric on $G$ as in Lemma* 2.1. *Let $F_1(x, y)$ and $F_2(x, y)$ be two left invariant Einstein–Randers metrics on $G$ described as in Theorem* 2.3 *with the navigation data $(h, W_1)$ and $(h, W_2)$, respectively. Then $F_1(x, y)$ and $F_2(x, y)$ are isometric if and only if $W_1$ and $W_2$ lie in the same orbit of* $\mathrm{Ad}(K')$.

Next we study the group of isometries of a left invariant Einstein–Randers metric on a simple Lie group. Recall that the group of isometries $I(M, F)$ of a Finsler space $(M, F)$ is a Lie group (see [5]). By Lemma 3.1 it is obvious that $I(M, F)$ is a closed subgroup of $I(M, h)$, where $F$ is a Randers metric whose navigation data is $(h, W)$. Suppose $G$ is a compact simple Lie group and $h$ is a left invariant Einstein-Riemannian metric as in Lemma 2.1. Let $F(x, y)$ be a left invariant Einstein–Randers metric with navigation data $(h, W)$, where $W \in \mathfrak{k}'$ is a left invariant vector field and $\mathfrak{k}' = V \oplus \mathfrak{k}$ as Lemma 2.2. Then $I(G, F)$ is a closed subgroup of $I(G, h)$. This implies that $I_0(G, F)$ is a closed subgroup of $I_0(G, h)$. Since $I_0(G, h)$ is of the form $G \times K'$, where $K'$ is the connected group generated by $\exp(\mathfrak{k}')$, and $G$ acts on $G$ by left translation, $K'$ acts on $G$ by right translation, we can identify $I_0(G, F)$ with a closed subgroup of $G \times K'$.

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***Corollary 3.3*** *Suppose $G$ is a connected compact simple Lie group. Let $F(x, y)$ be a left invariant Einstein–Randers metric on $G$ with the navigation data $(h, W)$ as in Theorem 2.3. Then $I_0(G, F)$ is of the form $G \times Z_0(\exp W)$, where $Z_0(\exp W)$ is the identity component of the centralizer of $\exp W$ in $K'$, $G$ acts on $G$ by left translation and $Z_0(\exp W)$ by right translation.*

**Proof** By Lemma 3.1 and the facts that left translations by elements of $G$ are isometries and that $W$ is a left invariant vector field, namely $L(g)_* W = W$, we deduce that $G$ is contained in $I_0(G, F)$. Now suppose $\varphi \in K'$ is an isometry of $(G, F)$ by right translation. Let $\varphi = R(g^{-1})(g \in K')$. Then by Theorem 3.2, $\mathrm{Ad}(g)W = W$. This implies that $g \exp W g^{-1} = \exp W$. Therefore, $g \in Z(\exp W)$. Thus $I_0(G, F)$ is of the form $G \times Z_0(\exp W)$, where $G$ acts on $G$ by left translation and $Z_0(\exp W)$ by right translation (see [7]). ■

## 4 Some Geometric Properties

We will now study some geometric properties of the metrics in Theorem 2.3. For the Einstein–Riemannian metric described in Lemma 2.1, the geodesic through the identity $e$ with initial vector $X \in \mathfrak{g}$ is just a little different from the one parameter group $\exp(tX)$. More precisely, the geodesic $\gamma(t)$ through $e$ is given by

$$\gamma(t) = \exp t\Big(X_{\mathfrak{a}} + \sum_{i=1}^{r+s} \alpha_i X_i\Big) \exp t\Big(\sum i = 1^{r+s}(1 - \alpha_i)X_i\Big),$$

where $X = X_{\mathfrak{a}} + \sum_{i=1}^{r+s} X_i$, $X_{\mathfrak{a}} \in \mathfrak{a}, X_i \in \mathfrak{k}_i$ and the constant $\alpha_i$ is as in Lemma 2.1 (see [7]). Now we suppose $F(x, y)$ is a Randers metric with the navigation data $(h, W)$ on the compact Lie group $G$. In [14], C. Robles found the relationship between the geodesics of $F(x, y)$ and the geodesics of $h$. She proved that any geodesic of $F$ must be of the form $\gamma'(t) = \varphi_t(\gamma(t))$, where $\gamma(t)$ is a geodesic of $h$ and $\varphi_t$ is the flow generated by $W$. But the flow generated by the left-invariant vector field $W$ is just the right translation $R(\exp tW)$. Hence we have the following proposition.

***Proposition 4.1*** *Suppose $F(x, y)$ is an Einstein–Randers metric with navigation data $(h, W)$ on a compact Lie group $G$ as in Theorem 2.3, Denote $e$ the identity of the group $G$. Then the geodesics through $e$ are given by*

$$\gamma(t) = \exp t\Big(X_{\mathfrak{a}} + \sum_{i=1}^{r+s} \alpha_i X_i\Big) \exp t\Big(\sum_{i=1}^{r+s}(1 - \alpha_i)X_i\Big) \exp tW,$$

*where $X = X_{\mathfrak{a}} + \sum_{i=1}^{r+s} X_i$, $X_{\mathfrak{a}} \in \mathfrak{a}, X_i \in \mathfrak{k}_i$.*

***Corollary 4.2*** *Suppose $F(x, y)$ is an Einstein–Randers metric with navigation data $(h, W)$ on a compact Lie group $G$ as in Theorem 2.3, $X = X_{\mathfrak{a}} + \sum_{i=1}^{r+s} X_i \in \mathfrak{g}$. If $[X_{\mathfrak{a}}, \sum_{i=1}^{r+s}(1-\alpha_i)X_i] = 0$ and $[X, W] = 0$, then the one parameter group $\exp t(X+W)$ is a geodesic of $(G, F)$.*

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**Proof** Since $[X_{\mathfrak{a}}, \sum_{i=1}^{r+s} X_i] = 0$, we have

$$\begin{aligned}\Big[X_{\mathfrak{a}} + \sum_{i=1}^{r+s}\alpha_i X_i, \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big] &= \Big[X_{\mathfrak{a}}, \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big] + \Big[\sum_{i=1}^{r+s}\alpha_i X_i, \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big] \\ &= \Big[X_{\mathfrak{a}}, \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big] + \Big[\sum_{i=1}^{r}\alpha_i X_i, \sum_{i=1}^{r}(1-\alpha_i)X_i\Big] \\ &= \Big[X_{\mathfrak{a}}, \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big] \\ &= 0.\end{aligned}$$

Then the geodesic in Proposition 4.1 is given by

$$\begin{aligned}\gamma(t) &= \exp t\Big(X_{\mathfrak{a}} + \sum_{i=1}^{r+s}\alpha_i X_i\Big) \exp t\Big(\sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big) \exp tW \\ &= \exp t\Big(X_{\mathfrak{a}} + \sum_{i=1}^{r+s}\alpha_i X_i + \sum_{i=1}^{r+s}(1-\alpha_i)X_i\Big) \exp tW \\ &= \exp tX \exp tW \\ &= \exp t(X+W).\end{aligned}$$

This implies that the one parameter group $\exp t(X+W)$ is a geodesic. ∎

We recall that a homogeneous Finsler manifold $(G/H, F)$ is called naturally reductive if there exists an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that

$$g_y([x,u]_{\mathfrak{m}}, v) + g_y(u, [x,v]_{\mathfrak{m}}) + 2C_y([x,y]_{\mathfrak{m}}, u, v) = 0,$$

where $y \neq 0, x, u, v \in \mathfrak{m}$, and $g_y$ is a bilinear symmetric form

$$g_y \colon T_pM \times T_pM \longrightarrow \mathbb{R}\ (p \in M, y \in T_pM);$$
$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(p, y + su + tv)]\ |_{s=t=0}\ (u, v \in T_pM);$$

and $C_y$ is the Cartan tensor

$$C_y(u,v,w) = \frac{1}{4}\frac{\partial^3}{\partial r \partial s \partial t}[F^2(y + ru + sv + tw)]\ |_{r=s=t=0}\ .$$

Just as in the Riemannian case, D. Latifi found in [10] that if $(G/H, F)$ is naturally reductive, then $(G/H, F)$ is a g.o. manifold. Using the definition of naturally reductive manifold, we find that the Einstein–Randers manifold $(G, F)$ in Theorem 2.1 is not naturally reductive.

***Proposition 4.3*** *The Einstein–Randers manifolds* $(G, F)$ *in Theorem* 2.1 *are not naturally reductive, unless* $W = 0$.

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**Proof** Suppose $(G,F)$ is naturally reductive and $W \neq 0$. Then

$$g_Y([X,U]_{\mathfrak{p}},V) + g_Y(U,[X,V]_{\mathfrak{p}}) + 2C_Y([X,Y]_{\mathfrak{p}},U,V) = 0,$$

where $Y \neq 0, X, U, V \in \mathfrak{p}$. The above equality yields $g_X([X,U]_{\mathfrak{p}},X) = 0$ when $X = Y = V \in \mathfrak{p}$, for all $U \in \mathfrak{p}$. Thus every $X \in \mathfrak{p}$ is a geodesic vector of $(G,F)$. This implies that every geodesic of $(G,F)$ through $e$ is of the form $\operatorname{Exp} tX$. But every geodesic of $(G,h)$ through $e$ is of the form $\operatorname{Exp} tX$ too, since $(G,h)$ is naturally reductive, where Exp denotes the exponential map of $G \times K$. Therefore, $(G,F)$ is affinely equivalent to $(G,h)$. In this case, $(G,F)$ must be Berwald manifold, which is a contradiction to Proposition 2.4. Thus, $(G,F)$ can not be naturally reductive when $W \neq 0$. ■

Next, we consider the curvature of the above Einstein–Randers metrics. Since the navigation data of $F(x,y)$ is $(h,W)$, we can expect that the curvature of $F(x,y)$ has some relation to the curvature of the Riemannian metric $h$. In [9], by studying the navigation problem on a Finsler manifold with respect to homothetic vector fields, L. Huang and X. Mo showed the following lemma; for details we refer to [9].

***Lemma 4.4*** *Let $(M,F)$ be a Finsler manifold and let $V$ be a homothetic vector field with dilation $\sigma$, and $F(x,V) < 1$. Let $\widetilde{F}$ be the Finsler metric produced by navigation problem. Then the flag curvature of $\widetilde{F}$ (resp. $F$), denoted by $\widetilde{K}(y,u)$ (resp. $K(y,u)$), satisfies*

$$\widetilde{K}(y,u) = K(\widetilde{y},u) - \sigma^2, \tag{4.1}$$

*where $\widetilde{y} = y - F(x,y)V$.*

By this Lemma, we can obtain some properties of the curvature $\widetilde{K}$ of $(G,F)$.

***Theorem 4.5*** *Suppose $F(x,y)$ is an Einstein–Randers metric with navigation data $(h,W)$ on a compact Lie group $G$ as in Theorem 2.3. If $u$ and $\widetilde{y} = y - F(x,y)W$ are orthonormal vectors with respect to the metric $h$, then*

$$\widetilde{K}(y,u) = \frac{1}{4}\|[\widetilde{y},u_{\mathfrak{k}}]\|_h^2 + \frac{1}{4}\sum_{i=1}^{r+s}\alpha_i^2\|[\widetilde{y}_i,u_{\mathfrak{a}}]\|_h^2. \tag{4.2}$$

*If $\widetilde{y} = \sum \widetilde{y}_i \in \mathfrak{k}$ with $\widetilde{y}_i \in \mathfrak{k}_i$ and $u = u_{\mathfrak{a}} + u_{\mathfrak{k}} \in \mathfrak{g}$, where $\|\cdot\|_h$ denotes the norm of the metric $h$, then*

$$\widetilde{K}(y,u) = \frac{1}{4}\sum_{i=1}^{r+s}\alpha_i^2\|[u_i,\widetilde{y}]\|_h^2$$

*for $\widetilde{y} \in \mathfrak{a}$, $u = \sum_{i=1}^{r+s} u_i \in \mathfrak{k}$, and*

$$\widetilde{K}(y,u) = \frac{1}{4}\|[\widetilde{y},u]_{\mathfrak{a}}\|_h^2 - \sum_{i=1}^{r+s}\Big(\frac{3}{4} - \frac{1}{\alpha_i}\Big)\|[\widetilde{y},u]_{\mathfrak{k}_i}\|_h^2 \tag{4.3}$$

*for $\widetilde{y}, u \in \mathfrak{a}$. Thus , if $\widetilde{y} \in \mathfrak{k}$, then $\widetilde{K}(y,u) \geq 0$ for any $u \in \mathfrak{g}$, with the equality holding if and only if $[\widetilde{y},u] = 0$. If $\alpha_i \leq 1$, then the flag curvature is everywhere non-negative.*

**Proof** Since $W$ is a Killing vector field, the constant $\sigma$ in Lemma 1.1 must be zero. By (4.1), we have $\widetilde{K}(y,u) = K(\widetilde{y},u)$. But the sectional curvature $K$ is (see [7])

$$K(X,Y) = \frac{1}{4}\|[X,Y_{\mathfrak{k}}]\|_h^2 + \frac{1}{4}\sum_{i=1}^{r+s}\alpha_i^2\|[X_i,Y_{\mathfrak{a}}]\|_h^2 \tag{4.4}$$

for $X = \sum X_i \in \mathfrak{k}$ with $X_i \in \mathfrak{k}_i$ and $Y = Y_{\mathfrak{a}} + Y_{\mathfrak{k}} \in \mathfrak{g}$, and

$$K(X,Y) = \frac{1}{4}\|[X,Y]_{\mathfrak{a}}\|_h^2 - \sum_{i=1}^{r+s}\Big(\frac{3}{4} - \frac{1}{\alpha_i}\Big)\|[X,Y]_{\mathfrak{k}_i}\|_h^2 \tag{4.5}$$

for $X, Y \in \mathfrak{a}$. Then (4.2) and (4.3) immediately follows from (4.4) and (4.5), respectively. For $\widetilde{y} \in \mathfrak{a}$, $u = \sum_{i=1}^{r+s} u_i \in \mathfrak{k}$, we have $\widetilde{K}(y,u) = K(\widetilde{y},u) = K(u,\widetilde{y})$. Hence 4.5 follows immediately from (4.4).

By (4.2), if $\tilde{y} \in \mathfrak{k}$, then $\widetilde{K}(\tilde{y},u) \geq 0$ for any $u \in \mathfrak{g}$. Moreover, $\widetilde{K}(y,u) = 0$ if and only if $[\widetilde{y}, u_{\mathfrak{k}}] = 0$ and $[\widetilde{y}_i, u_{\mathfrak{a}}] = 0$, or equivalently, $[\widetilde{y}, u] = 0$ since $\widetilde{y} = \sum \widetilde{y}_i \in \mathfrak{k}$. When $\alpha_i \leq 1$, the metric $h$ is normal homogeneous (see [7]), thus the sectional curvature of $(G,h)$ is nonnegative. Therefore, by (4.1), $\widetilde{K} \geq 0$. ■

It was proved by Robles that if a Randers space $(M,F)$ is Einstein and $\dim M \geq 3$, then $(M,F)$ has constant Ricci curvature (see [13]). But we do not know whether the metrics in Theorem 2.3 have constant flag curvature. In fact, by a theorem of D. Bao and C. Robles (see [2]), the Einstein–Randers manifold $(M,F)$ with navigation data $(h,W)$ has constant curvature if and only if $(M,h)$ has constant curvature, (see [4] for some examples of Randers metrics of this type on the Lie group $S^3$). Recall that if the rotation of the transverse edge $V$ about the flag $y$ leaves the flag curvature $K(y,V)$ unchanged, then we say that our Finsler manifold has *scalar curvature*.

***Proposition 4.6*** *Let $(G,F)$ be a Einstein–Randers manifold with navigation data $(h,W)$, where $G$ is a compact Lie group with* $\dim G \geq 3$. *Then $(G,F)$ has constant flag curvature if and only if $(G,F)$ is of scalar curvature.*

**Proof** Obviously, we only have to prove the "if" part. Suppose $(G,F)$ is of scalar curvature. Then by (4.1), $(G,h)$ is of scalar curvature. Since $h$ is Riemannian, and $\dim G \geq 3$, the Schur's lemma implies that $h$ has constant sectional curvature. Therefore, $(G,F)$ has constant flag curvature. ■

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*College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210003, P.R. China*
*e-mail*: wanghui0801@mail.nankai.edu.cn

*College of Mathematics, Nankai University, Tianjin 300071, P.R. China*
*e-mail*: dengsq@nankai.edu.cn