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VARIETIES OF ORTHOMODULAR LATTICES

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0. Introduction. In this paper we start investigating the lattice of varieties of orthomodular lattices. The varieties studied here are those generated by orthomodular lattices which are the horizontal sum of Boolean algebras. It turns out that these form a principal ideal in the lattice of all varieties of orthomodular lattices. We give a complete description of this ideal; in particular, we show that each variety in it is generated by its finite members. We furthermore show that each of these varieties is finitely based by exhibiting a (rather complicated) finite equational basis for each variety.

Our methods rely heavily on B. Jónsson's fundamental results in [**8**]. This, however, could be avoided by starting out with the equations given in sections 3 and 4. Some of our arguments were suggested by Baker [**1**].

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1. Generalities. Throughout this paper an orthomodular lattice (abbreviated: OML) is considered as a (universal) algebra $(L; \vee, \wedge, ', 0, 1)$ with two binary operations $\vee$ and $\wedge$, one unary operation $'$, and two nullary operations 0 and 1, such that $(L; \vee, \wedge)$ is a lattice with smallest element (zero) 0 and largest element (unit) 1, such that $'$ is an anti-monotone complementation on L and such that the orthomodular law $a \vee ((a \vee b) \wedge a') = a \vee b$ holds for all $a, b \in L$. For basic results and notations concerning OMLs see [**2**, p. 55 ff; **3**; **7**]. Regarding notions from universal algebra we follow the terminology of [**4**], with the only exception that we write $\mathbf{P}_U$ (instead of $\mathbf{P}_p$) for the operation of taking ultraproducts of a class of algebras.

A block [**5**] in an OML L is a maximal Boolean subalgebra of L. Let $\mathscr{B}(L)$ be the set of all blocks of L. Every element of L belongs to at least one block of L and every block contains 0 and 1. If $L \neq \{0, 1\}$, then every block of L is different from $\{0, 1\}$. A subset M of L is contained in some block B of L if and only if any two elements of M commute. An OML L is said to be the horizontal sum of its blocks [**10**] if and only if $A \cap B = \{0, 1\}$ holds for any two

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(different) blocks A and B of L. If $\mathscr{A}$ is a non-empty set of Boolean algebras, all having the same zero 0 and unit 1, such that $A \cap B = \{0, 1\}$ holds for any two (different) $A, B \in \mathscr{A}$ and such that if $\{0, 1\} \in \mathscr{A}$, then $\mathscr{A}$ consists of $\{0, 1\}$ only, then there exists exactly one OML L which is the horizontal sum of its blocks and which satisfies $\mathscr{A} = \mathscr{B}(L)$. If L is the horizontal sum of its blocks and if S is a subalgebra of L, then S is the horizontal sum of its blocks. If, moreover, $S \neq \{0, 1\}$, then

$$\mathscr{B}(S) = \{B \cap S \mid B \in \mathscr{B}(L), B \cap S \neq \{0, 1\}\}.$$

Conversely, if L_1 and L_2 are the horizontal sums of their blocks, then L_1 is isomorphic with a subalgebra of L_2 if and only if there exists a one-one mapping $f: \mathscr{B}(L_1) \rightarrow \mathscr{B}(L_2)$ such that for every $B \in \mathscr{B}(L_1)$, B is isomorphic with a subalgebra of $f(B)$. If B is finite, this simply means that $|B| \leqq |f(B)|$. ($|X|$ is the cardinal number of a set X.) If L is the horizontal sum of its blocks, then every finitely generated subalgebra of L is finite. If L is the horizontal sum of at least two blocks, then L is simple (i.e., has exactly two congruence relations); in particular it is directly and subdirectly irreducible. Let HOR be the class of all OMLs which are the horizontal sum of their blocks and which are simple. Then HOR consists of all OMLs which are the horizontal sum of at least two blocks and of all two-element Boolean algebras. Let [HOR] be the variety of OMLs generated by HOR. As usual we write aCb if the elements a and b of an OML commute and we write $a\tilde{C}b$ if they do not commute.

LEMMA 1. *For an* OML L *the following two conditions are equivalent:*
(1) L *is the horizontal sum of its blocks,*
(2) *for all* $a, b \in L$, *if* $a\tilde{C}b$, *then* $a \vee b = 1$ (*and hence* $a \vee b' = a' \vee b = a' \vee b' = 1$).

Proof. (1) $\Rightarrow$ (2). Given $a, b \in L$, the elements $a, a \vee b$ belong to some block A and the elements $b, a \vee b$ belong to some block B. If $a\tilde{C}b$, then $A \neq B$ and hence $A \cap B = \{0, 1\}$. Since $0 \neq a \vee b \in A \cap B$ it follows that $a \vee b = 1$.

(2) $\Rightarrow$ (1). Let B be a block and let x be an element of L. Define $B_1 = \{b \in B \mid bCx\}$ and $B_2 = \{b \in B \mid b\tilde{C}x\} \cup \{0, 1\}$. Then B_1 is a subalgebra of B and B_2 is closed under orthocomplementation. We show that $a, b \in B_2$ implies that $a \vee b \in B_2$ and hence that B_2 is also a subalgebra of B. If at least one of a, b is zero or one, or if $a \vee b\tilde{C}x$, this is obvious. If not, we have $a\tilde{C}x, b\tilde{C}x$ and $a \vee bCx$. By (2), this implies that $a \vee b = (a \vee b \vee x) \wedge (a \vee b \vee x') = 1 \in B_2$. Since $B_1 \cap B_2 = \{0, 1\}$ and $B_1 \cup B_2 = B$, it follows that $B_1 = \{0, 1\}$ or $B_2 = \{0, 1\}$, i.e., that either x commutes with all elements of B or with no element of $B - \{0, 1\}$. Let A and B be blocks of L and assume that there exists an element $x \neq 0, 1$ such that $x \in A \cap B$. Then every element of B commutes with $x \in A - \{0, 1\}$. By what we have just proved it follows that every element of A commutes with every element of B and hence that $A = B$, which was to be proved.

2. The lattice of subvarieties of [HOR]**.** Let HORF be the class of all finite algebras in HOR. We consider (possibly empty) subclasses $\mathscr{M}$ of HORF which satisfy the following condition:

(2.1) If $L \in \mathscr{M}$, if M is isomorphic with a subalgebra of L and if $M \in$ HOR, then $M \in \mathscr{M}$.

The "entity" of all classes $\mathscr{M} \subseteq$ HORF satisfying (2.1), partially ordered by class inclusion, form (modulo the foundations of set theory) a complete lattice $\mathscr{L}$. We show that this lattice is isomorphic with the lattice of subvarieties of [HOR].

THEOREM 1. *The mapping* $\mathscr{M} \rightarrow \mathbf{HSP}(\mathscr{M})$ ($\mathscr{M} \in \mathscr{L}$) *is a lattice isomorphism between* $\mathscr{L}$ *and the lattice of subvarieties of* [HOR]. *The inverse of this mapping is given by* $\mathscr{K} \rightarrow \mathscr{K} \cap$ HORF ($\mathscr{K}$ *a subvariety of* [HOR]).

Proof. If $\mathscr{M} \in \mathscr{L}$, then $\mathbf{HSP}(\mathscr{M})$ is clearly a subvariety of [HOR] and if $\mathscr{K}$ is a subvariety of [HOR], then $\mathscr{K} \cap$ HORF $\in \mathscr{L}$. Furthermore, both mappings are obviously monotone. It remains to show that their composites are the identity maps of $\mathscr{L}$ and of the lattice of subvarieties of [HOR], respectively.

In order to prove the first we have to show that $\mathscr{M} =$ HSP$(\mathscr{M}) \cap$ HORF holds for all $\mathscr{M} \in \mathscr{L}$. This is obvious if $\mathscr{M}$ is empty. Hence we may assume without loss of generality that $\mathscr{M} \neq \emptyset$; in particular, that $\mathscr{M}$ contains all two-element Boolean algebras. Clearly $\mathscr{M} \subseteq \mathbf{HSP}(\mathscr{M}) \cap$ HORF. To prove the converse, assume that there exists $L_0 \in \mathbf{HSP}(\mathscr{M}) \cap$ HORF with $L_0 \notin \mathscr{M}$. Since $L_0 \in$ HORF, it is subdirectly irreducible. Since L_0 is subdirectly irreducible and belongs to $\mathbf{HSP}(\mathscr{M})$, it follows from [**8**, Corollary 3.2], that L_0 belongs to $\mathbf{HSP}_U(\mathscr{M})$. The property of an OML L not to contain an isomorphic copy of L_0 as a subalgebra, is a first order property. Likewise, by Lemma 1, the property of an OML L to be the horizontal sum of its blocks, is a first order property. It follows that every $L \in \mathbf{SP}_U(\mathscr{M})$ is the horizontal sum of its blocks and that no $L \in \mathbf{SP}_U(\mathscr{M})$ is isomorphic with L_0. If $L \in \mathbf{SP}_U(\mathscr{M})$ contains at least two blocks, then L is simple and hence does not have L_0 as a homomorphic image. Since $L_0 \in \mathbf{HSP}_U(\mathscr{M})$, L_0 must be the homomorphic image of an OML L containing one block only, i.e., the homomorphic image of a Boolean algebra. Since $L_0 \in$ HOR, this means that L_0 is a two-element Boolean algebra, contradicting $L_0 \notin \mathscr{M}$. This proves that $\mathscr{M} = \mathbf{HSP}(\mathscr{M}) \cap$ HORF.

We complete the proof by showing that $\mathscr{K} = \mathbf{HSP}(\mathscr{K} \cap$ HORF) holds for every subvariety $\mathscr{K}$ of [HOR]. This is obvious if $\mathscr{K}$ is the trivial variety consisting of all one-element OMLs. Hence we may assume that $\mathscr{K}$ contains at least all Boolean algebras. Again it is obvious that $\mathscr{K} \supseteq \mathbf{HSP}(\mathscr{K} \cap$ HORF). In order to show the inverse inclusion it is enough to show that every subdirectly irreducible $L \in \mathscr{K}$ belongs to $\mathbf{HSP}(\mathscr{K} \cap$ HORF). Let $L \in \mathscr{K}$ be subdirectly irreducible. Then, as noted earlier, L is the horizontal sum of its blocks. Let F be a finitely generated (and hence finite) subalgebra of L. Then F is either Boolean or belongs to $\mathscr{K} \cap$ HORF. In both cases it belongs to

$\mathbf{HSP}(\mathscr{K} \cap \text{HORF})$. Since every finitely generated subalgebra of L belongs to $\mathbf{HSP}(\mathscr{K} \cap \text{HORF})$, L itself belongs to this class.

As a consequence of this theorem we note:

COROLLARY 1. *Every subvariety of* [HOR] *is generated by its finite members (even by its finite subdirectly irreducible members).*

To further investigate the lattice of subvarieties of [HOR] we represent the OMLs $L \in \text{HORF}$ by certain characteristic functions. Let N be the set of all natural numbers ($\geqq 0$) and let N_2 be the set of all natural numbers $n \geqq 2$. We consider the set Ω of all mappings $\psi: N_2 \to N$ which satisfy the following conditions:

(2.2) ψ is decreasing, i.e., $n \leqq m$ implies that $\psi(m) \leqq \psi(n)$,
(2.3) $\psi(n) = 0$ holds for all but finitely many n,
(2.4) $\psi(2) \neq 1$.

The set Ω with the argumentwise ordering is obviously a lattice. Given an OML $L \in \text{HORF}$, we define the characteristic function $\pi_L: N_2 \to N$ of L by putting $\pi_L(n)$ equal to the number of blocks B of L which satisfy $2^n \leqq |B|$. Obviously, $\pi_L \in \Omega$ and for every $\psi \in \Omega$ there exists up to isomorphism exactly one $L \in \text{HORF}$ such that $\psi = \pi_L$. Note that the function $\psi: N_2 \to \{0\}$ belongs to Ω and corresponds to the two-element Boolean algebra. Furthermore, it is easy to see that for OMLs L_1 and L_2 in HORF one has:

$$\pi_{L_1} \leqq \pi_{L_2} \tag{2.5}$$

if and only if L_1 is isomorphic with a subalgebra of L_2. Under the correspondence $L \to \pi_L$ the lattice $\mathscr{L}$ corresponds to the lattice $\mathscr{F}(\Omega)$ of all order-ideals of Ω. By an order-ideal we mean here a subset M of Ω satisfying:

$$\text{If } \psi \in M \text{ and } \varphi \leqq \psi, \text{ then } \varphi \in M. \tag{2.6}$$

Theorem 1 thus yields the following:

COROLLARY 2. *The mapping* $M \to \mathbf{HSP}(\{L \in \text{HORF} | \pi_L \in M\})\ (M \in \mathscr{F}(\Omega))$ *is a lattice isomorphism between* $\mathscr{F}(\Omega)$ *and the lattice of subvarieties of* [HOR]. *The inverse of this mapping is given by* $\mathscr{K} \to \{\pi_L | L \in \mathscr{K} \cap \text{HORF}\}$.

3. Some equations. The basic polynomial in our equations is

$$c(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y').$$

The elements x and y of an OML L commute if and only if $c(x, y) = 0$. From Lemma 1 it follows that if L is the horizontal sum of its blocks, then $c(x, y)$ takes in L the values 0 and 1 only.

LEMMA 2. *For a subdirectly irreducible* OML L *the following two conditions are equivalent:*
(1) $L \in \text{HOR}$,
(2) L *satisfies* $c(x, c(y, z)) = 0$.

Proof. (1) ⇒ (2). If $L \in$ HOR, then $c(y, z)$ takes in L the values 0 and 1 only, hence $c(x, c(y, z)) = 0$ holds for all $x, y, z \in L$.

(2) ⇒ (1). Let y, z be elements of L such that $y\tilde{C}z$. Then $c(y, z) \neq 0$ and by (2), $c(y, z)$ commutes with every $x \in L$ and is thus in the center of L. Since L is subdirectly (and hence directly) irreducible, the center of L consists of 0 and 1 only. It follows that $c(y, z) = 1$; in particular that $y \vee z = 1$. This by Lemma 1 gives $L \in$ HOR.

Let $d(x, y, z) = c'(y, z) \wedge c(x, (y \vee z) \wedge (y' \vee z'))$, where $c'(y, z)$ is the orthocomplement of $c(y, z)$. Let $n \geqq 2$ be a natural number and let $\bar{x}$ stand for the $2^{n-1} + 2$ variables $x, x_0, x_1, \ldots, x_{2^{n-1}}$. Define

$$p_n(\bar{x}) = \bigwedge_{0 \leqq i < j \leqq 2^{n-1}} d(x, x_i, x_j).$$

Note that for $L \in$ HOR the polynomial p_n takes the values 0 and 1 only and that, provided xCy, the expression $(x \vee y) \wedge (x' \vee y')$ is 0 if and only if $x = y$ and is 1 if and only if $x = y'$.

LEMMA 3. *Let L be the horizontal sum of at least two blocks and let $n \geqq 2$ be a natural number. Then the following two conditions are equivalent:*

(1) *every block of L contains at most 2^n elements,*

(2) *L satisfies $p_n(\bar{x}) = 0$.*

Proof. (1) ⇒ (2). Let $x, x_0, \ldots, x_{2^{n-1}}$ be arbitrary elements of L. We show that $d(x, x_i, x_j) = 0$ for some pair (i, j) with $0 \leqq i < j \leqq 2^{n-1}$. If $c'(x_i, x_j) = 0$ holds for at least one pair (i, j) there is nothing left to prove. If not, any two of the x_i commute and there exists a block B of L containing all x_i. Since this block B contains by assumption at most 2^n elements, there exist indices $i < j$ such that either $x_i = x_j$ or $x_i = x_j'$. For these indices

$$c(x, (x_i \vee x_j) \wedge (x_i' \vee x_j')) = 0,$$

i.e., our equation is satisfied.

(2) ⇒ (1). Let B be an arbitrary block of L. Since L contains at least two blocks there exists $x \in L - B$ and we can choose elements $x_i (0 \leqq i \leqq 2^{n-1})$ in $B - \{0, 1\}$. Since $c'(x_i, x_j) = 1$ for all i, j, our equation $p_n(\bar{x}) = 0$ implies that there exist indices $i < j$ such that $c(x, (x_i \vee x_j) \wedge (x_i' \vee x_j')) = 0$. Since $(x_i \vee x_j) \wedge (x_i' \vee x_j')$ belongs to B and x belongs to $L - B$, it follows that $(x_i \vee x_j) \wedge (x_i' \vee x_j')$ is either 0 or 1, i.e., that either $x_i = x_j$ or $x_i = x_j'$. Since the x_i are arbitrary elements of $B - \{0, 1\}$ it follows that B has at most $2.2^{n-1} + 2 < 2^{n+1}$ elements, i.e., that B has at most 2^n elements.

We next define for every pair (k, n) of natural numbers with $1 \leqq k$ and $2 \leqq n$ a polynomial q_{kn} in $(k + 1)(2^{n-2} + 1)$ variables which for notational convenience we write $x_i, x_{il} (0 \leqq i \leqq k, 1 \leqq l \leqq 2^{n-2})$. Let $\bar{x}_i$ stand for the variables $x_i, x_{i1}, x_{i2}, \ldots, x_{i2^{n-2}}$ and let

$$q_{n-1}(\bar{x}_i) = \bigwedge_{1 \leqq l < m \leqq 2^{n-2}} d(x_i, x_{il}, x_{im}) \quad \text{for } 0 \leqq i \leqq k.$$

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Let z stand for the $(k + 1)(2^{n-2} + 1)$ variables above. Then we define

$$q_{kn}(z) = \bigwedge_{0 \leqq i \leqq k} q_{n-1}(\bar{x}_i) \wedge \bigwedge_{\substack{0 \leqq i < j \leqq k \\ 1 \leqq l, m \leqq 2^{n-2}}} c(x_{il}, x_{jm}).$$

Note again that in an OML L which is the horizontal sum of its blocks, the polynomial q_{kn} takes the values 0 and 1 only.

LEMMA 4. *Let L be an* OML *which is the horizontal sum of at least two blocks and let k and n be natural numbers such that $1 \leqq k$ and $2 \leqq n$. Then the following conditions are equivalent:*

(1) *L has at most k blocks B with $2^n \leqq |B|$,*

(2) *L satisfies $q_{kn}(z) = 0$.*

Proof. (1) $\Rightarrow$ (2). Let x_i, $x_{il}(0 \leqq i \leqq k, 1 \leqq l \leqq 2^{n-2})$ be arbitrary elements of L. If at least one of the elements $c'(x_{il}, x_{im})$ or $c(x_{il}, x_{jm})$ is 0 there is nothing left to prove. If not, we have

$$x_{il} C x_{im} (0 \leqq i \leqq k, 1 \leqq l < m \leqq 2^{n-2}), \tag{3.1}$$

$$x_{il} \tilde{C} x_{jm} (0 \leqq i < j \leqq k, 1 \leqq l, m \leqq 2^{n-2}). \tag{3.2}$$

From (3.1) it follows that for every $i = 0, 1, \ldots, k$ there exists a block B_i containing all x_{il}. Since $1 \leqq k$ and $2 \leqq n$ it follows from (3.2) that $B_i \neq B_j$ if $i \neq j$ and that all x_{il} are different from 0 and 1. By condition 1, there exists an index i $(0 \leqq i \leqq k)$ such that $|B_i| \leqq 2^{n-1}$. For this index i there exist indices l and m with $1 \leqq l < m \leqq 2^{n-2}$ such that either $x_{il} = x_{im}$ or $x_{il} = x_{im}'$. In both cases we have $c(x_i, (x_{il} \vee x_{im}) \wedge (x_{il}' \vee x_{im}')) = 0$. This means that our equation is satisfied.

(2) $\Rightarrow$ (1). If L contains at most k blocks there is nothing to prove. If not, let $B_0, B_1, \ldots, B_k$ be arbitrary pairwise different blocks of L. If $2^n \leqq |B_i|$ choose elements $x_{i1}, x_{i2}, \ldots, x_{i2^{n-2}} \in B_i - \{0, 1\}$ in such a way that $x_{il} \neq x_{im}$ and $x_{il} \neq x_{im}'$ hold for all l, m with $1 \leqq l < m \leqq 2^{n-2}$ and choose $x_{il} \in B_i - \{0, 1\}$ arbitrarily otherwise. Since L contains at least two blocks we may choose $x_i \in L - B_i (i = 0, 1, \ldots, k)$. Our equation then implies that there exist i, l, m $(0 \leqq i \leqq k, 1 \leqq l < m \leqq 2^{n-2})$ with $c(x_i, (x_{il} \vee x_{im}) \wedge (x_{il}' \vee x_{im}')) = 0$. This as before implies that $x_{il} = x_{im}$ or that $x_{il} = x_{im}'$ which in view of the choice of the x_{il} gives $|B_i| \leqq 2^{n-1}$.

4. Equational bases for the subvarieties of [HOR]**.** For a given $\psi \in \Omega$ with $\psi(2) \neq 0$, let n_0 be the smallest natural number $n \geqq 2$ for which $\psi(n + 1) \leqq 1$ and let n_1 be the smallest natural number $n \geqq 2$ for which $\psi(n + 1) = 0$. We introduce a polynomial e_ψ depending on the following variables: $x_i^s (2 \leqq s \leqq n_0; 0 \leqq i \leqq \psi(s) - 1)$, $x^s_{il}(2 \leqq s \leqq n_0; 0 \leqq i \leqq \psi(s) - 1;$ $1 \leqq l \leqq 2^{s-2})$, $x^s(n_0 < s \leqq n_1)$ and $x_k^s(n_0 < s \leqq n_1; 0 \leqq k \leqq 2^{s-2})$. To simplify notation we let v_s stand for the variables x_i^s, $x^s_{il}(0 \leqq i \leqq \psi(s) - 1;$

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$1 \leqq l \leqq 2^{s-2}$) if $2 \leqq s \leqq n_0$ and we let v_s stand for the variables $x^s, x_0{}^s, x_1{}^s, \ldots, x^s{}_{2^s-2}$ if $n_0 < s \leqq n_1$. We then define

$$e_\psi(v_s; 2 \leqq s \leqq n_1) = \left(\bigwedge_{2 \leqq s \leqq n_0} q_{\psi(s)-1,s}(v_s) \right) \wedge \left(\bigwedge_{n_0 < s \leqq n_1} p_{s-1}(v_s) \right).$$

LEMMA 5. *Assume that* $\psi \in \Omega$, $\psi(2) \geqq 2$, $L \in$ HORF *and* L *not Boolean. Then the following conditions are equivalent:*

(1) $\psi \nleqq \pi_L$,

(2) L *satisfies* $e_\psi = 0$.

Proof. (1) $\Rightarrow$ (2). If $\psi \nleqq \pi_L$, there exists a natural number s such that $\pi_L(s) \leqq \psi(s) - 1$. If $2 \leqq s \leqq n_0$ we have, by Lemma 4, $q_{\psi(s)-1,s}(v_s) = 0$. If $n_0 < s \leqq n_1$ we have, by Lemma 3, $p_{s-1}(v_s) = 0$. In both cases we have $e_\psi = 0$.

(2) $\Rightarrow$ (1). If $\psi \leqq \pi_L$, again by Lemma 3 and Lemma 4, the variables can be chosen in such a way that $e_\psi(v_s; 2 \leqq s \leqq n_1) = 1$, i.e., that (2) is violated.

We are now in a position to give equational bases for all subvarieties of [HOR]. We start out with the variety [HOR] itself. From Lemma 2, follows immediately:

THEOREM 2. *The variety* [HOR] *consists of exactly those* OMLs *which satisfy* $c(x, c(y, z)) = 0$.

The next theorem characterizes arbitrary subvarieties of [HOR].

THEOREM 3. *Let* $\mathscr{K} \subseteq$ [HOR] *be a variety of* OMLs *different from the variety consisting of all one-element* OMLs. *Define* $M = \{\pi_L | L \in \mathscr{K} \cap \text{HORF}\}$. *Then for every orthomodular lattice L the following two conditions are equivalent:*

(1) $L \in \mathscr{K}$,

(2) L *satisfies* $c(x, c(y, z)) = 0$ *and* $e_\psi = 0$ *for all minimal elements* ψ *of* $\Omega - M$.

Proof. Take $\mathscr{M} \in \mathscr{L}$ (the lattice used in Theorem 1), $\mathscr{M} \neq \emptyset$ and put $M = \{\pi_L | L \in \mathscr{M}\}$. By Theorem 1 and Theorem 2 it is enough to show that an OML $L \in$ HORF belongs to $\mathscr{M}$ if and only if it satisfies $e_\psi = 0$ for all minimal elements ψ of $\Omega - M$. If $L \in \mathscr{M}$ and if ψ is a minimal element of $\Omega - M$ then $\psi \nleqq \pi_L$. But then L satisfies $e_\psi = 0$. This follows from Lemma 5 if L is not Boolean and is obvious otherwise. Assume conversely that $L \in$ HORF and $L \notin \mathscr{M}$. Then $\pi_L \notin M$. Since Ω satisfies the descending chain condition there exists a minimal element ψ of $\Omega - M$ with $\psi \leqq \pi_L$. Since $\mathscr{M} \neq \emptyset$ it follows that $\psi(2) \geqq 1$ and that L contains at least two blocks. By Lemma 5, L does not satisfy $e_\psi = 0$.

The equations given in Theorem 3 to characterize the subvarieties of [HOR] are by no means the most economic ones in every special case. We mention only one important example.

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It is easy to see that a lattice $L \in$ HORF is modular if and only if all its blocks have at most 4 elements, i.e., if their characteristic function satisfies $\psi_L(3) = 0$. Such an OML is completely determined by the number of its blocks. But it follows from Lemma 4, and could easily be checked directly, that an OML $L \in$ HORF has at most n blocks if and only if it satisfies

$$\bigwedge_{0 \leqq i < j \leqq n} c(x_i, x_j) = 0.$$

We thus obtain that the variety of modular ortholattices generated by the $(2n + 2)$-element ortholattice of dimension 2 is characterized by modularity and the equations

$$c(x, c(y, z)) = 0$$

and

$$\bigwedge_{0 \leqq i < j \leqq n} c(x_i, x_j) = 0.$$

The lattice-varieties generated by these lattices have been characterized in [**9**].

5. The finite basis problem. We show now that there are only finitely many equations occurring in Theorem 3. We start out with the following:

LEMMA 6. *Every infinite subset S of Ω has comparable elements.*

Proof. Define $\alpha: N_2 \to N \cup \{\infty\}$ by $\alpha(n) = \sup\{\psi(n) | \psi \in S\}$. Assume first that $\alpha(n) = \infty$ for all n. Take $\psi \in S$ arbitrarily. Then $\psi(n) = 0$ for some n. Since $\alpha(n) = \infty$ there exists $\varphi \in S$ such that $\varphi(n) > \psi(2)$. It follows that $\psi < \varphi$. Assume next that $\alpha(n) < \infty$ holds for at least one n. For every such n there exists $k \in N$ such that $\psi(n) = k$ holds for infinitely many $\psi \in S$. Let n_0 be the smallest number $n \in N_2$ for which such a number k exists and let k_0 be the smallest k. Define $S_0 = \{\psi \in S | \psi(n_0) = k_0\}$. Starting with (S_0, k_0) we define recursively a sequence (S_n, k_n) $(n = 0, 1, \ldots)$ as follows. If $n \geqq 1$ then k_n is the smallest natural number k such that $\psi(n_0 + n) = k$ holds for infinitely many $\psi \in S_{n-1}$ and $S_n = \{\psi \in S_{n-1} | \psi(n_0 + n) = k_n\}$. Then the S_n form a decreasing sequence of infinite subsets of S and the k_n form a decreasing sequence of natural numbers. It follows that there exists $n_1 \geqq 0$ such that $k_n = k_{n_1}$ for all $n \geqq n_1$. Take $\psi \in S_{n_1}$. Then $\psi(n_0 + n) = 0$ for some $n > n_1$. Assume that $n_0 = 2$. Then $k_{n_1} \neq 0$ since S_{n_1} is infinite. Therefore $\varphi > \psi$ holds for all $\varphi \in S_n$. If $n_0 > 2$ then by definition of n_0, $\varphi(n_0 - 1) \leqq \psi(2)$ holds for finitely many $\varphi \in S_n$ only. Hence there exists $\varphi \in S_n$ with $\varphi(n_0 - 1) > \psi(2)$. For every such φ, one clearly has $\psi < \varphi$.

Any two minimal elements of a subset S of Ω are incomparable. Hence by Theorem 3 and Lemma 6 we have:

THEOREM 4. *Every subvariety of* [HOR] *is finitely based.*

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