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ON THE FINITENESS AND UNIQUENESS OF CERTAIN 2-TAME *N*-GROUPS

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Unlike ring modules certain faithful N-groups are unique. The main theorem is that if N is a 2-tame ring-free near-ring where $N/J(N)$ has DCCR, then all faithful 2-tame N-groups are finite and N-isomorphic. The finiteness of such an N-group follows easily from the fact it has a composition series. It is then shown that the length of a composition series depends only on N. This fact is used at key points in the proof. The situations where the N-group has or has not a minimal submodule require different analysis. The first case makes use of other interesting results and the second makes strong use of the inductive assumption.

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Throughout this paper all near-rings are left distributive, zero-symmetric and have an identity. Also all N-groups will be unitary. In this and other regards we shall be making use of conventions, notation and definitions taken from [5]. The purpose of this paper is to prove the following theorem.

Theorem 1. *Let N be a 2-tame ring-free near-ring. If $N/J(N)$ has DCCR, then all faithful 2-tame N-groups are finite and N-isomorphic.*

Although the definition of all terms used in the statement of this theorem can be found in [5], it remains desirable, for the purposes of completeness, to briefly explain their meaning.

A near-ring N is said to have *DCCR* if it satisfies the descending chain condition on right ideals. In the above theorem it is only required that $N/J(N)$ satisfies this condition but strong consequences follow. The meaning associated with the statement 'N is ring-free' is that no non-zero homomorphic image of N is a ring.

A tame N-group V is one in which all N-subgroups are submodules. This is equivalent (see 2.1 of [5]) to requiring that, for any given v and w in V and α in N, we can find β in N such that $(v+w)\alpha - v\alpha = w\beta$. This characterization of a tame N-group leads to the definition of n-tame, where n is any cardinal (see 9.169 of [3]). For example, a 2-tame N-group V is one where, for given v in V and α in N, we may find β in N, such that $(v+w_i)\alpha - v\alpha = w_i\beta$ for any pair (w_1, w_2) of elements of V. Furthermore, a near-ring N with a faithful tame (2-tame) N-group is called tame (2-tame).

On p. 242 of [4], we define the centre $Z(V)$ of an N-group V. Also the useful concept of a central sum is defined in p. 243. The N-group V is said to be a central sum of the submodules V_i, $i=1,2$, if $V=V_1+V_2$ and $(v_1+v_2)\alpha=v_1\alpha+v_2\alpha$, for all v_i, $i=1,2$, in V_i and α in N. If this is the case, then $V_1\cap V_2\leqq Z(V)$.

The proof of the above theorem will be accomplished in the sequence of results that follow. The easiest part of the proof is to establish that the N-groups involved are finite.

According to 5.32 of [3], the semi-simple near-ring $N/J(N)$ is a direct sum $R_1\oplus R_2\oplus\cdots\oplus R_k$ ($k\geqq 1$ an integer) of minimal right ideals. If U is a minimal N-group, then it is a minimal $N/J(N)$-group. It follows that $UR_i\neq\{0\}$ for some i in $\{1,\dots k\}$ and thus $uR_i=U$ for some u in U. From this it is easily verified that the map taking ρ in R_i to $u\rho$ in uR_i is an N-isomorphism of R_i onto U. Thus the number of N-isomorphism types of minimal N-groups cannot exceed k. Let this finite number be denoted by $m(N)$.

Proposition 2. *Under the assumptions of Theorem* 1 *the number* $m(N)$ *of distinct* N*-isomorphism types of minimal* N*-groups is finite*

The next lemma is one of the main steps in establishing the finiteness of a faithful 2-tame N-group.

Lemma 3. *Let* N *be as in the statement of Theorem* 1. *If* V *is a faithful* 2*-tame* N*-group, then* V *has a composition series of length* $\leqq m(N)$.

Proof. Suppose V is non-zero and does not have a composition series. A series of submodules.

$$V_0=\{0\}<V_1<\cdots<V_r=V \tag{a}$$

of V, with $r\geqq 1$ an integer, will have a proper refinement and the series obtained in turn has a proper refinement, *etc.* It follows readily that, if V does not have a composition series, or there exists a composition series of length $>m(N)$, then the series (a) can be chosen so that $r>m(N)$.

Now take v_i in V_i but not in V_{i-1} for $i=1,\dots,r$. Using Zorn's lemma we can find a submodule X_i of V_i containing V_{i-1} and maximal for excluding v_i. Let $Y_i=v_iN+X_i$. It follows that for each $i=1,\dots,r$, $V_{i-1}\leqq X_i<Y_i\leqq V_i$ and Y_i/X_i is a minimal factor of V. Since $r>m(N)$, there exists j and k, $j<k$, in $\{1,\dots r\}$ such that Y_k/X_k is N-isomorphic (by σ say) to Y_j/X_j. It is also clear that

$$X_j<Y_j\leqq V_{k-1}\leqq X_k<Y_k.$$

Let κ be the natural N-homomorphism of Y_k/X_j onto

$$(Y_k/X_j)/(X_k/X_j).$$

Since this N-group is N-isomorphic (by δ say) to Y_k/X_k, $\kappa\delta\sigma$ is an N-endomorphism of Y_k/X_j onto Y_j/X_j. By 7.3 of [5], $1-\kappa\delta\sigma$ is an N-endomorphism of Y_k/X_j and, by 1.4 of

[4], Y_k/X_j is a central sum of the submodules $(Y_k/X_j)\kappa\delta\sigma$ and $(Y_k/X_j)(1-\kappa\delta\sigma)$. Now $(Y_k/X_j)\kappa\delta\sigma = Y_j/X_j$. Also $ker\,\kappa\delta\sigma = ker\,\kappa = X_k/X_j$ and

$$(X_k/X_j)(1-\kappa\delta\sigma)=(X_k/X_j).$$

Thus

$$(Y_k/X_j)\kappa\delta\sigma \cap (Y_k/X_j)(1-\kappa\delta\sigma) \geqq Y_j/X_j.$$

It follows, by 1.3 of [4], that Y_j/X_j is a central submodule of Y_k/X_j. Thus $N/(0\colon Y_j/X_j)(\neq\{0\})$ is a ring. This contradiction implies V has a composition series of length $\leqq m(N)$. The lemma is entirely proved.

Our next requirement is a result that depends on 8.4 of [5] (see also 1.5 of [4]). A relatively straightforward proof, based on 8.4 of [5], can be found in [6] (see also 4.61 of [3]).

Theorem 4. *Suppose the non-ring N is 2-tame and primitive on V. If N has DCCR, then V is finite.*

The proof that the N-groups of Theorem 1 are finite now follows readily.

Lemma 5. *Let N be as in the statement of Theorem 1. If V is a faithful 2-tame N-group, then V and N are finite.*

Proof. If $V\neq\{0\}$, then by Lemma 3, there exists an integer $r\geqq 1$ and composition series

$$V_0=\{0\}<V_1<V_2<\cdots<V_r=V$$

of submodules of V. Since V_i/V_{i-1}, $i=1,\dots,r$, are minimal N-groups the primitive non-rings $N/(0\colon V_i/V_{i-1})$ are homomorphic images of $N/J(N)$. Therefore they have *DCCR*. Clearly they are 2-tame on V_i/V_{i-1} for $i=1,\dots,r$. Thus, by 4, each V_i/V_{i-1}, $i=1,\dots,r$, is finite. Thus V is finite. Since N can be regarded as a subnear-ring of $M_0(V)$, N must be finite also. The lemma is completely proved.

We now come to a much more difficult aspect, that of proving that all N-groups, as in the statement of Theorem 1, are in fact N-isomorphic.

At this stage it is possible to sharpen Lemma 3. The more precise information given in the next lemma will be required later.

Lemma 6. *Let N be as in the statement of Theorem 1. If V is a faithful 2-tame N-group, then the length of a composition series of V is $m(N)$.*

Proof. Suppose

$$V_0=\{0\}<V_1<V_2<\cdots<V_r=V$$

is a composition series of V ($r\geqq 1$ being an integer). By Lemma 3, we have $r\leqq m(N)$. If it is shown that for any minimal N-group U there exists a minimal factor Y/X of V, N-isomorphic to U, then, since Y/X is N-isomorphic to some V_i/V_{i-1} (i in $\{1,\ldots,r\}$), it will follow that $r\geqq m(N)$ and $r=m(N)$.

As in the explanation proceeding Proportion 2 we have that U *is* N-isomorphic to a minimal right ideal of $N/J(N)$. This right ideal is clearly of the form $R/J(N)$, where $R>J(N)$ is a right ideal of N. By Lemma 5, N is finite and out of all minimal factors of N, N-isomorphic to the minimal factor $R/J(N)$, we may choose one H/K where H is minimal. Now if $K_1<H$ is a right ideal of N such that $K_1\nleqq K$, then $K_1+K=H$ and H/K is N-isomorphic to the minimal factor $K_1/K_1\cap K$. Thus $K_1\leqq K$. Since V is faithful there exists v in V such that $(0{:}\,v)\cap H<H$. Consequently $(0{:}\,v)\cap H\leqq K$.

Let δ be the obvious N-homomorphism of H into vN. Let κ be the natural N-homomorphism of vH onto vH/vK. Since $ker\,\delta\kappa$ consists of all ρ in H such that $v\rho$ is in vK, it contains K. However, $ker\,\delta\kappa$ is an N-subgroup of H and therefore, by 4.2 of [**5**], $ker\,\delta\kappa=K$ or $ker\,\delta\kappa=H$. Suppose $ker\,\delta\kappa=H$. In this case $vH=vK$ and for each α in H there exists β in K such that $v\alpha=v\beta$. This implies $H\leqq K+(0{:}\,v)$, and therefore $H=K+(0{:}\,v)\cap H$. However, from above $(0{:}\,v)\cap H\leqq K$ and we have the contradiction that $H=K$. Thus $ker\,\delta\kappa=K$ and vH/vK is N-isomorphic to H/K which is N-isomorphic to U. Thus V has a minimal factor N-isomorphic to U and, according to the explanation given at the beginning of the proof $r=m(N)$. The lemma is completely proved.

More detailed information on the structural properties of tame N-groups of a ring-free near-ring N is now developed.

Theorem 7. *Suppose N is a ring-free near-ring. If V is a tame N-group with a composition series, then there exists v in V such that $vN=V$.*

Proof. Clearly V satisfies both ascending and descending chain conditions on submodules. It follows that if V is not cyclic (i.e. monogenic), then there exists a minimal non-cyclic submodule U of V. Obviously $U\neq\{0\}$. Suppose the sum of any two proper submodules H_i, $i=1,2$, of U is such that $H_1+H_2<U$. If H is a maximal submodule of U, then $H+H_2=H$ and $H_2\leqq H$. In this case H must be the unique maximal submodule of U. Take v in $U\backslash H$. Clearly vN is not contained in H. This implies $vN=U$. Thus there exist proper submodules H_i, $i=1,2$, of U, such that $H_1+H_2=U$. Out of all such pairs of submodules choose one K_i, $i=1,2$, with $K_1\cap K_2$ minimal. Now $K_1+K_2=U$ and there exists v_i, $i=1,2$, in K_i such that $v_iN=K_i$. Since the v_i, $i=1,2$, are contained in U, it follows that

$$(v_1+v_2)N+v_iN\leqq U.$$

However, $(v_1+v_2)N+v_1N$ contains v_2N. Similarly $(v_1+v_2)N+v_2N$ contains v_1N. Since $K_1+K_2=U$, it follows that

$$(v_1+v_2)N+v_iN=U$$

for $i=1$ and 2. Since $N/(0:V)$ is ring-free, it follows by 3.24 of [2] that the lattice of submodules of V is distributive. Thus

$$(v_1+v_2)N+(v_1+v_2)N\cap v_1N+(v_1+v_2)N\cap v_2N+v_1N\cap v_2N=U.$$

Clearly

$$(v_1+v_2)N\cap v_iN\leqq(v_1+v_2)N$$

for $i=1,2$, and it therefore follows that

$$(v_1+v_2)N+v_1N\cap v_2N=U. \tag{b}$$

However, if

$$(v_1+v_2)N\cap v_1N\cap v_2N<v_1N\cap v_2N,$$

then we have a contradiction to the choice of K_i $(=v_iN)$, $i=1,2$. Thus

$$(v_1+v_2)N\geqq v_1N\cap v_2N$$

and, by (b), $U=(v_1+v_2)N$. This contradiction to the nature of U establishes that the N-group V is cyclic. Theorem 7 is entirely proved.

Suppose N is as in the statement of Theorem 1. In proving that all faithful 2-tame N-groups are N-isomorphic there are two cases to be distinguished. These are the situations where N has more than one minimal ideal and where N has a unique minimal ideal. We deal with the second case first. This property of N can be recovered from a faithful 2-tame N-group.

Lemma 8. *Let N be as in the statement of Theorem 1. A faithful 2-tame N-group has a unique minimal submodule if, and only if, N has a unique minimal ideal.*

Proof. Let V be a faithful 2-tame N-group. Suppose V has a unique minimal submodule U. By Lemma 5, N has DCC on N-subgroups and therefore has a minimal ideal T say. From 3.54 of [3], T can be expressed as a direct sum $R_1\oplus R_2\oplus\cdots\oplus R_k$ ($k\geqq 1$ an integer) of minimal right ideals. Also, by 4.2 of [5], these minimal right ideals are minimal right N-subgroups of N. If x is in V, then $xT=\sum_{i=1}^{k}xR_i$. Furthermore, each $xR_i=\{0\}$ or is N-isomorphic to R_i. Thus, if $xR_i\neq\{0\}$, then it is a minimal N-subgroup of V and coincides with U. It follows that $xT=\{0\}$ or $xT=U$.

Now suppose T_1 and T_2 are two distinct minimal ideals of N. By Theorem 7, $V=vN$ for some v in V. Since $VT_i\neq\{0\}$ for $i=1,2$, we have, from above, $vT_1=vT_2=U$. However, since the sum T_1+T_2 is direct, U is a central sum of vT_i, $i=1,2$. By 1.3 of [4], this implies U is a ring module and $N/(0{:}\,U)$ is a ring. Thus N has a unique minimal ideal.

Suppose on the other hand N has a unique minimal ideal. Clearly $V \neq \{0\}$ and, by 5, V has minimal submodules. To obtain a contradiction assume, that U_i, $i=1,2$, are distinct minimal submodules of V. If $(U_1{:}\,V)=\{0\}$, then N is faithful on the 2-tame N-group V/U_1. By Lemma 6 this implies that V/U_1 has a composition series of length $m(N)$. However, since N is faithful on V, V also has a composition series of length $m(N)$. This contradiction means $(U_1{:}\,V)\neq\{0\}$. Similarly $(U_2{:}\,V)\neq\{0\}$. Clearly

$$(U_1{:}\,V)\cap(U_2{:}\,V)\leqq(0{:}\,V)=\{0\}.$$

By Lemma 5, N has minimal ideals T_i, $i=1,2$, contained in $(U_i{:}\,V)$. However, since $T_1\cap T_2=\{0\}$ the T_i are distinct. This contradiction implies V has a unique minimal submodule. The lemma is completely proved.

Let N be as in the statement of Theorem 1. We come now to the main step in showing that, when N has a unique minimal ideal, all faithful 2-tame N-groups are N-isomorphic. In order to state this lemma it is convenient at this stage to introduce the centralizer of a right ideal of a ring-free near-ring.

If N is a ring-free near-ring and R a right ideal of N, then, by Zorn's lemma, there exists a right ideal H of N maximal for the property that $H\cap R=\{0\}$. Furthermore, H is unique since if H_1 is a right ideal of N such that $H_1\cap R=\{0\}$, then, by 3.24 of [2], $R\cap(H_1+H)=\{0\}$. Thus $H_1+H=H$ and $H_1\leqq H$. We shall denote the right ideal H of N by $C_N(R)$. This will be called the centralizer of R in N. This name is given to H since, as is easily verified, it is the unique right ideal of N maximal for the property that $(\rho+h)\alpha=\rho\alpha+h\alpha$ for all ρ in R, h in H and α in N.

Lemma 9. *Let N be as in the statement of Theorem 1. Suppose N has a unique minimal ideal T and $R\leqq T$ is a minimal right ideal of N. If V is a faithful 2-tame N-group, then V is N-isomorphic to $N/C_N(R)$.*

Proof. Since V is a faithful N-group there exists v in V such that $vR\neq\{0\}$. We first show that $vN=V$. Clearly $vNT\supseteq vR\neq\{0\}$. Thus $T\cap(0{:}\,vN)=\{0\}$. However, by Lemma 5, N is finite and if $(0{:}\,vN)\neq\{0\}$, then $(0{:}\,vN)$ contains a minimal ideal. This contradiction implies $(0{:}\,vN)=\{0\}$. Thus vN is a faithful 2-tame N-group and $\{0\}\leqq vN\leqq V$. Since by Lemma 6, vN and V both have composition series of length $m(N)$, it follows that $vN=V$.

Now let δ be the obvious N-homomorphism of N onto vN. Since $ker\,\delta=(0{:}\,v)$ the lemma will follow if it is shown that $(0{:}\,v)=C_N(R)$. By 4.2 of [5], R is a minimal N-group, since $vR\neq\{0\}$ the map taking ρ in R to $v\rho$ is an N-isomorphism of R onto vR. Thus vR is a minimal N-group. However, the sum $R+C_N(R)$ is direct and, $vR+vC_N(R)$ is a central sum of vR and $vC_N(R)$ (see §.1 of [4]). If $vC_N(R)\neq\{0\}$, then by Lemma 5, $vC_N(R)$ contains a minimal N-subgroup of V. However, by Lemma 8, this coincides with vR. In this case it follows, by 1.3 of [4], that vR is central in $vR+vC_N(R)(=vC_N(R))$.

This yields the contradiction that $N/(0{:}\,vR)(\neq\{0\})$ is a ring. Thus $vC_N(R)=\{0\}$ and $C_N(R)\leqq(0{:}\,v)$. If $C_N(R)<(0{:}\,v)$, then, by the maximality of $C_N(R)$, $(0{:}\,v)\geqq R$. This contradiction to the fact that $vR\neq\{0\}$ implies $(0{:}\,v)=C_N(R)$. The lemma is completely proved.

As will be seen later the above result in fact deals with the situation where N has a faithful 2-tame N-group V with a unique minimal N-subgroup. Results that follow are directed toward providing insight as to what happens when V has more than one minimal N-subgroup.

Lemma 10. *Let N be a ring-free near-ring and V a tame N-group. If W_i, $i=1,2$, are minimal N-isomorphic N-subgroups of V, then $W_1=W_2$.*

Proof. Suppose $W_1\neq W_2$. In this case the sum W_1+W_2 is direct. Suppose δ is an N-isomorphism of W_1 onto W_2. Let Δ be the subset of $W_1\oplus W_2$ consisting of all $w+w\delta$ where w is in W_1. Clearly $\Delta\neq\{0\}$. Furthermore the difference of two elements of Δ is in Δ and, if α is in N and w_1 in W_1, then $(w_1+w_1\delta)\alpha=w_1\alpha+w_1\alpha\delta$ is again in Δ. Thus Δ is an N-subgroup of V and consequently a submodule. If for x_1 in W_1, $x_1+x_1\delta$ is in W_1, then $x_1\delta=0$ and $x_1+x_1\delta=0$. Thus $\Delta\cap W_1=\{0\}$ and similarly $\Delta\cap W_2=\{0\}$. It follows, from Section 1 of [4], that Δ is a central submodule of $W_1\oplus W_2$. Thus if $W_1\neq W_2$, $N/(0{:}\,\Delta)(\neq\{0\})$ is a ring. This contradiction implies $W_1=W_2$ and the lemma follows.

Let N be as in the statement of Theorem 1 and V a faithful 2-tame N-group. The requirement that V has a minimal N-subgroup of given N-isomorphism type is, in fact, equivalent to a condition on N.

Lemma 11. *Let N be as in the statement of Theorem* 1, *V a faithful* 2*-tame N-group and U a minimal N-group. We have that V has a minimal N-subgroup N-isomorphic to U if and only if, N contains a minimal right ideal N-isomorphic to U.*

Proof. Suppose X is a minimal N-subgroup of V, N-isomorphic to U. If $(X{:}\,V)=\{0\}$, then V/X is a faithful 2-tame N-group. By Lemma 6, V/X has a composition series of length $m(N)$. However, by Lemma 6, we have the contradiction that V also has a composition series of length $m(N)$. We conclude that $(X{:}\,V)\neq\{0\}$. Now, from Lemma 5, there exists a minimal right ideal $R\leqq(X{:}\,V)$. Since V is faithful we may find v in V such that $vR\neq\{0\}$. Now $vR\leqq X$, and thus $vR=X$. The map taking ρ in R to $v\rho$ in X is an N-isomorphism of R onto X. Thus R is a minimal right ideal N-isomorphic to X and therefore to U.

Suppose on the other hand R is a minimal right ideal of N, N-isomorphic to U. Since V is faithful we can find v in V such that $vR\neq\{0\}$. The map that takes ρ in R to $v\rho$ in vR is an N-isomorphism of R onto vR. Thus U is N-isomorphic to the minimal N-subgroup vR of V. The proof of Lemma 11 is complete.

To proceed further we need some straightforward results on complete reducibility.

A submodule U of an N-group V is said to be *completely reducible* in V (c.f. 15.1 of

[1]) if, for every submodule U_1 of V contained in U, there can be found a submodule U_2 of V such that $U_1 \oplus U_2 = U$.

It is an elementary fact that:

Proposition 12. *If a submodule U of an N-group V is completely reducible in V, then any submodule $H \leqq U$ of V is completely reducible in V.*

The techniques used to prove 15.3 of [1] are available to show that:

Proposition 13. *A non-zero submodule of an N-group V is completely reducible in V if, and only if, it is a sum of minimal submodules of V.*

The *socle*, soc N of a near-ring N is defined to be a sum of all minimal right ideals of N when such right ideals exist. Otherwise soc N is taken to be $\{0\}$. Proposition 12 and Proposition 13 have been stated in order to present the following corollary:

Corollary 14. *Let N be a near-ring. A non-zero right ideal of N contained in* soc N *is a sum of minimal right ideals.*

Let N be a near-ring and U a minimal N-group. It will be of use to have notation for specifying certain right ideals of N contained in soc N. In this regard $S(U)$ is taken to be the sum $\sum R_i$ over all minimal right ideals R_i of N, N-isomorphic to U provided such right ideals exist. If there are no such right ideals of N, then $S(U)$ is taken to be $\{0\}$.

Lemma 15. *Let N be as in the statement of Theorem 1 and V a faithful 2-tame N-group. If U is a minimal N-subgroup of V, then $S(U) = (U: V)$.*

Proof. It will first be shown that $S(U) \leqq (U: V)$. If $S(U) = \{0\}$, then this inclusion holds. We may therefore suppose (see Lemma 11) that there exists minimal right ideals of N, N-isomorphic to U. Let R be such a right ideal. It is easily seen that for each v in V, $vR = \{0\}$ or vR is N-isomorphic to the N-subgroup U of V. This follows since if $vR \neq \{0\}$, then the map taking ρ in R to $v\rho$ in vR is an N-isomorphism of R onto vR. However, by Lemma 10, $vR = U$ when $vR \neq \{0\}$. Thus for all v in V, $vR \leqq U$ and $R \leqq (U: V)$. Since this is true for any such R, $S(U) \leqq (U: V)$.

It remains to show that $(U: V) \leqq S(U)$. Since

$$V(U: V)J(N) \subseteq U \cdot J(N) = \{0\}$$

it follows that $(U: V)J(N) = \{0\}$. However, by Lemma 5, N has $DCCR$. By 5.3 of [5], it follows that $(U: V) \leqq \operatorname{soc} N$. Clearly we may assume $(U: V) \neq \{0\}$. Now Corollary 14 implies that $(U: V)$ is a sum $\sum R_i$, $i \in I$, of minimal right ideals of N. However for each R_i, $i \in I$, there exists v in V such that $vR_i \neq \{0\}$. Since $R_i \leqq (U: V)$, $vR_i = U$ and the map taking ρ in R_i to $v\rho$ is an N-isomorphism of R_i onto U. Thus R_i is in $S(U)$ and $(U: V) \leqq S(U)$. The lemma is entirely proved.

The completion of the proof of Theorem 1 will be given once information concerning the embedding of certain N-groups into direct sums has been obtained. This accounts for the next three lemmas. The result that follows covers well known properties of certain subdirect products (the proof is omitted). All the information covered will be required.

Lemma 16. *Let V be an N-group and $U_j, j=1,2$, submodules of V such that $U_1 \cap U_2=\{0\}$. Let Y be the external direct sum of V/U_1 and V/U_2 and let $H_1=(V/U_1,\{0\})$ and $H_2=(\{0\},V/U_2)$. Furthermore let $X_1=((U_1+U_2)/U_1,\{0\})$ and $X_2=(\{0\},(U_1+U_2)/U_2)$ and δ be the map of V into Y given by $v\delta=(v+U_1,v+U_2)$ for all v in V. We have that H_j and $X_j, j=1,2$, are submodules of Y and Y is the internal direct sum $H_1 \oplus H_2$. Furthermore, δ is an N-group embedding of V into Y, such that $V\delta+H_j=Y$ and $V\delta \cap H_j=X_j$ for $j=1,2$.*

Under conditions similar to those of Lemma 16 information is required as to how certain 2-tame N-groups are embedded in $H_1 \oplus H_2$. The next lemma facilitates the proof of a much more useful result i.e. Lemma 19.

Lemma 17. *Let N be a ring-free near-ring. Suppose the N-group Y is a direct sum $H_1 \oplus H_2$ of submodules H_j, $j=1,2$. If T_k, $k=1,2$, are 2-tame N-subgroups of Y such that $T_k+H_j=Y$ and $T_k \cap H_j=\{0\}$ for $k=1,2$, and $j=1,2$, then $T_1=T_2$.*

Proof. Let $\pi_j, j=1,2$, be the projection of Y onto H_j. Let π_{kj} be the restriction of π_j to T_k, $j=1,2$, $k=1,2$. Since $\ker\pi_1=H_2$, $\ker\pi_2=H_1$ and $T_k \cap H_j=\{0\}$, we see that π_{kj} is an N-isomorphism of T_k into H_j. However for $k=1,2$,

$$H_2=Y\pi_2=(T_k+H_1)\pi_2=T_k\pi_2$$

and similarly

$$H_1=(T_k+H_2)\pi_1=T_k\pi_1.$$

Thus π_{kj} is an N-isomorphism of T_k, $k=1,2$, onto H_j, $j=1,2$. Now π_{11} is an N-isomorphism of T_1 onto H_1, π_{21}^{-1} an N-isomorphism of H_1 onto T_2, π_{22} an N-isomorphism of T_2 onto H_2 and π_{12}^{-1} an N-isomorphism of H_2 onto T_1. Thus $\pi_{11}\pi_{21}^{-1}\pi_{22}\pi_{12}^{-1}$ ($=\delta$ say) is an N-automorphism of T_1. Since T_1 is 2-tame it is, by 1.4 of [4], a central sum of $T_1\delta$ and $T_1(1-\delta)$ (here 1 is the identity on T_1). However, $T_1\delta=T_1$ and, by 1.3 of [4], $T_1(1-\delta)$ is in the centre of T_1. Thus if $T_1(1-\delta)\neq\{0\}$, then $N/(0\colon T_1(1-\delta))$ is a non-zero ring. Since N is ring-free, we have $T_1(1-\delta)=\{0\}$ and thus

$$\pi_{11}\pi_{21}^{-1}\pi_{22}\pi_{12}^{-1}=1. \tag{c}$$

Now a typical element x_2 of T_2 is of the form $x_2\pi_{21}+x_2\pi_{22}$. However, since $\pi_{11}\pi_{21}^{-1}$ is an N-isomorphism of T_1 onto T_2, there exists x_1 in T_1 such that $x_1\pi_{11}\pi_{21}^{-1}=x_2$. Thus

$$x_2=x_1\pi_{11}\pi_{21}^{-1}\pi_{21}+x_1\pi_{11}\pi_{21}^{-1}\pi_{22}$$

which by (c) yields

$$x_2=x_1\pi_{11}+x_1\pi_{12}=x_1.$$

Thus, x_2 is in T_1 and $T_2 \leqq T_1$. Similarly $T_1 \leqq T_2$. Thus $T_1=T_2$ and the lemma is proved.

To prove the straightforward extension of this lemma, that was mentioned above, an elementary proposition is in order.

Proposition 18. *Suppose the N-group Y is a direct sum $H_1 \oplus H_2$ of submodules H_i, $i=1,2$. If W is a submodule of H_1, then it is a submodule of Y.*

Material has been covered that provides a key result in establishing Theorem 1.

Lemma 19. *Let N be a ring-free near-ring. Suppose the N-group Y is a direct sum $H_1 \oplus H_2$ of submodules H_j, $j=1,2$. If T_k, $k=1,2$, are 2-tame N-subgroups of Y such that $T_1 \cap H_j = T_2 \cap H_j$ for $j=1,2$, and $T_k + H_j = Y$ for $k=1,2$, $j=1,2$, then $T_1 = T_2$.*

Proof. Let π_1 be the projection of T onto H_1. Since $ker\,\pi_1 = H_2$ and $T_1 + H_2 = Y$ we have $H_1 = T_1\pi_1$ and H_1 is an N-homomorphic image of a 2-tame N-group. Thus H_1 is 2-tame and, by Proposition 18, $T_1 \cap H_1 (= T_2 \cap H_1)$ is a submodule of Y. Similarly $T_1 \cap H_2 (= T_2 \cap H_2)$ is a submodule of Y. Let X be the submodule

$$T_1 \cap H_1 \oplus T_1 \cap H_2 = T_2 \cap H_1 \oplus T_2 \cap H_2$$

of Y. Set $\bar{Y} = Y/X$, $\bar{H}_j = (H_j + X)/X$ for $j=1,2$, and $\bar{T}_k = T_k/X$ for $k=1,2$. Clearly the $\bar{T}_k, k=1,2$, are 2-tame N-subgroups of $\bar{Y}$. If it is shown that $\bar{Y} = \bar{H}_1 \oplus \bar{H}_2$, $\bar{T}_k \cap \bar{H}_j = \{0\}$ and $\bar{T}_k + \bar{H}_j = \bar{Y}$ for $k=1,2$, and $j=1,2$, then it will follow, by Lemma 17, that $\bar{T}_1 = \bar{T}_2$. This will in turn imply $T_1 = T_2$.

Firstly $\bar{H}_1 + \bar{H}_2 = \bar{Y}$, since $H_1 + H_2 = Y$. Also the sum $\bar{H}_1 + \bar{H}_2$ is direct, since

$$\begin{aligned}(H_1+X)\cap(H_2+X) &= (H_1 + H_2 \cap T_1) \cap (H_2 + H_1 \cap T_1)\\ &= H_1 \cap (H_2 + H_1 \cap T_1) + H_2 \cap T_1\\ &= H_1 \cap T_1 + H_2 \cap T_1 = X.\end{aligned}$$

Secondly we must show $\bar{T}_k \cap \bar{H}_j = \{0\}$ for $k=1,2$, and $j=1,2$. Since for $k=1,2$,

$$T_k \cap (H_1 + T_k \cap H_2) = X$$

it follows that $\bar{T}_k \cap \bar{H}_1 = \{0\}$. Similarly $\bar{T}_k \cap \bar{H}_2 = \{0\}$ and $\bar{T}_k \cap \bar{H}_j = \{0\}$ for $k=1,2$, and $j=1,2$. Finally it is clearly true that since $T_k + H_j = Y$ for $k=1,2$, $j=1,2$, that

$\bar{T}_k + \bar{H}_j = \bar{Y}$. From comments above it follows that $T_1 = T_2$. The proof of Lemma 19 is complete.

The proof of Theorem 1 is now accomplished in the concluding analysis that follows.

We shall be assuming that N is a near-ring satisfying the conditions of Theorem 1. Furthermore, we shall assume that V_i, $i=1,2$, are two faithful 2-tame N-groups. The fact that the V_i, $i=1,2$, are finite follows from Lemma 5. The remaining requirement is to show that V_1 is N-isomorphic to V_2. By Lemma 6, V_i, $i=1$, 2, must both have a composition series of length $m(N)$. It will be shown by induction on $m(N)$ that V_1 is N-isomorphic to V_2. If $m(N)=0$, then, by Lemma 5, V_i, $i=1,2$, cannot have any minimal N-groups. This can only happen if the V_i, $i=1,2$, are $\{0\}$ and $N=\{0\}$. In this case V_1 is N-isomorphic to V_2.

We may therefore assume that $m(N)>0$ and thus, by Lemma 6, both V_i, $i=1,2$, are non-zero N-groups. Also V_1 clearly has a minimal N-subgroup. We now use Lemma 9 to exclude that situation where this minimal N-subgroup is unique. If such an N-subgroup is unique, then, by Lemma 8, N has a unique minimal ideal (T say). By 5, there exists a minimal right ideal $R \leqq T$ of N. However, Lemma 9 implies that both V_i, $i=1,2$, are N-isomorphic to $N/C_N(R)$ and V_1 is therefore N-isomorphic to V_2.

The proof of Theorem 1 has been reduced to showing that when V_i, $i=1,2$, are non-zero and V_1 has more than one minimal N-subgroup then V_1 is N-isomorphic to V_2. In the remainder of the proof a certain amount of extra notation will be required. In supplying this requirement we shall also obtain certain useful equations (see (d) below). Let U_{j1}, $j=1,2$, be two distinct minimal N-subgroups of V_1. By Lemma 10, U_{11} is not N-isomorphic to U_{21}. Also, by Lemma 11, N has a minimal right ideal N-isomorphic to U_{11}. Again, by Lemma 11, we can find a minimal N-subgroup U_{12} of V_2, N-isomorphic to U_{11}. Similarly V_2 contains a minimal N-subgroup U_{22}, N-isomorphic to U_{21}. Since U_{12} and U_{22} are not N-isomorphic, they are clearly distinct. Now take Y_i, $i=1,2$, to be the external direct sum of V_i/U_{1i} and V_i/U_{2i}. If H_{1i} is taken to be $(V_i/U_{1i},\{0\})$ and H_{2i} is taken as $(\{0\}, V_i/U_{2i})$, then, by 16, the $H_{ji}, j=1,2$, $i=1,2$, are submodules of Y_i, $i=1,2$, and $Y_i = H_{1i} \oplus H_{2i}$. Furthermore, by Lemma 16, we can define two N-group embeddings δ_i, $i=1,2$, of V_i into Y_i by setting $v_i\delta_i = (v_i + U_{1i}, v_i + U_{2i})$, for all v_i in V_i. If X_{1i} is taken as $((U_{1i}+U_{2i})/U_{1i},\{0\})$, i$=1,2$, and X_{2i} as $(\{0\},(U_{1i}+U_{2i})/U_{2i})$ then, by Lemma 16, X_{ji}, $j=1,2$, $i=1,2$, are submodules of Y_i and

$$V_i\delta_i + H_{ji} = Y_i \quad \text{and} \quad V_i\delta_i \cap H_{ji} = X_{ji}. \tag{d}$$

We now show how the equations (d) can be used in conjunction with certain other equations. If it is shown that there exists an N-isomorphism τ of Y_1 onto Y_2 having the properties

$$H_{j1}\tau = H_{j2} \quad \text{and} \quad X_{j1}\tau = X_{j2} \tag{e}$$

then, using (d), Theorem 1 will follow from Lemma 19. The actual construction of τ, which depends on the induction assumption, is postponed until the next paragraph.

It is sufficient for the moment to show that the existence of such an N-isomorphism shows V_1 is N-isomorphic to V_2. By (d), we have that for $j=1,2$, $V_1\delta_1+H_{j1}=Y_1$ and $V_1\delta_1\cap H_{j1}=X_{j1}$ for $j=1,2$. By (e), it follows on applying τ to these results, we obtain $V_1\delta_1\tau+H_{j2}=Y_2$, and $V_1\delta_1\tau\cap H_{j2}=X_{j2}$ for $j=1,2$. However, by (d) this implies

$$V_1\delta_1\tau+H_{j2}=V_2\delta_2+H_{j2} \quad \text{and} \quad V_1\delta_1\tau\cap H_{j2}=V_2\delta_2\cap H_{j2} \quad \text{for} \quad j=1,2.$$

Thus if in Lemma 19 we take $Y=Y_2$, $H_1=H_{12}$, $H_2=H_{22}$, $T_1=V_1\delta_1\tau$ and $T_2=V_2\delta_2$, then we obtain $V_1\delta_1\tau=V_2\delta_2$. This follows because the N-groups V_1 and V_2 are 2-tame and, since δ_2 and $\delta_1\tau$ are N-homomorphisms, $V_1\delta_1\tau$ and $V_2\delta_2$ are also 2-tame N-groups. Thus $V_1\delta_1\tau=V_2\delta_2$, where V_i, $i=1,2$, is N-isomorphic to $V_i\delta_i$ (the δ_i are embeddings). Since τ is an N-isomorphism, it follows that V_1 is N-isomorphic to V_2. It only remains to show that there exists an N-isomorphism τ of Y_1 onto Y_2 such that (e) holds.

The proof of the existence of τ will use the induction assumption and previous results. Let $S(U_{1i})$, $i=1,2$, be the right ideals defined as in Lemma 15. Since U_{11} is N-isomorphic to U_{12}, $S(U_{11})=S(U_{12})$. Thus, by Lemma 15, we have that $(U_{11}:V_1)=(U_{12}:V_2)$. Thus V_i/U_{1i}, $i=1,2$, are faithful 2-tame $N/(U_{11}:V_1)$-groups (i.e. $N/(U_{12}:V_2)$-groups). The near-ring $N/(U_{11}:V_1)$ (i.e. $N/(U_{12}:V_2)$) is ring-free. Furthermore, both the N-groups V_i/U_{1i}, $i=1,2$, have a composition series of length $m(N)-1$. By induction, we may assume that V_1/U_{11} is $N/(U_{11}:V_1)$-isomorphic (i.e. $N/(U_{12}:V_2)$-isomorphic) to V_2/U_{12}. Since $(U_{11}:V_1)(=(U_{12}:V_2))$ annihilates V_i/U_{1i}, $i=1,2$, we have that V_1/U_{11} is N-isomorphic (by λ_1 say) to V_2/U_{12}. Similarly, V_1/U_{21} is N-isomorphic (by λ_2 say) to V_2/U_{22}. We can now find an N-isomorphism τ of Y_1 into Y_2 as in the previous paragraph. For any (x_1,x_2) in Y_1, x_1 in V_1/U_{11} and x_2 in V_1/U_{21}, we define the map τ of Y_1 into Y_2 by setting $(x_1,x_2)\tau=(x_1\lambda_1,x_2\lambda_2)$. Clearly τ is an N-isomorphism of Y_1 onto Y_2. Since H_{11} is simply the submodule (see Lemma 16) $(V_1/U_{11}, \{0\})$ of Y_1, it follows that $H_{11}\tau=H_{12}$. Similarly $H_{21}\tau=H_{22}$. Thus the first set of equations in (e) hold. It remains to show that $X_{j1}\tau=X_{j2}$ for $j=1,2$. Now $X_{11}=((U_{11}+U_{21})/U_{11},\{0\})$ and this submodule of Y_1 is clearly N-isomorphic to U_{21}. Thus $X_{11}\tau$ is a submodule of Y_2, N-isomorphic to U_{21}. However $X_{11}\tau\leqq H_{11}\tau=H_{12}$. Thus $X_{11}\tau$ is a submodule of H_{12}, N-isomorphic to U_{21} (i.e. to U_{22}). Since H_{12} is N-isomorphic to V_2/U_{12}, H_{12} is a 2-tame N-group. Since $X_{12}=((U_{12}+U_{22})/U_{12}, \{0\})$ is N-isomorphic to U_{22} it follows, by Lemma 10, that $X_{11}\tau=X_{12}$. Similarly $X_{21}\tau=X_{22}$. The second set of equations in (e) also hold. From the previous paragraph V_1 is N-isomorphic to V_2 and Theorem 1 is completely proved.

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