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ON A RELATION BETWEEN THE "SQUARE" FUNCTIONAL EQUATION AND THE "SQUARE" MEAN-VALUE PROPERTY

BY
HIROSHI HARUKI

1. **Introduction.** We consider the following functional equation

$$f(x+t, y+t)+f(x-t, y+t) +f(x-t, y-t)+f(x+t, y-t) = 4f(x, y), \tag{1}$$

where $f=f(x, y)$ is a real-valued function of two real variables x, y on the whole xy-plane and t is a real variable.

With regard to the geometric meaning of (1), the equation is called the "square" functional equation.

The following theorem was proved in [1] by using distributions and analytic function theory (see also [3]):

THEOREM A. *Let f be a real-valued continuous function of two real variables x, y on the whole xy-plane; it satisfies* (1) *on the whole xy-plane, if and only if it is a harmonic polynomial of degree* 4, *i.e.*

$$f(x, y) = axy(x^2-y^2)+b(3x^2y-y^3) +c(x^3-3xy^2)+dxy+e(x^2-y^2)+fx+gy+h,$$

where a, b, c, d, e, f, g, h are arbitrary real constants.

A real-valued function U of two real variables x, y is said to have the Gauss mean-value property on the whole xy-plane if for every (x_0, y_0) the value $U(x_0, y_0)$ is the mean of the values of U over an arbitrary circle whose center is (x_0, y_0). Every function harmonic on the whole xy-plane possesses the Gauss mean-value property. Conversely (due to Koebe), if a function U is continuous on the whole xy-plane and has the Gauss mean-value property on the whole xy-plane, then U is harmonic on the whole xy-plane. Now we replace the circle by an arbitrary square whose sides are parallel to the coordinate axes. A real-valued function U of two real variables x, y is said to have the "square" mean-value property on the whole xy-plane if for every (x_0, y_0) the value $U(x_0, y_0)$ is the mean of the values of U over an arbitrary square whose center is (x_0, y_0) and whose sides are parallel

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to the coordinate axes, i.e.

$$\frac{1}{8l}\int_{ABCD} f(x, y)\, ds = f(G), \tag{2}$$

where $ABCD$ is an arbitrary square with center at G whose sides are parallel to the coordinate axes and $2l$ stands for the length of one of the sides of $ABCD$.

The purpose of this note is to prove the following

THEOREM. *If f is a real-valued continuous function of two real variables x, y on the whole xy-plane, then* (1) *is equivalent to* (2).

2. **A proof that (1) implies (2).** First of all we shall explain a notation which will be used in this proof. Suppose that $ABCD$ is an arbitrary square whose sides are parallel to the coordinate axes. We divide each of the four sides of this square into 2^n equal parts where n is an arbitrary natural number and denote the arithmetic mean of the $2^n\times 4=2^{n+2}$ values of f at these 2^{n+2} vertices of these 2^{n+2} division points by $M(ABCD, 2^{n+2}, f)$.

We shall prove

$$M(ABCD, 2^{n+2}, f) = f(G), \tag{3}$$

where G is the center of the square $ABCD$.

The proof depends on induction on n. Since (1) holds, we have (see [1, p. 43])

$$f(x+t, y)+f(x, y+t)+f(x-t, y)+f(x, y-t) = 4f(x, y). \tag{4}$$

By (1), (4) the result is true for $n=1$. Suppose that P, Q, R, S are the four middle points of the four sides AB, BC, CD, DA of the square $ABCD$ and that our theorem is true for $n=m$. We divide each of the four sides of the square $ABCD$ into 2^{m+1} equal parts. Considering the inductive hypothesis in the four squares $APGS$, $PBQG$, $GQCR$, $SGRD$, we have

$$M(APGS, 2^{m+2}, f) = f(G_1), \tag{5}$$

$$M(PBQG, 2^{m+2}, f) = f(G_2), \tag{6}$$

$$M(GQCR, 2^{m+2}, f) = f(G_3), \tag{7}$$

$$M(SGRD, 2^{m+2}, f) = f(G_4), \tag{8}$$

where G_1, G_2, G_3, G_4 are the four centers of the four squares $APGS$, $PBQG$, $GQCR$, $SGRD$, respectively.

Observing that the quadrilateral $G_1G_2G_3G_4$ is a square with center at G whose sides are parallel to the coordinate axes, by (1) we have

$$f(G_1)+f(G_2)+f(G_3)+f(G_4) = 4f(G). \tag{9}$$

By adding (5), (6), (7), (8), (9) side by side we have

$$\begin{aligned}(10)\qquad & M(APGS, 2^{m+2}, f)+M(PBQG, 2^{m+2}, f)+M(GQCR, 2^{m+2}, f)\\ & +M(SGRD, 2^{m+2}, f) = 4f(G).\end{aligned}$$

Now we note that each of the division points for $M(APGS, 2^{m+2}, f)$, $M(PBQG, 2^{m+2}, f)$, $M(GQCR, 2^{m+2}, f)$, $M(SGRD, 2^{m+2}, f)$ is a division point for $M(ABCD, 2^{m+3}, f)$; considering the overlapping division points and using (4), we have

$$\begin{aligned}& M(APGS, 2^{m+2}, f)+M(PBQG, 2^{m+2}, f)+M(GQCR, 2^{m+2}, f)\\ (11)\qquad & +M(SGRD, 2^{m+2}, f) = 2M(ABCD, 2^{m+3}, f)+\frac{1}{2^{m+2}}4f(G)\\ & +\frac{1}{2^{m+2}}4f(G)+\frac{1}{2^{m+2}}2(2^m-1)4f(G).\end{aligned}$$

By (10), (11) we have

$$M(ABCD, 2^{m+3}, f) = f(G).$$

Thus (3) is proved.

As $n \to +\infty$ in (3), by the continuity of f we have (2).

3. **A proof that (2) implies (1).** (See [2].) We shall use the following:

LEMMA. *Suppose that f is a real-valued continuous function of two real variables x, y on the whole xy-plane. If f satisfies* (2), *then*

$$\frac{1}{4l^2}\iint_{ABCD} f(x, y)\,dx\,dy = f(G).$$

Proof. Suppose that $A_1B_1C_1D_1$ is a square whose sides are parallel to the sides of $ABCD$ and whose center is G. Then we have

$$\begin{aligned}(12)\qquad \frac{1}{4l^2}\iint_{ABCD} f(x, y)\,dx\,dy = \frac{1}{4l^2}\Big(&\iint_{\triangle GAB} f(x, y)\,dx\,dy+\iint_{\triangle GBC} f(x, y)\,dx\,dy\\ &+\iint_{\triangle GCD} f(x, y)\,dx\,dy+\iint_{\triangle GDA} f(x, y)\,dx\,dy\Big).\end{aligned}$$

Using the well-known theorem concerning repeated integration in each of the four integrals of the right side of (12), the right side of (12) is equal to

$$\frac{1}{4l^2}\int_0^l\left(\int_{A_1B_1C_1D_1} f(x, y)\,ds\right)dh$$

where we denote the differential of the arc length by ds and denote the length of

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one of the sides of $A_1B_1C_1D_1$ by $2h$. Hence, by (12) we have

$$\frac{1}{4l^2}\iint\limits_{ABCD} f(x,y)\,dx\,dy = \frac{1}{4l^2}\int_0^l \left(\int_{A_1B_1C_1D_1} f(x,y)\,ds\right) dh. \tag{13}$$

By hypothesis we have

$$\int_{A_1B_1C_1D_1} f(x,y)\,ds = 8hf(G). \tag{14}$$

Hence, by (13), (14) the lemma is proved.

Proof that (2) implies (1). We denote the four middle points of the four sides AB, BC, CD, DA of $ABCD$ by P, Q, R, S, respectively. Furthermore, we denote the four centers of the four squares $APGS$, $PBQG$, $GQCR$, $SGRD$ by G_1, G_2, G_3, G_4, respectively.

By hypothesis and by the above lemma we have

$$\frac{1}{l^2}\iint\limits_{APGS} f(x,y)\,dx\,dy = f(G_1), \tag{15}$$

$$\frac{1}{l^2}\iint\limits_{PBQG} f(x,y)\,dx\,dy = f(G_2), \tag{16}$$

$$\frac{1}{l^2}\iint\limits_{GQCR} f(x,y)\,dx\,dy = f(G_3), \tag{17}$$

$$\frac{1}{l^2}\iint\limits_{SGRD} f(x,y)\,dx\,dy = f(G_4), \tag{18}$$

$$\frac{1}{4l^2}\iint\limits_{ABCD} f(x,y)\,dx\,dy = f(G). \tag{19}$$

Adding (15), (16), (17), (18), and using (19), we have

$$f(G_1)+f(G_2)+f(G_3)+f(G_4) = 4f(G). \tag{20}$$

Since G is the center of the square $G_1G_2G_3G_4$ whose sides are parallel to the coordinate axes and we can consider that $G_1G_2G_3G_4$ is an arbitrary square whose sides are parallel to the coordinate axes, by (20) we have (1).

COROLLARY TO THEOREM. *Suppose that f is a real-valued continuous function of two real variables x, y on the whole xy-plane. The function f satisfies* (2), *if and only if*

$$f(x,y) = axy(x^2-y^2)+b(3x^2y-y^3)+c(x^3-3xy^2)$$
$$+dxy+e(x^2-y^2)+fx+gy+h,$$

where a, b, c, d, e, f, g, h are arbitrary real constants.

Proof. By Theorem A and the above theorem the proof is clear.

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REFERENCES

1. J. Aczél, H. Haruki, M. A. McKiernan, and G. N. Sakovič, *General and regular solutions of functional equations characterizing harmonic polynomials*, Aequationes Math. **1** (1968), 37–53.

2. H. Haruki, *On a certain definite integral mean value problem* (in Japanese), Sûgaku, **20** (1968), 165–166.

3. H. Światak, *On the regularity of the distributional and continuous solutions of the functional equations* $\sum_{i=1}^{k} a_i(x, t)f(x+\varphi_i(t))=b(x, t)$, Aequationes Math. **1** (1968), 6–19.

UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO

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