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DENJOY-BOCHNER ALMOST PERIODIC FUNCTIONS

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Abstract

The special Denjoy-Bochner integral (the D*B-integral) which are generalisations of Lebesgue-Bochner integral are discussed in [7, 6, 5]. Just as the concept of numerical almost periodicity was extended by Burkill [3] to numerically valued D*- or D-integrable function, we extend the concept of almost periodicity for Banach valued function to Banach valued D*B-integrable function. For this purpose we introduce as in [3] a distance in the space of all D*B-integrable functions with respect to which the D*B-almost periodicity is defined. It is shown that the D*B-almost periodicity shares many of the known properties of the almost periodic Banach valued function [1, 4].

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1. Definitions and terminology

For the definition of almost periodicity for numerical valued and Banach valued functions we refer to [2] and [1, 4] respectively. Throughout the paper **R** and **C** will denote the real line and the complex plane and **X** will denote a fixed complex Banach space with norm $\|\cdot\|$. For a function f defined on **R**, f_η will denote the translation of f by the number η; that is, $f_\eta(x) = f(x + \eta)$.

DEFINITION 1.1 [3]. Let $\mathfrak{D}^*$ be the class of all functions f: $\mathbf{R} \to \mathbf{C}$ such that f is D*-integrable on each closed interval $[a, b] \subset \mathbf{R}$. For $f, g \in \mathfrak{D}^*$ the D* distance

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between f and g is defined to be

$$\rho_{D^*}(f, g) = \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left| (D^*) \int_x^{x+h} \{f(t) - g(t)\}\, dt \right|.$$

A function $f \in \mathfrak{D}^*$ is *almost periodic in the sense of the* D* *distance* (or simply D* a.p.) if, given $\varepsilon > 0$ there is a relatively dense set $\{\tau\}$ such that

$$\rho_{D^*}(f_\tau, f) < \varepsilon$$

for all $\tau \in \{\tau\}$.

DEFINITION 1.2 [7, 6, 5]. A function f: $[a, b] \to \mathbf{X}$ is said to be *special Denjoy-Bochner integrable* or D*B-integrable in $[a, b]$ if there is a function F: $[a, b] \to \mathbf{X}$ such that F is strongly ACG_* on $[a, b]$ and $\mathrm{AD}_s F = f$ almost everywhere in $[a, b]$ where $\mathrm{AD}_s F$ stands for the strong approximate derivative of F. The function F is then called an indefinite D*B-integral of f on $[a, b]$ and $F(b) - F(a)$ is called its definite D*B-integral on $[a, b]$ and is denoted by

$$(D^*B) \int_a^b f(\xi)\, d\xi.$$

DEFINITION 1.3. Let $\mathfrak{D}^*\mathfrak{B}$ be the class of all functions f: $\mathbf{R} \to \mathbf{X}$ such that f is D*B-integrable on each closed interval $[a, b] \subset \mathbf{R}$. For $f, g \in \mathfrak{D}^*\mathfrak{B}$ the D*B distance between f and g is defined to be

$$\rho_{D^*B}(f, g) = \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| (D^*B) \int_x^{x+h} \{f(t) - g(t)\}\, dt \right\|.$$

A function $f \in \mathfrak{D}^*\mathfrak{B}$ is said to be *almost periodic in the sense of the* D*B-*distance* (or, simply D*B a.p.) if, given $\varepsilon > 0$ there is a relatively dense set $\{\tau\} = \{\tau; f, \varepsilon\}$ such that

$$\rho_{D^*B}(f_\tau, f) < \varepsilon$$

for all $\tau \in \{\tau\}$. Clearly every almost periodic function f: $\mathbf{R} \to \mathbf{X}$ is D*B a.p.

REMARK. This definition of the D*B-distance, of course, does not guarantee that

$$\rho_{D^*B}(f, g) < \infty$$

for all $f, g \in \mathfrak{D}^*\mathfrak{B}$. We shall, however, prove that every D*B a.p. function f is D*B-bounded, that is

$$\rho_{D^*B}[f] = \rho_{D^*B}(f, \theta) < \infty$$

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from which it will follow that for all D*B a.p. functions f and g

$$\rho_{D^*B}(f, g) < \infty.$$

DEFINITION 1.4. A continuous function ϕ: $\mathbf{R} \times [0,1] \to \mathbf{X}$ is called *almost periodic in* $x \in \mathbf{R}$ *uniformly with respect to* $h \in [0,1]$ if to arbitrary $\varepsilon > 0$ corresponds a relatively dense set $\{\tau\}$ such that

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \|\phi(x+\tau, h) - \phi(x, h)\| < \varepsilon$$

for all $\tau \in \{\tau\}$.

The following result for integration by parts for the D*B-integral, which will be needed later, is proved in [5].

THEOREM 1.5. *Let* f: $[a, b] \to \mathbf{X}$ *be* D*B-*integrable and*

$$F(\xi) = \int_a^\xi f(t)\,dt.$$

Let g: $[a, b] \to \mathbf{R}$ *be L-integrable and let*

$$G(\xi) = \int_a^\xi g(t)\,dt.$$

Then fG *is* D*B-*integrable over* $[a, b]$ *and*

$$\int_a^b fG = [FG]_a^b - \int_a^b Fg.$$

2. Properties of D*B a.p. functions

THEOREM 2.1. *If a function* f *is* D*B a.p. *then*

$$F(x) = \int_0^x f(t)\,dt$$

is uniformly continuous.

Since the D*B-integral,

$$F(x) = \int_0^x f(t)\,dt,$$

is continuous and since a continuous Banach valued function is uniformly continuous on a closed interval the theorem can be proved by the usual process.

THEOREM 2.2. *If* f *is* D*B a.p. *then the function* ϕ: $\mathbf{R} \times [0,1] \to \mathbf{X}$ *defined by*

$$\phi(x, h) = \int_x^{x+h} f(t)\,dt$$

is almost periodic in $x \in \mathbf{R}$, *uniformly with respect to* $h \in [0,1]$.

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PROOF. We first show that the function ϕ is continuous. Let $\varepsilon > 0$ be arbitary. Since by Theorem 2.1 $F(x) = \int_0^x f(t)\,dt$ is uniformly continuous, there is a $\delta > 0$ such that $\|F(x_1) - F(x_2)\| < \varepsilon/2$ whenever $|x_1 - x_2| < \delta$ for all $x_1, x_2 \in \mathbf{R}$. Now, let $(x_0, h_0) \in \mathbf{R} \times [0, 1]$ be arbitrary. Then

$$\begin{aligned}\|\phi(x_0, h_0) - \phi(x, h)\| &= \left\| \int_{x_0}^{x_0+h_0} f(t)\,dt - \int_x^{x+h} f(t)\,dt \right\| \\ &= \|F(x_0 + h_0) - F(x_0) - F(x + h) + F(x)\| \\ &\leqslant \|F(x_0 + h_0) - F(x + h)\| + \|F(x) - F(x_0)\| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon\end{aligned}$$

whenever $|x - x_0| < \delta/2$, $|h - h_0| < \delta/2$. Hence $\phi(x, h)$ is continuous on $\mathbf{R} \times [0, 1]$.

Now, since f is D*B a.p., corresponding to $\varepsilon > 0$ there is a relatively dense set $\{\tau\}$ such that $\rho_{\mathrm{D^*B}}(f_\tau, f) < \varepsilon$ for all $\tau \in \{\tau\}$. Hence

$$\sup_{\substack{0 \leqslant h \leqslant 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} f(t + \tau)\,dt - \int_x^{x+h} f(t)\,dt \right\| < \varepsilon,$$

that is,

$$\sup_{\substack{0 \leqslant h \leqslant 1 \\ -\infty < x < \infty}} \left\| \int_{x+\tau}^{x+\tau+h} f(t)\,dt - \int_x^{x+h} f(t)\,dt \right\| < \varepsilon$$

from which it follows that

$$\sup_{\substack{0 \leqslant h \leqslant 1 \\ -\infty < x < \infty}} \|\phi(x + \tau, h) - \phi(x, h)\| < \varepsilon,$$

which completes the proof.

LEMMA 2.3. *Let $\mathcal{C}_{\mathbf{X}}[0, 1]$ be the Banach space of all continuous functions y: $[0, 1] \to \mathbf{X}$ with norm*

$$\|y\|_{\mathcal{C}_{\mathbf{X}}} = \sup_{0 \leqslant h \leqslant 1} \|y(h)\|$$

and let ϕ: $\mathbf{R} \times [0, 1] \to \mathbf{X}$ be a continuous function. Then the function Φ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ defined by

$$\Phi(x) = \phi(x, \cdot)$$

is almost periodic if and only if the function ϕ is almost periodic in $x \in \mathbf{R}$, uniformly with respect to $h \in [0, 1]$.

PROOF. Since $\|\Phi(x)\|_{\mathcal{C}_{\mathbf{X}}} = \sup_{0 \leq h \leq 1} \|\phi(x, h)\|$ we have

$$\|\Phi(x + \tau) - \Phi(x)\|_{\mathcal{C}_{\mathbf{X}}} = \sup_{0 \leq h \leq 1} \|\phi(x + \tau, h) - \phi(x, h)\|$$

and so the result follows.

LEMMA 2.4. *If the continuous functions* ϕ: $\mathbf{R} \times [0, 1] \to \mathbf{X}$ *and* ψ: $\mathbf{R} \times [0, 1] \to \mathbf{X}$ *are almost periodic in* $x \in \mathbf{R}$ *uniformly with respect to* $h \in [0, 1]$ *then* $\phi + \psi$ *is so.*

PROOF. Let $\mathcal{C}_{\mathbf{X}}[0, 1]$ be as in Lemma 2.3 and let Φ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ and Ψ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ be defined by

$$\Phi(x) = \phi(x, \cdot), \quad \Psi(x) = \psi(x, \cdot).$$

Then by Lemma 2.3, Φ and Ψ are almost periodic and so is the sum $\Phi + \Psi$, and hence by Lemma 2.3, $\phi + \psi$ is almost periodic in $x \in \mathbf{R}$ uniformly with respect to $h \in [0, 1]$.

THEOREM 2.5. *If f and g are* D*B a.p. *then so is* $f + g$.

PROOF. By Theorem 2.2, the functions $\phi(x, h) = \int_x^{x+h} f(t)\, dt$ and $\psi(x, h) = \int_x^{x+h} g(t)\, dt$ are almost periodic in $x \in \mathbf{R}$ uniformly with respect to $h \in [0, 1]$. Hence by Lemma 2.4, $\phi(x, h) + \psi(x, h)$ is almost periodic in $x \in \mathbf{R}$ uniformly with rspect to $h \in [0, 1]$. So, given $\varepsilon > 0$, there is a relatively dense set $\{\tau\}$ such that

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \|\phi(x + \tau, h) + \psi(x + \tau, h) - \phi(x, h) - \psi(x, h)\| < \varepsilon$$

for all $\tau \in \{\tau\}$. Hence

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_{x+\tau}^{x+\tau+h} \{f(t) + g(t)\}\, dt - \int_x^{x+h} \{f(t) + g(t)\}\, dt \right\| < \varepsilon,$$

that is,

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} [\{f(t + \tau) + g(t + \tau)\} - \{f(t) + g(t)\}]\, dt \right\| < \varepsilon,$$

that is,

$$\rho_{\mathrm{D^*B}}((f + g)_\tau, f + g) < \varepsilon$$

for all $\tau \in \{\tau\}$. Hence $f + g$ is D*B a.p.

THEOREM 2.6. *If f is* D*B a.p. *then f is* D*B *bounded, that is,*

$$\rho_{\mathrm{D^*B}}[f] = \rho_{\mathrm{D^*B}}(f, \theta) < \infty.$$

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PROOF. Letting $\phi(x, h) = \int_x^{x+h} f(t)\,dt$ and constructing the function Φ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ as in Lemma 2.3 we see Φ is almost periodic. Then by [1, page 5, property IV], the range of Φ is relatively compact and hence

$$\sup_{-\infty < x < \infty} \|\Phi(x)\|_{\mathcal{C}_{\mathbf{X}}} < \infty.$$

Hence by the definition of $\|\cdot\|_{\mathcal{C}_{\mathbf{X}}}$

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \|\phi(x, h)\| < \infty,$$

that is,

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} f(t)\,dt \right\| < \infty,$$

that is,

$$\rho_{\mathrm{D^*B}}(f, \theta) < \infty.$$

THEOREM 2.7. *If f is* D*B a.p. *then f is uniformly continuous with respect to the metric $\rho_{\mathrm{D^*B}}$; that is, for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\rho_{\mathrm{D^*B}}(f_\eta, f) < \varepsilon$$

for all η satisfying $|\eta| < \delta$.

PROOF. Since f is D*B a.p. by Theorem 2.2 and Lemma 2.3 the function Φ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ defined by $\Phi(x) = \phi(x, \cdot)$ is almost periodic, where $\phi(x, h) = \int_x^{x+h} f(t)\,dt$. By [1, page 5, property III], Φ is uniformly continuous. So, for arbitrary $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{-\infty < x < \infty} \|\Phi(x + \eta) - \Phi(x)\|_{\mathcal{C}_{\mathbf{X}}} < \varepsilon$$

for all η satisfying $|\eta| < \delta$. That is,

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \|\phi(x + \eta, h) - \phi(x, h)\| < \varepsilon,$$

that is,

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} \{f(t + \eta) - f(t)\}\,dt \right\| < \varepsilon,$$

that is,

$$\rho_{D^*B}(f_\eta, f) < \varepsilon$$

whenever $|\eta| < \delta$.

THEOREM 2.8. *If $\{f_n\}$ is a sequence of* D*B a.p. *functions such that $f_n \to f$ with respect to the metric ρ_{D^*B} then f is* D*B a.p.

PROOF. Let $\varepsilon > 0$ be arbitrary. Then there is N such that $\rho_{D^*B}(f_n, f) < \varepsilon/3$ for all $n \geqslant N$. Since f_N is D*B a.p. so there is a relatively dense set $\{\tau\}$ for which $\rho_{D^*B}((f_N)_\tau, f_N) < \varepsilon/3$. Hence

$$\begin{aligned}\rho_{D^*B}(f_\tau, f) &\leqslant \rho_{D^*B}(f_\tau, (f_N)_\tau) + \rho_{D^*B}((f_N)_\tau, f_N) + \rho_{D^*B}(f_N, f)\\ &= \rho_{D^*B}(f, f_N) + \rho_{D^*B}((f_N)_\tau, f_N) + \rho_{D^*B}(f_N, f)\\ &< \varepsilon.\end{aligned}$$

Thus f is D*B a.p.

THEOREM 2.9. *If f is* D*B a.p. *and $u(x)$ is almost periodic numerical valued function with its derivative $u'(x)$ uniformly continuous then $f(x)u(x)$ is* D*B a.p.

The proof of the theorem is similar to that of the corresponding theorem of [3]. In fact all the arguments of [3] will apply in this case taking into account the fact that the integration by parts formula for integral is given in Theorem 1.5.

LEMMA 2.10. *If f is* D*B a.p. *then x^*f is* D* a.p. *for every $x^* \in \mathbf{X}^*$, where $\mathbf{X}^*$ is the conjugate space of the Banach space* **X**.

PROOF. Take any $x^* \in \mathbf{X}^*$ and $\varepsilon > 0$. Then there corresponds a relatively dense set $\{\tau\} = \{\tau; f, \varepsilon(\|x^*\| + 1)^{-1}\}$ such that

$$\sup_{\substack{0 \leqslant h \leqslant 1\\ -\infty < x < \infty}} \left\| \int_x^{x+h} \{f(t+\tau) - f(t)\}\, dt \right\| < \varepsilon(\|x^*\| + 1)^{-1}$$

for all $\tau \in \{\tau\}$. Now since f is D*B a.p., f is D*B-integrable on each closed interval $[a, b]$ and so by a result of [5] x^*f is D*-integrable on each $[a, b]$ and therefore $x^*f \in \mathfrak{D}^*$. Moreover

$$x^* \int_x^{x+h} f(t)\, dt = \int_x^{x+h} x^* f(t)\, dt$$

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for all $x \in \mathbf{R}$ and $h \in [0, 1]$. Hence for all $\tau \in \{\tau\}$

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left| \int_x^{x+h} \{x^* f(t+\tau) - x^* f(t)\}\, dt \right|$$

$$= \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left| \int_x^{x+h} x^* \{f(t+\tau) - f(t)\}\, dt \right|$$

$$= \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left| x^* \int_x^{x+h} \{f(t+\tau) - f(t)\}\, dt \right|$$

$$\leq \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \|x^*\| \left\| \int_x^{x+h} \{f(t+\tau) - f(t)\}\, dt \right\|$$

$$= \|x^*\| \sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} \{f(t+\tau) - f(t)\}\, dt \right\|$$

$$< \|x^*\| \varepsilon (\|x^*\| + 1)^{-1} < \varepsilon.$$

This completes the proof of the lemma.

LEMMA 2.11. *If* $x^* f$ *is* D* a.p. *for all* $x^* \in \mathbf{X}^*$ *and if*

$$F(t) = \int_0^t f(x)\, dx$$

is bounded then F *is weakly almost periodic* (*that is,* $x^* F$ *is almost periodic for all* $x^* \in \mathbf{X}^*$).

PROOF. The function $F(t)$ being bounded $x^* F(t)$ is also bounded for all $x^* \in \mathbf{X}^*$ and since

$$x^* F(t) = (\mathrm{D}^*) \int_0^t x^* f(x)\, dx,$$

$x^* F$ is almost periodic by [3], that is, F is weakly almost periodic.

THEOREM 2.12. *If* f *is* D*B a.p. *and if*

$$F(t) = \int_0^t f(x)\, dx$$

is such that the range of F *is relatively compact then* F *is almost periodic.*

PROOF. By Lemma 2.10, $x^* f$ is D* a.p. for all $x^* \in \mathbf{X}^*$. The range of F being relatively compact (that is, its closure being compact) F is bounded. Hence by Lemma 2.11, F is weakly almost periodic. So by [1, page 45, property X] F is almost periodic.

THEOREM 2.13. *The class of all* D*B a.p. *functions is identical with the* D*B-*closure of the set of all trigonometric polynomials*

$$P(t) = \sum_{r=1}^{n} a_r e^{i\lambda_r t}$$

where $a_r \in \mathbf{X}$, $\lambda_r \in \mathbf{R}$.

The theorem can be proved in the same way as the corresponding theorem of D a.p. functions of [3].

THEOREM 2.14. *If f is* D*B a.p. *and is uniformly continuous then f is almost periodic.*

PROOF. Let ϕ: $\mathbf{R} \to \mathbf{R}$ be a nonnegative function with support $[0, 1]$ having continuous derivative ϕ' such that $\int_0^1 \phi(t)\,dt = 1$. For a fixed n let $\phi_n(x) = n\phi(nx)$. Then ϕ_n is a nonnegative function with support $[0, 1/n]$ having continuous derivative ϕ_n' and $\int_0^{1/n} \phi_n(t)\,dt = 1$. Let

$$f_n(x) = \int_0^{1/n} f(t + x)\phi_n(t)\,dt.$$

Then we shall show that f_n is almost periodic for each n. Let n be fixed and let $\varepsilon > 0$ be arbitrary. Let

$$M = \sup_{0 \leq x \leq 1} |\phi_n(x)|, \qquad M' = \sup_{0 \leq x \leq 1} |\phi_n'(x)|.$$

Since f is D*B a.p. there is a relatively dense set $\{\tau\}$ such that

$$\rho_{D^*B}(f_\tau, f) < \varepsilon(M + M')^{-1} \tag{2.1}$$

for all $\tau \in \{\tau\}$. Let $\tau \in \{\tau\}$. Then writing $F(x) = \int_0^x f(t)\,dt$ and $\psi(x) = F(x + \tau) - F(x)$ we have employing Theorem 1.5

$$\begin{aligned}
(2.2)\quad & \|f_n(x + \tau) - f_n(x)\| \\
&= \left\| \int_0^{1/n} \{f(t + x + \tau) - f(t + x)\}\phi_n(t)\,dt \right\| \\
&= \Big\| \left[\phi_n(t)\{F(t + x + \tau) - F(t + x)\}\right]_{t=0}^{1/n} \\
&\qquad\qquad - \int_0^{1/n} \{F(t + x + \tau) - F(t + x)\}\phi_n'(t) \Big\| \\
&= \left\| \left[\phi_n(t)\psi(x + t)\right]_{t=0}^{1/n} - \int_0^{1/n} \psi(x + t)\phi_n'(t)\,dt \right\| \\
&= \left\| \left[\phi_n(t)\{\psi(x + t) - \psi(x)\}\right]_0^{1/n} - \int_0^{1/n} \{\psi(x + t) - \psi(x)\}\phi_n'(t)\,dt \right\| \\
&= \left\| \phi_n\left(\frac{1}{n}\right)\left\{\psi\left(x + \frac{1}{n}\right) - \psi(x)\right\} - \int_0^{1/n} \{\psi(x + t) - \psi(x)\}\phi_n'(t)\,dt \right\|.
\end{aligned}$$

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Now let $t \in [0, 1/n]$. Then from (2.1)

$$\|\psi(x+t)-\psi(x)\| = \left\|\int_x^{x+t}\{f(\xi+t)-f(\xi)\}\,d\xi\right\| \leqslant \rho_{\mathrm{D^*B}}(f_\tau, f) < \varepsilon(M+M')^{-1}.$$

Hence from (2.2)

$$\|f_n(x+\tau)-f_n(x)\| < M\varepsilon(M+M')^{-1} + M'\varepsilon(M+M')^{-1} = \varepsilon.$$

Since $\tau \in \{\tau\}$ is arbitrary, f_n is almost periodic for each n.

Now since f is uniformly continuous, for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|f(t+x)-f(x)\| < \varepsilon$$

whenever $|t| < \delta$. Choose N such that $1/N \leqslant \delta$. Then when $n \geqslant N$ we have

$$\begin{aligned}\|f_n(x)-f(x)\| &= \left\|\int_0^{1/n} f(t+x)\phi_n(t)\,dt - \int_0^{1/n} f(x)\phi_n(t)\,dt\right\| \\ &= \left\|\int_0^{1/n}\{f(t+x)-f(x)\}\phi_n(t)\,dt\right\| \\ &\leqslant \int_0^{1/n}\|f(t+x)-f(x)\|\phi_n(t)\,dt \\ &< \varepsilon.\end{aligned}$$

Thus $\{f_n\}$ converges uniformly to f. Since each f_n is almost periodic, by [1, page 6, property V] f is almost periodic.

3. Mean values and Fourier series

THEOREM 3.1. *If f is* D*B a.p. *then the mean value*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt = M(f)$$

exists; further

$$\lim_{T\to\infty}\frac{1}{T}\int_a^{a+T} f(t)\,dt = M(f)$$

uniformly with respect to $a \in \mathbf{R}$.

PROOF. Since

$$\frac{1}{T}\int_a^{a+T} e^{i\lambda t}\,dt = \begin{cases} 1 & \text{if } \lambda = 0, \\ \dfrac{1}{i\lambda T}\left[e^{i\lambda(a+T)} - e^{i\lambda a}\right] & \text{if } \lambda \neq 0, \end{cases}$$

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it is clear that

$$\lim_{T\to\infty}\frac{1}{T}\int_a^{a+T} e^{i\lambda t}\,dt = \begin{cases} 1 & \text{if } \lambda = 0, \\ 0 & \text{if } \lambda \neq 0 \end{cases}$$

uniformly with respect to $a \in \mathbf{R}$ and hence for any trigonometric polynomial P,

$$P(t) = \sum_{r=1}^{n} a_r e^{i\lambda_r t} \qquad (a_r \in \mathbf{X}, \lambda_r \in \mathbf{R}),$$

$$\lim_{T\to\infty}\frac{1}{T}\int_a^{a+T} P(t)\,dt = M(P)$$

exists uniformly with respect to $a \in \mathbf{R}$. Let $\varepsilon > 0$ be arbitrary. By Theorem 2.13 there is a trigonometric polynomial P such that $\rho_{\mathrm{D^*B}}(f, P) < \varepsilon$. Hence

$$\sup_{\substack{0 \leqslant h \leqslant 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} \{f(t) - P(t)\}\,dt \right\| < \varepsilon. \tag{3.1}$$

Now corresponding to ε there is T_ε which is independent of a, such that

$$\left\| \frac{1}{T'}\int_a^{a+T'} P(t)\,dt - \frac{1}{T''}\int_a^{a+T''} P(t)\,dt \right\| < \varepsilon \tag{3.2}$$

for all $T', T'' > T_\varepsilon$.

Set $T_0 = \max[T_\varepsilon, 2]$ and let $T_1, T_2 > T_0$. Then there is a positive integer N such that $N - 1 < T_1 \leqslant N$. Putting $h = T_1/N$, since $N > 2$, we have $\frac{1}{2} < h \leqslant 1$. Now by (3.1) we have

$$\begin{aligned} \left\| \frac{1}{T_1}\int_a^{a+T_1} \{f(t) - P(t)\}\,dt \right\| &= \left\| \frac{1}{Nh}\int_a^{a+Nh} \{f(t) - P(t)\}\,dt \right\| \\ &= \left\| \frac{1}{Nh}\sum_{n=1}^{N}\int_{a+(n-1)h}^{a+nh} \{f(t) - P(t)\}\,dt \right\| \\ &\leqslant \frac{1}{Nh}\sum_{n=1}^{N}\left\| \int_{a+(n-1)h}^{a+nh} \{f(t) - P(t)\}\,dt \right\| \\ &< \frac{1}{Nh} N\varepsilon \\ &< 2\varepsilon \end{aligned} \tag{3.3}$$

since $1/h < 2$. Similarly for $T_2 > T_0$,

$$\left\| \frac{1}{T_2}\int_a^{a+T_2} \{f(t) - P(t)\}\,dt \right\| < 2\varepsilon. \tag{3.4}$$

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Since $T_0 \geqslant T_\varepsilon$ we have from (3.2), (3.3) and (3.4) when $T_1, T_2 > T_0$,

$$\begin{aligned}\left\| \frac{1}{T_1}\int_a^{a+T_1} f(t)\,dt - \frac{1}{T_2}\int_a^{a+T_2} f(t)\,dt \right\| &\leqslant \left\| \frac{1}{T_1}\int_a^{a+T_1} f(t)\,dt - \frac{1}{T_1}\int_a^{a+T_1} P(t)\,dt \right\| \\ &\quad + \left\| \frac{1}{T_1}\int_a^{a+T_1} P(t)\,dt - \frac{1}{T_2}\int_a^{a+T_2} P(t)\,dt \right\| \\ &\quad + \left\| \frac{1}{T_2}\int_a^{a+T_2} f(t)\,dt - \frac{1}{T_2}\int_a^{a+T_2} P(t)\,dt \right\| \\ &< 5\varepsilon.\end{aligned}$$

Thus since **X** is complete and since T_0 is independent of a,

$$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(t)\,dt = M(f)$$

exists uniformly with respect to $a \in \mathbf{R}$, completing the proof.

Now if f is D*B a.p. then since $u(x) = e^{-i\lambda x}$ is numerically valued almost periodic function and $u'(x)$ is uniformly continuous, by Theorem 2.9 $f(x)e^{-i\lambda x}$ is D*B a.p. for all $\lambda \in \mathbf{R}$ and consequently

$$M\{f(x)e^{-i\lambda x}\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x)e^{-i\lambda x}\,dx$$

exists for every $\lambda \in \mathbf{R}$. For a D*B a.p. function f we shall write

$$a(\lambda) = a(\lambda; f) = M\{f(x)e^{-i\lambda x}\}.$$

THEOREM 3.2. *If f is* D*B a.p. *then $a(\lambda; f)$ differs from the zero element θ of* **X** *for only an enumerable set of values of λ.*

PROOF. Let

$$F(x) = \int_0^x f(t)\,dt.$$

Then for a given $h \in [0, 1]$ we have, by integrating by parts by Theorem 1.5.

$$\begin{aligned}\frac{1}{T}\int_0^T \{f(x+h) - f(x)\}e^{-i\lambda x}\,dx &= \frac{1}{T}\left[\{F(x+h) - F(x)\}e^{-i\lambda x}\right]_0^T \\ &\quad + \frac{i\lambda}{T}\int_0^T \{F(x+h) - F(x)\}e^{-i\lambda x}\,dx.\end{aligned} \tag{3.5}$$

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Also

$$\frac{1}{T}\int_0^T \{f(x+h)-f(x)\}e^{-i\lambda x}\,dx = \frac{1}{T}\left[e^{i\lambda h}\int_h^{T+h} f(t)e^{-i\lambda t}\,dt - \int_0^T f(t)e^{-i\lambda t}\,dt\right]. \tag{3.6}$$

Now by Theorem 2.2 the function $F(x+h)-F(x)$ is almost periodic. Let its Fourier coefficients be $\alpha_h(\lambda)$. Then applying Theorem 3.1 we get from (3.5) and (3.6), by letting $T\to\infty$ since $F(x+h)-F(x)$ is bounded,

$$(e^{i\lambda h}-1)a(\lambda; f) = i\lambda\alpha_h(\lambda). \tag{3.7}$$

So, if $\lambda \neq 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

$$a(\lambda; f) = \frac{i\lambda}{e^{i\lambda h}-1}\alpha_h(\lambda).$$

Since $\alpha_h(\lambda)\neq\theta$ for at most enumerable number of λ, $a(\lambda)\neq\theta$ for these enumerable λ and probably for $\lambda = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. Thus $a(\lambda)$ differs from θ for at most an enumerable set of values of λ. This completes the proof of the theorem.

Let $\{\lambda_n\}$ be the enumerable set such that $a(\lambda_n)\neq\theta$. Putting $a_n = a(\lambda_n)$ we say that $\Sigma a_n e^{i\lambda_n x}$ is the Fourier series of f and write

$$f \sim \sum_n a_n e^{i\lambda_n x}.$$

LEMMA 3.3. *If f is* D*B a.p. *and $x^* \in \mathbf{X}^*$ then*

$$x^*a(\lambda; f) = a(\lambda; x^*f).$$

PROOF.

$$\begin{aligned} x^*a(\lambda; f) &= x^*M\{f(x)e^{-i\lambda x}\} \\ &= x^* \lim_{T\to\infty}\frac{1}{T}(\mathrm{D^*B})\int_0^T f(x)e^{-i\lambda x}\,dx \\ &= \lim_{T\to\infty}\frac{1}{T}x^*(\mathrm{D^*B})\int_0^T f(x)e^{-i\lambda x}\,dx \end{aligned}$$

since x^* is continuous. Now since a Denjoy-Bochner integrable function is Denjoy-Pettis integrable with integrals equal [5], we have

$$x^*(\mathrm{D^*B})\int_0^T f(x)e^{-i\lambda x}\,dx = (\mathrm{D^*})\int_0^T x^*f(x)e^{-i\lambda x}\,dx$$

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and hence

$$
\begin{aligned}
x^*a(\lambda; f) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T x^*f(x)e^{-i\lambda x}\,dx \\
&= M\{x^*f(x)e^{-i\lambda x}\} \\
&= a(\lambda; x^*f).
\end{aligned}
$$

THEOREM 3.4 (Uniqueness Theorem). *If two* D*B a.p. *functions f and g have same Fourier series then*

$$\rho_{\mathrm{D^*B}}(f, g) = 0.$$

PROOF. Let $x^* \in \mathbf{X}^*$ be arbitrarily chosen. By Lemma 2.10 x^*f and x^*g are D* a.p. scalar functions and by Lemma 3.3 they have same Fourier series. As the corresponding theorem of [3] it can be shown that $\rho_{\mathrm{D^*}}(x^*f, x^*g) = 0$, that is,

$$\sup_{\substack{0\le h\le 1\\ -\infty<x<\infty}} \left|(\mathrm{D^*})\int_x^{x+h}\{x^*f(t) - x^*g(t)\}\,dt\right| = 0.$$

Now by our previous remark

$$x^*(\mathrm{D^*B})\int_x^{x+h}\{f(t) - g(t)\}\,dt = (\mathrm{D^*})\int_x^{x+h} x^*\{f(t) - g(t)\}\,dt$$

and hence

$$\sup_{\substack{0\le h\le 1\\ -\infty<x<\infty}} \left|x^*(\mathrm{D^*B})\int_x^{x+h}\{f(t) - g(t)\}\,dt\right| = 0.$$

Therefore,

$$x^*(\mathrm{D^*B})\int_x^{x+h}\{f(t) - g(t)\}\,dt = 0$$

for all $x \in \mathbf{R}$ and $h \in [0, 1]$. Since x^* is arbitrary, by Hahn-Banach Theorem

$$(\mathrm{D^*B})\int_x^{x+h}\{f(t) - g(t)\}\,dt = \theta$$

for all $x \in \mathbf{R}$ and $h \in [0, 1]$. Therefore

$$\sup_{\substack{0\le h\le 1\\ -\infty<x<\infty}} \left\|\int_x^{x+h}\{f(t) - g(t)\}\,dt\right\| = 0,$$

that is,

$$\rho_{\mathrm{D^*B}}(f, g) = 0.$$

4. Bochner-Fejer summability of Fourier series

We shall show that if f be D*B a.p. then the Fourier series of f is Bochner-Fejer summable to f with respect to the metric ρ_{D^*B} defined on the space of all D*B a.p. functions. For this purpose we shall use the 'Bochner-Fejer Kernel' and the 'Bochner-Fejer Polynomials' the details of which are discussed in [2, pages 46-50], [1, page 26] and [4, page 153].

Let f be D*B a.p. and let $f(t) \sim \Sigma a_k e^{i\lambda_k t}$. Let $\beta_1, \beta_2, \ldots$ be a basis of the sequence $\{\lambda_k\}$ of the Fourier exponents of f. For each positive integer m we consider the Bochner-Fejer Kernel

$$K_m(t) = \sum\left(1 - \frac{|\nu_1|}{(m!)^2}\right)\cdots\left(1 - \frac{|\nu_m|}{(m!)^2}\right)\exp\left(-\frac{it}{m!}\sum_{k=1}^{m}\nu_k\beta_k\right) \tag{4.1}$$

and the Bochner-Fejer polynomial for f

$$\begin{aligned} \sigma_m(t) &= \sigma_m(t; f) \\ &= \sum\left(1 - \frac{|\nu_1|}{(m!)^2}\right)\cdots\left(1 - \frac{|\nu_m|}{(m!)^2}\right) \\ &\quad\times a\left(\frac{1}{m!}\sum_{k=1}^{m}\nu_k\beta_k; f\right)\exp\left(\frac{it}{m!}\sum_{1}^{m}\nu_k\beta_k\right), \end{aligned} \tag{4.2}$$

where the first summations in (4.1) and (4.2) extend to all ν_j, $|\nu_j| \leqslant (m!)^2$, $j = 1, 2, \ldots, m$, and $a(\lambda; f)$ in (4.2) is defined by

$$a(\lambda; f) = M\{fe^{-i\lambda x}\}.$$

If, however, the basis contains a finite number of elements $\beta_1, \beta_2, \ldots, \beta_p$ then we take

$$\begin{aligned} \sigma_m(t) = \sum&\left(1 - \frac{|\nu_1|}{(m!)^2}\right)\cdots\left(1 - \frac{|\nu_p|}{(m!)^2}\right) \\ &\times a\left(\frac{1}{m!}\sum_{k=1}^{p}\nu_k\beta_k; f\right)\exp\left(\frac{it}{m!}\sum_{k=1}^{p}\nu_k\beta_k\right), \end{aligned}$$

the summation being extended to $|\nu_j| \leqslant (m!)^2$, $j = 1, 2 \cdots p$ with similar modification for $K_m(t)$. It can be verified that

$$\sigma_m(t; f) = \lim_{T\to\infty}\int_0^T K_m(u) f(u + t)\,du.$$

In what follows we need the function

$$\phi(x, h) = \int_x^{x+h} f(t)\,dt, \qquad x \in \mathbf{R}, h \in [0, 1].$$

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For fixed $h \in [0, 1]$ this is a function of x alone which is almost periodic by Theorem 2.2. Therefore for arbitrary but fixed $h \in [0, 1]$, the $\sigma_m(x; \phi)$ will have the same meaning as given in (4.2).

THEOREM 4.1. *Let f be* D*B a.p. *and let*

$$f(t) \sim \sum a_k e^{i\lambda_k t}.$$

Then the sequence of trigonometric polynomials $\{\sigma_m(t; f)\}$ *converges to f with respect to the metric* ρ_{D^*B} *as* $m \to \infty$.

We shall complete the proof of the theorem in three lemmas.

LEMMA 4.2. *If f is* D*B a.p. *then*

$$\sigma_m(x; \phi) \to \phi(x, h)$$

as $m \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$ *and* $h \in [0, 1]$ *where* $\phi(x, h) = \int_x^{x+h} f(t)\, dt$.

PROOF. By Theorem 2.2 $\phi(x, h)$ is almost periodic in $x \in \mathbf{R}$ uniformly with respect to $h \in [0, 1]$. Hence by Lemma 2.3 the Banach valued function Φ: $\mathbf{R} \to \mathcal{C}_{\mathbf{X}}[0, 1]$ defined by $\Phi(t) = \phi(t, \cdot)$ is almost periodic. If

$$\Phi(t) \sim \sum b_n e^{i\lambda_n t}$$

then $b_n \in \mathcal{C}_{\mathbf{X}}[0, 1]$ and

$$b_n = \lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t}\, dt \tag{4.3}$$

uniformly with respect to a (see [4, page 146]). By the definition of Φ we can write

$$\Phi(t) e^{-i\lambda_n t} = \phi(t, \cdot) e^{-i\lambda_n t}$$

and so

$$\frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t}\, dt = \frac{1}{T} \int_a^{a+T} \phi(t, \cdot) e^{-i\lambda_n t}\, dt.$$

Hence from (4.3)

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_a^{a+T} \Phi(t) e^{-i\lambda_n t}\, dt - b_n \right\|_{\mathcal{C}_{\mathbf{X}}} = 0$$

uniformly with respect to a. That is

$$\lim_{T \to \infty} \sup_{0 \leq h \leq 1} \left\| \frac{1}{T} \int_a^{a+T} \phi(t, h) e^{-i\lambda_n t}\, dt - b_n(h) \right\| = 0$$

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uniformly with respect to a. Hence

$$\lim_{T\to\infty}\frac{1}{T}\int_a^{a+T}\phi(t,h)e^{-i\lambda_n t}\,dt = b_n(h)$$

uniformly with respect to a and h. So, $b_n(h)$ are the Fourier coefficients of $\phi(t, h)$ and the Fourier exponents of $\Phi(t)$ and $\phi(t, h)$ will remain the same. Now it is proved in [1, page 26] that

$$\lim_{m\to\infty}\sigma_m(t;\Phi) = \Phi(t) \tag{4.4}$$

uniformly with respect to t, where $\sigma_m(t; \Phi)$ is defined as in (4.2) and the limit in (4.4) is taken with respect to the Banach space in which $\Phi(t)$ lies and so (4.4) becomes

$$\|\sigma_m(t;\Phi) - \Phi(t)\|_{\mathcal{C}_X} \to 0$$

as $m \to \infty$ uniformly with respect to t. That is

$$\sup_{0\le h\le 1}\|\sigma_m(t;\phi) - \phi(t,h)\| \to 0$$

as $m \to \infty$ uniformly with respect to t. Thus

$$\sigma_m(t;\phi) \to \phi(t,h)$$

as $m \to \infty$ uniformly with respect to t and h.

LEMMA 4.3. *If f is* D*B a.p. *then for each $h \in [0, 1]$*

$$\int_x^{x+h}\sigma_m(t;f)\,dt = \sigma_m(x;\phi).$$

Integrating (4.2) and using (3.7) the proof can be completed.

LEMMA 4.4. *If f is* D*B a.p. *then $\sigma_m(t; f) \to f(t)$ as $m \to \infty$ with respect to the metric $\rho_{\mathrm{D^*B}}$.*

PROOF. Let $\phi(x, h) = \int_x^{x+h} f(t)\,dt$. Then by Lemma 4.2

$$\sigma_m(x;\phi) \to \phi(x,h)$$

as $m \to \infty$ uniformly with respect to $x \in \mathbf{R}$ and $h \in [0, 1]$. So,

$$\sup_{\substack{0\le h\le 1\\ -\infty<x<\infty}}\|\sigma_m(x;\phi) - \phi(x,h)\| \to 0$$

as $m \to \infty$. Hence by Lemma 4.3

$$\sup_{\substack{0\le h\le 1\\ -\infty<x<\infty}}\left\|\int_x^{x+h}\sigma_m(t;f)\,dt - \int_x^{x+h}f(t)\,dt\right\| \to 0$$

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as $m \to \infty$. So,

$$\sup_{\substack{0 \leq h \leq 1 \\ -\infty < x < \infty}} \left\| \int_x^{x+h} \{\sigma_m(t; f) - f(t)\}\, dt \right\| \to 0,$$

that is,

$$\rho_{D^*B}(\sigma_m(t; f), f) \to 0$$

as $m \to \infty$. This completes the proof of Theorem 4.1.

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