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ON THE STRUCTURE OF $Q_2(G)$ FOR FINITELY GENERATED GROUPS

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1. Introduction. Let G be a group, ZG its integral group ring and $\Delta = \Delta(G)$ the augmentation ideal of ZG. Denote by G_i the ith term of the lower central series of G. Following Passi [**3**], we set $Q_n(G) = \Delta^n/\Delta^{n+1}$. It is well-known that $Q_1(G) \simeq G/G_2$ (see, for example [**1**]). In [**3**] Passi shows that if G is an abelian group then $Q_2(G) \simeq Sp^2(G)$, the second symmetric power of G. What is $Q_2(G)$ in general? We find a clue in [**4**] where Sandling shows that if G is any finite group then the canonical homomorphism φ: $G_2/G_3 \rightarrow \Delta^2/\Delta^3$ given by $gG_3 \rightarrow (g - 1) + \Delta^3$ is a split monomorphism; thus $Q_2(G) \simeq G_2/G_3 \oplus M$ for some abelian group M. Comparing this with Passi's result it is tempting to conjecture that for any group G,

$$Q_2(G) \simeq G_2/G_3 \oplus Sp^2(G/G_2).$$

The object of this paper is, first, to extend Sandling's result to finitely generated groups and, secondly, to verify the above conjecture for such groups.

In § 2, we develop the necessary machinery for handling the problem. This is an extension to finitely generated groups of tools developed in [**1**] for finite groups. Some of these ideas have previously appeared in [**2**]. In § 3, we prove the results mentioned above.

2. Definitions and preliminary results. Let A be an abelian group. Then $Sp^2(A) = A \otimes_z A/J$, where J is the subgroup of $A \otimes_z A$ generated by all elements $x \otimes y - y \otimes x$, $x, y \in A$, is called the second symmetric power of A. The image of $x \otimes y$ in $Sp^2(A)$ will be denoted by $x \vee y$. The mapping $A \times A \rightarrow Sp^2(A)$ given by $(x, y) \rightarrow x \vee y$ is bilinear and symmetric and is universal with respect to these properties.

Let G be a finitely generated nilpotent group of class c. For each $g \in G$, $g \neq 1$, set $w(g) = k$ if and only if $g \in G_k$, $g \notin G_{k+1}$; $w(g)$ is called the *weight* of g. For convenience set $w(1) = \infty$. Since $[G_i, G_j] \leqq G_{i+j}$ for all i and j we have $w([g, h]) \geqq w(g) + w(h)$ for all $g, h \in G$. For each $g \in G$ define $o^*(g)$ to be the order of the coset $gG_{w(g)+1}$, that is, $o^*(g)$ is the order of the image of g in the quotient $G_{w(g)}/G_{w(g)+1}$.

Each quotient G_k/G_{k+1} is a finitely generated abelian group and thus there exist elements $\bar{x}_{k1}, \bar{x}_{k2}, \ldots, \bar{x}_{k\mu(k)}$ (where $\bar{x}_{ki} = x_{ki}G_{k+1}$) such that each element

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$\bar{g} \in G_k/G_{k+1}$ can be written uniquely in the form

$$\bar{g} = \bar{x}_{k1}{}^{e(1)}\bar{x}_{k2}{}^{e(2)} \dots \bar{x}_{k\mu(k)}{}^{e(\mu(k))}$$

where $0 \leqq e(i) < o^*(x_{ki})$ if $o^*(x_{ki}) < \infty$.

Set

$$\Phi_0 = \{x_{ki} : k = 1, 2, \dots, c; i = 1, 2, \dots, \mu(k)\}.$$

Order Φ_0 by putting $x_{.j} < x_{kl}$ if $i < k$ or $i = k$ and $j < l$. Enlarge Φ_0 to a set Φ by adjoining to Φ_0 all $x_{ij}{}^{-1}$ for which $o^*(x_{ij}) = \infty$; extend the order on Φ_0 to Φ by putting $x_{ij}{}^{-1}$ immediately after x_{ij}. Let $|\Phi| = m$. Reindex Φ by the integers $1, 2, \dots, m$ so that $x_i < x_j$ if and only if $i < j$. Then every element $g \in G$ can be written uniquely in the form

$$g = x_1{}^{e(1)}x_2{}^{e(2)} \dots x_m{}^{e(m)} \tag{1}$$

where

(i) $0 \leqq e(i) < o^*(x_i)$ for all i,

(ii) if $x_{i+1} = x_i{}^{-1}$ then $e(i)e(i+1) = 0$.

The set Φ will be called a *positive uniqueness basis* for G. For each $x_i \in \Phi$, set $d(i) = o^*(x_i)$.

Let G be a finitely generated nilpotent group, Φ a positive uniqueness basis for G and $|\Phi| = m$. By an *m-sequence* $\alpha = (e(1), e(2), \dots, e(m))$ we mean an ordered m-tuple of non-negative integers. The set S_m of all m-sequences is ordered lexicographically; S_m is then well ordered. An m-sequence $\alpha = (e(1), e(2), \dots, e(m))$ is *basic* (with respect to Φ) if (i) $0 \leqq e(i) < d(i)$ for all i and (ii) if $x_{i+1} = x_i{}^{-1}$ then $e(i)e(i+1) = 0$. It follows from the uniqueness of the expression (1) above that there is a one-one correspondence between the elements of G and the basic m-sequences.

The weight $W(\alpha)$ of an m-sequence $\alpha = (e(1), e(2), \dots, e(m))$ is defined to be

$$W(\alpha) = \sum_{i=1}^{m} w(x_i)e(i).$$

Given an m-sequence $\alpha = (e(1), e(2), \dots, e(m))$ we define the proper product $P(\alpha) \in ZG$ to be

$$P(\alpha) = \prod_{i=1}^{m} (x_i - 1)^{e(i)}$$

where the factors occur in order of increasing i from left to right. If $W(\alpha) = k$ then $P(\alpha) \in \Delta^k$. If α is basic then $P(\alpha)$ is called a *basic product*. Note that if $\alpha = (0, 0, \dots, 0)$ then $P(\alpha) = 1$.

Since

$$x_i{}^{e(i)} = (1 + (x_i - 1))^{e(i)} = 1 + \sum_{j=1}^{e(i)} \binom{e(i)}{j} (x_i - 1)^j,$$

from (1) we obtain

$$(2) \quad g = 1 + e(1)(x_1 - 1) + \ldots + e(m)(x_m - 1)$$
$$+ \text{ a } Z\text{-linear combination of basic products of higher degree,}$$

which we can rewrite as

$$(2') \quad g - 1 = e(1)(x_1 - 1) + \ldots + e(m)(x_m - 1)$$
$$+ \text{ a } Z\text{-linear combination of basic products of higher degree.}$$

LEMMA 1. *The basic products form a free Z-basis for ZG. The non-identity basic products form a free Z-basis for* Δ.

Proof. It follows from (2) (respectively (2′)) that the basic products (respectively basic products $\neq 1$) span ZG (respectively Δ). The lemma will follow if we can show linear independence. Suppose $\sum r_\alpha P(\alpha) = 0$ is a non-trivial linear relation among the basic products. Among the basic m-sequences α for which $r_\alpha \neq 0$ there is a maximal one, say $\beta = (f(1), f(2), \ldots, f(m))$. If we multiply out all the $P(\alpha)$ and collect terms we obtain a linear combination of group elements each expressed in its unique form (1). It follows from the maximality of β that the element $x_1^{f(1)}x_2^{f(2)} \ldots x_m^{f(m)}$ occurs with coefficient $r_\beta \neq 0$. But this contradicts the fact that the elements of G are linearly independent in ZG.

LEMMA 2. *Let* $x_{i(1)}, x_{i(2)}, \ldots, x_{i(s)} \in \Phi$ *and let* $k = \sum_{j=1}^{s} w(x_{i(j)})$, $\mu = \min\{i(j) : 1 \leqq j \leqq s\}$. *Then the product*

$$(x_{i(1)} - 1)(x_{i(2)} - 1) \ldots (x_{i(s)} - 1)$$

can be written as a Z-linear combination of proper products $P(\alpha)$ *such that for each such* $\alpha = (e(1), e(2), \ldots, e(m))$

(i) $W(\alpha) \geqq k$,

(ii) $j < \mu$ *implies* $e(j) = 0$.

(The process of replacing such a product by a linear combination of proper products satisfying (i) and (ii) will be called *straightening*.)

COROLLARY. *The ideal* Δ^k *is spanned over Z by all proper products* $P(\alpha)$ *with* $W(\alpha) \geqq k$.

The proofs of Lemma 2 and its corollary are the same as those given in [**1**, Lemma 4] for the case of a finite group G; we refer the reader to the proofs given there.

LEMMA 3. *The ideal* Δ^2 *has a free Z-basis consisting of the elements.*

(i) $d(i)(x_i - 1)$, *where* $w(x_i) = 1$, $d(i) < \infty$;

(ii) $(x_i - 1) + (x_{i+1} - 1)$, *where* $w(x_i) = 1$, $x_{i+1} = x_i^{-1}$;

(iii) $P(\alpha)$, *where* α *is basic*, $W(\alpha) \geqq 2$.

Proof. If $w(x_i) = 1$ and $d(i) < \infty$ then $x_i^{d(i)} \in G_2$ and since

$$x_i^{d(i)} = (1 + (x_i - 1))^{d(i)} \equiv 1 + d(i)(x_i - 1) \bmod \Delta^2$$

we have

$$d(i)(x_i - 1) \equiv x^{d(i)} - 1 \equiv 0 \bmod \Delta^2.$$

If $w(x_i) = 1$ and $d(i) = \infty$ then, if $x_{i+1} = x_i^{-1}$,

$$(x_i - 1) + (x_{i+1} - 1) = -(x_i - 1)(x_{i+1} - 1) \in \Delta^2.$$

Thus elements of types (i), (ii) and (iii) are all in Δ^2.

Let $\psi : \Delta \to G/G_2$ be the canonical mapping determined by $g - 1 \mapsto gG_2$ for all $g \in G$. It is well-known (see [**1**] for example) that $\operatorname{Ker} \psi = \Delta^2$. Let $\gamma \in \Delta^2$. Then, by Lemma 1, we can write γ uniquely in the form

$$\gamma = a(1)(x_1 - 1) + \ldots + a(k)(x_k - 1) + \text{a } Z\text{-linear combination of elements of type (iii)},$$

where $w(x_1) = \ldots = w(x_k) = 1$. Since $\gamma \in \operatorname{Ker} \psi$ we have

$$\psi(\gamma) = \prod_{i=1}^{k} x_i^{a(i)} G_2 = G_2$$

and so $\prod_{i=1}^{k} x_i^{a(i)} \equiv 1 \bmod G_2$. By the uniqueness of the expression (1) it follows that $a(i) = b_i d(i)$ for some integer b_i if $d(i) < \infty$ and that $a(i) = a(i + 1)$ if $d(i) = \infty$ and $x_{i+1} = x_i^{-1}$. Thus we have $a(i)(x_i - 1) = b_i d(i)(x_i - 1)$ if $d(i) < \infty$ and

$$a(i)(x_i - 1) + a(i)(x_{i+1} - 1) = a(i)((x_i - 1) + (x_{i+1} - 1))$$

if $d(i) = \infty$ and $x_{i+1} = x_i^{-1}$. It follows then that γ can be written uniquely as a Z-linear combination of elements of types (i), (ii) and (iii).

3. The main results.

We are now in position to prove

THEOREM 1. *Let G be any finitely generated group. Then the canonical homomorphism*

$$\varphi : G_2/G_3 \to \Delta^2/\Delta^3$$

defined by $gG_3 \mapsto (g - 1) + \Delta^3$ *is a split monomorphism.*

THEOREM 2. *If G is any finitely generated group then*

$$Q_2(G) \simeq G_2/G_3 \oplus Sp^2(G/G_2).$$

Proof of Theorem 1. By passing to quotients by G_3 we may assume $G_3 = 1$. Then G_2 is abelian and $\varphi : g \mapsto (g - 1) + \Delta^3$. We define a homomorphism $\sigma : \Delta^2 \to G_2$ by defining it on the basis given in Lemma 3 as follows:

$$\begin{array}{lll} d(i)(x_i - 1) & \mapsto x_i^{d(i)} & w(x_i) = 1, d(i) < \infty \\ (x_i - 1) + (x_{i+1} - 1) & \mapsto 1 & w(x_i) = 1, x_{i+1} = x_i^{-1} \\ x_i - 1 & \mapsto x_i & w(x_i) = 2 \\ P(\alpha) & \mapsto 1 & \text{other basic } \alpha, w(\alpha) \geqq 2, \end{array}$$

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where $\Phi = \{x_i\}$ is a fixed positive uniqueness basis for the finitely generated nilpotent group G.

If $g \in G_2$ then we can write g in its unique form

(1) $$g = x_{i(1)}{}^{e(1)}x_{i(2)}{}^{e(2)} \ldots x_{i(s)}{}^{e(s)} \text{ with each } w(x_{i(j)}) = 2.$$

Hence, from (2′),

$$g - 1 = e(1)(x_{i(1)} - 1) + \ldots + e(s)(x_{i(s)} - 1) + \text{basic products of weight} \geqq 3.$$

Thus from the definition of σ,

$$\sigma(g - 1) = x_{i(1)}{}^{e(1)} \ldots x_{i(s)}{}^{e(s)} = g.$$

Therefore $\sigma(g - 1) = g$ for all $g \in G_2$.

We claim that σ vanishes on Δ^3. In view of the Corollary to Lemma 2, it suffices to show that σ vanishes on all proper products $P(\alpha)$ with $W(\alpha) \geqq 3$. We show this by induction over the well ordered set S_m of m-sequences. Suppose $W(\alpha) \geqq 3$ and $\sigma(P(\beta)) = 1$ for all $\beta < \alpha$ with $W(\beta) \geqq 3$. If α is basic then σ vanishes on $P(\alpha)$ by definition. So assume α is not basic. Then $P(\alpha)$ is either of the form

$$\text{(I)} \quad P(\alpha_1)(x - 1)(x^{-1} - 1)P(\alpha_2)$$

or

$$\text{(II)} \quad P(\alpha_1)(x - 1)^d P(\alpha_2)$$

for some $x = x_i \in \Phi$, $d = d(i)$, and suitable products $P(\alpha_1)$ and $P(\alpha_2)$.

Case (I): In this case we have

$$P(\alpha) = -P(\alpha_1)(x - 1)P(\alpha_2) - P(\alpha_1)(x^{-1} - 1)P(\alpha_2).$$

If $W(\alpha_1) + W(\alpha_2) \geqq 2$ then both terms on the right are proper products $P(\beta)$, $\beta < \alpha$ and $W(\beta) \geqq 3$ and, by the induction hypothesis, we are done. So we may assume $W(\alpha_1) + W(\alpha_2) \leqq 1$. Suppose $W(\alpha_1) + W(\alpha_2) = 0$, that is, $P(\alpha) = (x - 1)(x^{-1} - 1) = -((x - 1) + (x^{-1} - 1))$. If $w(x) = 1$ then $\sigma(P(\alpha)) = 1$ by definition; if $w(x) = 2$ then $\sigma(P(\alpha)) = x^{-1} \cdot (x^{-1})^{-1} = 1$. Suppose $W(\alpha_1) + W(\alpha_2) = 1$, say $P(\alpha_1) = y - 1$, $P(\alpha_2) = 1$, $w(y) = 1$. Then

$$\begin{aligned} P(\alpha) &= (y - 1)(x - 1)(x^{-1} - 1) \\ &= -(y - 1)(x - 1) - (y - 1)(x^{-1} - 1). \end{aligned}$$

If $y < x < x^{-1}$ then both terms on the right are basic and σ vanishes on both terms by definition. If $y = x < x^{-1}$ then

$$P(\alpha) = -(x - 1)^2 + ((x - 1) + (x^{-1} - 1))$$

and again, σ vanishes on both terms by definition. The case $P(\alpha_1) = 1$, $P(\alpha_2) = y - 1$, $w(y) = 1$, is handled similarly.

Case (II): In this case we have

$$P(\alpha) = \sum_{j=1}^{d-1} \binom{d}{j} P(\alpha_1)(x-1)^j P(\alpha_2) + P(\alpha_1)(x^d - 1)P(\alpha_2).$$

We replace $x^d - 1$ in the last term by its basic form (2′) and straighten the resulting terms. If $W(\alpha_1) + W(\alpha_2) \geqq 2$ then this expresses $P(\alpha)$ as a linear combination of proper products $P(\beta)$ with $\beta < \alpha$ and $W(\alpha) \geqq 3$. By the induction hypothesis σ vanishes on each term and so σ vanishes on $P(\alpha)$. We may therefore assume $W(\alpha_1) + W(\alpha_2) \leqq 1$. Suppose $W(\alpha_1) + W(\alpha_2) = 0$; then $P(\alpha) = (x-1)^d = \sum_{j=1}^{d-1} \binom{d}{i}(x-1)^j + (x^d - 1)$. If $w(x) = 1$ then $P(\alpha) = -d(x-1) + (x^d - 1) +$ elements of Ker σ and so, since $\sigma(g-1) = g$ for all $g \in G_2$, $\sigma(P(\alpha)) = x^{-d} \cdot x^d = 1$. If $w(x) = 2$ then $x^d = 1$ and so $P(\alpha) = -d(x-1) +$ elements of Ker σ. Therefore $\sigma(P(\alpha)) = x^{-d} = 1$. Now suppose $W(\alpha_1) + W(\alpha_2) = 1$, say $P(\alpha_1) = 1$, $P(\alpha_2) = y - 1$, $w(y) = 1$. Then $P(\alpha) = (x-1)^d(y-1)$ and $w(x) = 1$. We can write this as

$$P(\alpha) = -\sum_{j=1}^{d-1} \binom{d}{j} (x-1)^j (y-1) + (y-1)(x^d - 1)$$

since $x^d \in G_2 \leqq C(G)$. If we replace $x^d - 1$ by its basic form (2′) then, by definition of σ,

$$P(\alpha) \equiv -d(x-1)^{d-1}(y-1) \bmod \operatorname{Ker} \sigma.$$

If $x \neq y$ then $(x-1)^{d-1}(y-1)$ is basic and σ also vanishes on this term. If $x = y$ then $P(\alpha) \equiv -d(x-1)^d \bmod \operatorname{Ker}\sigma$. But $\sigma((x-1)^d) = 1$ as shown above. Hence $\sigma(P(\alpha)) = 1$. The case $P(\alpha_1) = y - 1$, $P(\alpha_2) = 1$, $w(y) = 1$ is handled similarly.

Thus we have shown by induction over the well ordered set S_m that σ vanishes on Δ^3. It follows that σ induces a homomorphism $\bar{\sigma} : \Delta^2/\Delta^3 \to G_2$ with the property that $\bar{\sigma}((g-1) + \Delta^3) = g$ for all $g \in G_2$. Therefore $\bar{\sigma}\varphi$ is the identity on G_2 and, consequently, φ is a split monomorphism.

Proof of Theorem 2. It follows from Theorem 1 that

$$Q_2(G) \simeq G_2/G_3 \oplus \operatorname{Coker}(\varphi).$$

Let $\eta : G \to G/G_2$ be the natural map and let $\tilde{\eta} : ZG \to Z(G/G_2)$ be its linear extension. Then $\tilde{\eta}$ is a ring homomorphism with kernel $I_G(G_2)$, the (right) ideal of ZG generated by all $g - 1$, $g \in G_2$. Let

$$\bar{\eta} : ZG/I_G(G_2) \to Z(G/G_2)$$

be the induced isomorphism.

The ideal $I_G(G_2)$ is spanned over Z by the elements $(g-1)h$, $g \in G_2$, $h \in G$. Now

$(g-1)h + \Delta^3 = (g-1) + (g-1)(h-1) + \Delta^3 = (g-1) + \Delta^3 \in \operatorname{Im}(\varphi)$. It follows that

$$\operatorname{Im}(\varphi) = I_G(G_2) + \Delta^3/\Delta^3$$

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and, therefore, that

$$\operatorname{Coker}(\varphi) \simeq \Delta^2/I_G(G_2) + \Delta^3.$$

On the other hand,

$$\bar{\eta}^{-1}(\Delta^2(G/G_2)) = \Delta^2(G)/I_G(G_2)$$

and

$$\bar{\eta}^{-1}(\Delta^3(G/G_2)) = \Delta^3(G) + I_G(G_2)/I_G(G_2).$$

Thus

$$Q_2(G/G_2) \simeq \Delta^2(G)/\Delta^3(G) + I_G(G_2) \simeq \operatorname{Coker}(\varphi).$$

Combining this with the above we see that

$$Q_2(G) \simeq G_2/G_3 \oplus Q_2(G/G_2).$$

By the result of Passi [**3**] mentioned in § 1,

$$Q_2(G/G_2) \simeq Sp^2(G/G_2)$$

and Theorem 2 follows.

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