Can. J. Math., Vol. XXVIII, No. 1, 1976, pp. 207–210

SEQUENTIAL COMPACTNESS OF X IMPLIES A COMPLETENESS PROPERTY FOR $C(X)$

M. RAJAGOPALAN AND R. F. WHEELER

A locally convex Hausdorff topological vector space is said to be *quasi-complete* if closed *bounded* subsets of the space are complete, and *von Neumann complete* if closed *totally bounded* subsets are complete (equivalently, compact). Clearly quasi-completeness implies von Neumann completeness, and the converse holds in, for example, metrizable locally convex spaces. In this note we obtain a class of locally convex spaces for which the converse fails. Specifically, let X be a completely regular Hausdorff space, and let $C_c(X)$ denote the space of continuous real-valued functions on X, endowed with the compact-open topology. We prove

THEOREM 1. *If X is sequentially compact, then $C_c(X)$ is von Neumann complete.*

A space X is said to be a k_R-space if a real-valued function on X is necessarily continuous when its restrictions to compact subsets are continuous. Any k-space is a k_R-space, but the converse is not true. It is well-known (see [**12**]) that $C_c(X)$ is quasi-complete (or complete) if and only if X is a k_R-space. Thus if X is sequentially compact, but not a k_R-space, then $C_c(X)$ is von Neumann complete but not quasi-complete. We give a simple example of such an X. A second example shows that "sequentially compact" may not be replaced by "countably compact" in Theorem 1.

1. Some background. The first example of a von Neumann complete non-quasi-complete space seems to have been given by Ptak [**11**, pp. 64-67]: if X_0 is the space of countable ordinals, then the space of continuous real-valued functions with compact support on X_0, endowed with the compact-open topology, has the desired properties. (The authors thank Robert Anderson for providing a translation of this material.) Almost twenty years later Dazord and Jourlin [**3**] made a systematic study of von Neumann complete locally convex spaces (calling them p-semi-reflexive spaces); see also Brauner [**1**]. Shortly thereafter Haydon [**7**] found a complicated example of a $C_c(X)$ space which is von Neumann complete but not quasi-complete.

Let $\mathscr{T}$, $\mathscr{C}$, and $\mathscr{E}$ be the collections of subsets of $C(X)$ which are, respectively, totally bounded in the compact-open topology, relatively compact in the compact-open topology, and pointwise bounded and equicontinuous. Then

Received May 27, 1975.

$\mathscr{E} \subset \mathscr{C} \subset \mathscr{T}$. Von Neumann completeness of $C_c(X)$ is the condition $\mathscr{T} = \mathscr{C}$; those spaces X for which $\mathscr{T} = \mathscr{E}$ were called "infra-k_R-spaces" by Buchwalter [**2**]. Haydon [**5**, Corollary 3.2] proved the surprising result that $\mathscr{T} = \mathscr{C}$ if and only if $\mathscr{T} = \mathscr{E}$. Consequently, $C_c(X)$ is von Neumann complete but not quasi-complete precisely when X is infra-k_R but not k_R.

2. The proofs.

Proof of Theorem 1. Let X be sequentially compact. By Haydon's result (quoted above), it must be shown that $\mathscr{T} = \mathscr{E}$. Suppose $A \subset C(X)$ is totally bounded in the compact-open topology, but not equicontinuous at a point x_0 of X. Then there is a positive ϵ_0 such that for every neighborhood U of x_0, there exist $f_U \in A$ and $x_U \in U$ such that $|f_U(x_U) - f_U(x_0)| \geqq \epsilon_0$. By induction sequences (U_n), (x_n), and (f_n) can be constructed such that (1) U_n is a neighborhood of x_0 (let $U_1 = X$), $x_n \in U_n$, and $f_n \in A$; (2) $|f_n(x_n) - f_n(x_0)| \geqq \epsilon_0$; and (3) if $x \in U_n$, $|f_i(x) - f_i(x_0)| < \epsilon_0/4$ for $1 \leqq i \leqq n - 1$.

Now $(f_n(x_0))$ is a bounded sequence of real numbers, hence there is a real number L and a subsequence (f_{n_k}) such that $f_{n_k}(x_0) \to L$. Since X is sequentially compact, a subsequence of (x_{n_k}) converges to a point y_0 of X. Thus without loss of generality we may assume that $f_n(x_0) \to L$ and $x_n \to y_0$. Then $K = \{x_n\}_{n=1}^{\infty} \cup \{y_0\}$ is compact. Choose n_0 such that $|f_n(x_0) - L| < \epsilon_0/4$ for $n \geqq n_0$. Then if $n_0 \leqq n_1 < n_2$,

$$\begin{aligned}
&\sup\{|f_{n_1}(x) - f_{n_2}(x)| : x \in K\} \geqq |f_{n_1}(x_{n_2}) - f_{n_2}(x_{n_2})| \\
&= |f_{n_1}(x_{n_2}) - f_{n_1}(x_0) + f_{n_1}(x_0) - L + L - f_{n_2}(x_0) + f_{n_2}(x_0) - f_{n_2}(x_{n_2})| \\
&\geqq |f_{n_2}(x_0) - f_{n_2}(x_{n_2})| - |f_{n_1}(x_{n_2}) - f_{n_1}(x_0)| - |f_{n_1}(x_0) - L| - |L - f_{n_2}(x_0)| \\
&> \epsilon_0 - 3\epsilon_0/4 = \epsilon_0/4.
\end{aligned}$$

Thus A is not totally bounded in the compact-open topology, a contradiction. Hence A is equicontinuous.

This result remains true under the weaker assumption that every infinite subset of X has infinitely many points in common with some compact subset of X. See [**10**] for a discussion of this concept.

Example 1. A completely regular, T_2, scattered, sequentially compact, non-k_R-space.

Let ω_1 and ω_2 be the least ordinals of cardinal $\aleph_1$ and $\aleph_2$, respectively. Let X be the subspace $([1, \omega_1) \times [1, \omega_2)) \cup \{(\omega_1, \omega_2)\}$ of $[1, \omega_1] \times [1, \omega_2]$. Then X is completely regular, T_2, and scattered. Since $[1, \omega_1)$ and $[1, \omega_2)$ are sequentially compact, so is X. Finally, we show that (ω_1, ω_2) is an isolated point of every compact subset A of X which contains it. If not, let (ω_1, ω_2) be a cluster point of $B = A \cap ([1, \omega_1) \times [1, \omega_2))$. Then given $\alpha \in [1, \omega_1)$, there exists $(x_\alpha, y_\alpha) \in B$ so that $x_\alpha > \alpha$. There is a $\lambda \in [1, \omega_2)$ so that $y_\alpha \leqq \lambda$ for all $\alpha \in [1, \omega_1)$. Now $[1, \omega_1) \times [1, \lambda]$ is closed in X. Hence $F = A \cap ([1, \omega_1) \times [1, \lambda])$ is compact. Then $\pi_1(F)$ should be a compact subset of $[1, \omega_1)$ where

$\pi_1 : [1, \omega_1) \times [1, \lambda] \rightarrow [1, \omega_1)$ is the projection map. However, $\pi_1(F) \supset \{x_\alpha : \alpha \in [1, \omega_1)\}$ which is unbounded in $[1, \omega_1)$, a contradiction. Thus (ω_1, ω_2) is an isolated point of A. Now the function $f : X \rightarrow R$ which is 1 at (ω_1, ω_2) and 0 elsewhere is continuous on compact sets but not continuous. Hence X is not a k_R-space.

This example was suggested by ideas found in [**8**] and [**9**]. The final example, which is related to constructions of Novak [**4**, p. 245] and Haydon [**6**, Ex. 2.5], shows that Theorem 1 does not hold if "sequentially compact" is replaced by "countably compact."

Example 2. A completely regular, T_2, countably compact space which is not an infra-k_R-space.

It suffices to exhibit an infinite, countably compact subset X of βN in which compact sets are finite, because $A = \{f \in C(X) : \sup |f(x)| \leqq 1\}$ is then totally bounded in the compact-open topology, but not equicontinuous. Now βN has 2^c infinite compact subsets, each of cardinal 2^c. Well-order them as $(K_\alpha)_{\alpha<\Gamma}$, where Γ is the least ordinal of cardinal 2^c. Also there are 2^c countably infinite subsets of βN: similarly, well-order them as $(C_\alpha)_{\alpha<\Gamma}$.

Define a subset X of βN as follows: Choose a point p_1 of $\bar{C}_1 \backslash C_1$ (closure taken in βN). Let q_1 be a point of K_1 distinct from p_1. Suppose $(p_\alpha)_{\alpha<\beta}$, $(q_\alpha)_{\alpha<\beta}$ have been chosen, where $\beta < \Gamma$. Now choose $p_\beta \in \bar{C}_\beta \backslash C_\beta$ such that $p_\beta \notin \{q_\alpha\}_{\alpha<\beta}$ (possible, since card $(\bar{C}_\beta \backslash C_\beta) = 2^c$ and card $\beta < 2^c$). Then choose $q_\beta \in K_\beta$ such that $q_\beta \notin \{p_\alpha\}_{\alpha \leqq \beta}$. This completes the inductive procedure.

Let $X = \{p_\alpha\}_{\alpha<\Gamma}$. Then X is countably compact, indeed every sequence of distinct points in βN has a cluster point in X. But if K is an infinite compact subset of βN, then $K = K_\beta$ for some $\beta < \Gamma$, and $q_\beta \in K \backslash X$ ($q_\beta \neq p_\alpha$ for $\alpha \leqq \beta$ by choice of q_β; $q_\beta \neq p_\alpha$ for $\beta < \alpha$ by choice of p_α). Thus every compact subset of X is finite.

References

1. K. Brauner, *A class of topologically free locally convex spaces and related group Hopf algebras*, Doctoral Dissertation, University of California, Berkeley, 1972.

2. H. Buchwalter, *Fonctions continues et mesures sur un espace completement regulier*, Springer-Verlag Lecture Notes *331* (1973), 183–202.

3. J. Dazord and M. Jourlin, *Sur les precompacts d'un espace localement convexe*, C. R. Acad. Sci. Paris *274* (1972), 463–466, 550–553.

4. J. Dugundji, Topology (Allyn & Bacon, Boston, 1966).

5. R. Haydon, *Compactness in $C_s(T)$ and applications*, Publ. Dep. Math. Lyon *9* (1972), 105–113.

6. ——— *On compactness in spaces of measures and measure-compact spaces*, Proc. Lond. Math. Soc. *29* (1974), 1–16.

7. ——— *Sur un probleme de H. Buchwalter*, C. R. Acad. Sci. Paris *275* (1972), 1077–1080.

8. V. Kannan and M. Rajagopalan, *On scattered spaces*, Proc. Amer. Math. Soc. *43* (1974), 402–408.

9. K. Malliha Devi, P. Meyer, and M. Rajagopalan, *When does countable compactness imply sequential compactness?*, Gen. Top. and its Applications, to appear.

10. N. Noble, *Countably compact and pseudocompact products*, Czech. Math. J. *19* (*94*) (1969), 390–397.
11. V. Ptak, *Compact subsets of convex topological linear spaces*, Czech. Math. J. *4* (*79*) (1954), 51–74 (Russian. English summary).
12. S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J. *25* (1958), 265–282.

Memphis State University,
Memphis, Tennessee;
Northern Illinois University,
DeKalb, Illinois