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FRIEZE PATTERNS IN THE HYPERBOLIC PLANE

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Abstract. It is well known that in the Euclidean plane there are seven distinct frieze patterns, i.e. seven ways to generate an infinite design bounded by two parallel lines. In the hyperbolic plane, this can be generalized to two types of frieze patterns, those bounded by concentric horocycles and those bounded by concentric equidistant curves. There are nine such frieze patterns; as in the Euclidean case, their symmetry groups are $\mathscr{C}_\infty, \mathscr{D}_\infty, \mathscr{C}_\infty \times \mathscr{D}_1$, and $\mathscr{D}_\infty \times \mathscr{D}_1$.

1. **Euclidean frieze patterns.** In the Euclidean plane, a *frieze pattern* or *pattern on a strip* is an infinite design bounded by two parallel lines ([6] pp. 81–83). The number of such distinct frieze patterns is seven. Their symmetry groups are: the infinite cyclic group $\mathscr{C}_\infty$ (generated by a translation or a glide reflection); the infinite dihedral group $\mathscr{D}_\infty$ (generated by two parallel line reflections, or two point reflections, or a point reflection and a line reflection); the direct product $\mathscr{C}_\infty \times \mathscr{D}_1$ (where $\mathscr{D}_1$ is the dihedral group of order 2 generated by reflection in an axis of the translation generating $\mathscr{C}_\infty$); and the direct product $\mathscr{D}_\infty \times \mathscr{D}_1$ (where $\mathscr{D}_1$ is again generated by a reflection, this time in a line perpendicular to the lines whose reflections generate $\mathscr{D}_\infty$). See [3] pp. 48–49, [4] pp. 16–20.

It is easy to see that these are the only possibilities of direct products because:
(1) two line reflections commute if and only if the lines are perpendicular;
(2) a line reflection and a point reflection commute if and only if the point and line are incident.
([3] p. 45).

Table 1 shows the seven distinct frieze patterns in the Euclidean plane, their symmetry groups and generators. This table is essentially that of Coxeter ([3] p. 48). Fejes Toth ([4] pp. 16–20) uses a different notation.

If we consider the two parallel lines bounding a frieze pattern as "circles" with parallel diameters (or equivalently, concentric circles with centre at infinity), these frieze patterns are infinite generalizations of the ornaments bounded by two concentric circles. Here the symmetry groups are the finite cyclic groups $\mathscr{C}_n$, generated by a rotation about a point through angle $2\pi/n$, and the finite dihedral groups $\mathscr{D}_n$, generated by reflections in two lines intersecting in an angle π/n, with $n=1, 2, 3, \ldots$. Figure 1 illustrates a finite frieze pattern with symmetry group

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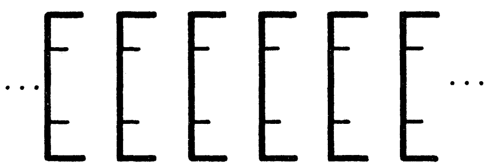
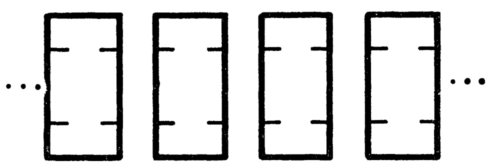
Table 1. Euclidean frieze patterns

Typical Pattern	Generators	Symmetry Group
	1 translation	$\mathscr{C}_\infty$
	1 glide reflection	$\mathscr{C}_\infty$
	2 line reflections	$\mathscr{D}_\infty$
	2 point reflections	$\mathscr{D}_\infty$
	1 line reflection and 1 point reflection	$\mathscr{D}_\infty$
	1 translation and 1 line reflection	$\mathscr{C}_\infty \times \mathscr{D}_1$
	3 line reflections	$\mathscr{D}_\infty \times \mathscr{D}_1$

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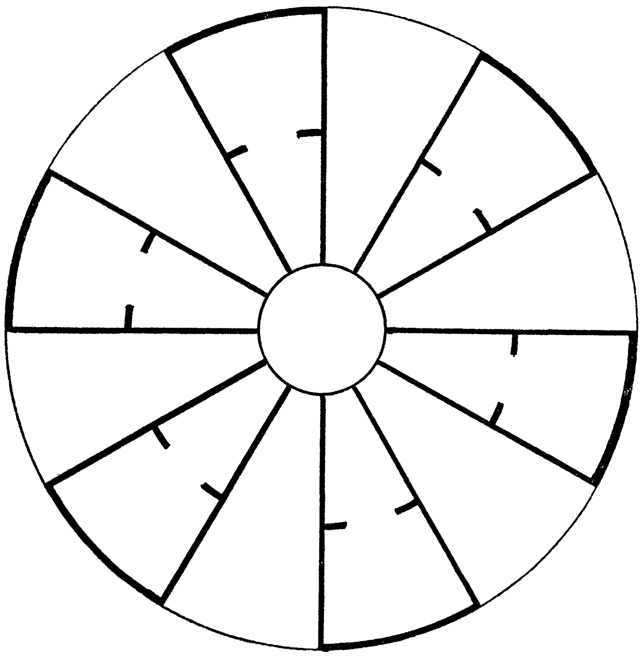

Figure 1

$\mathscr{D}_6$, in either the Euclidean or hyperbolic plane. The *fundamental region* of the pattern, or figure whose images under the symmetry group form the pattern, is the letter "F". The stem lies on a straight line, while the bars are on concentric circles. The concentric circles are marked in lightly, the pattern itself heavily. Cf. [3] pp. 30–31 and [4] pp. 14–15.

2. **Hyperbolic frieze patterns.** In the hyperbolic plane, these same finite patterns exist, but the infinite frieze patterns can be generalized to those of two types:

I frieze patterns bounded by two horocycles (i.e. concentric "circles" whose centre is on the absolute; their diameters are parallels);

II frieze patterns bounded by two equidistant curves (i.e. concentric "circles" whose centre is an ultra-infinite point; their diameters are ultra-parallels).

See [3] pp. 268–270 for a lucid description of horocycles, equidistant curves, and hyperbolic isometries, and [2] Chapters X and XI for a more detailed account.

For frieze patterns of type I, the group $\mathscr{C}_\infty$ is generated by a parallel displacement, defined as the product of reflections in two parallel lines, while the group $\mathscr{D}_\infty$ is generated by two parallel line reflections. Thus $\mathscr{C}_\infty$ is again exhibited as a subgroup of index 2 in $\mathscr{D}_\infty$.

There are no frieze patterns of type I exhibiting the symmetry groups $\mathscr{C}_\infty \times \mathscr{D}_1$, $\mathscr{D}_\infty \times \mathscr{D}_1$. For there is no axis for a parallel displacement, and so no commuting line reflection; and since two parallel lines have no common perpendicular, there is no line reflection commuting with two parallel line reflections. We are using here

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results (1) and (2) which hold in hyperbolic as well as Euclidean geometry, being theorems of absolute geometry ([1] pp. 37–38).

Frieze patterns of type II are the Euclidean frieze patterns again, but now bounded by equidistant curves instead of Euclidean parallel lines. We could, however, replace one equidistant curve by a hyperbolic line, namely the unique perpendicular to the ultra-parallel diameters, as in [3] p. 269. $\mathscr{C}_\infty$ can be generated by a translation or a glide reflection whose axis is this unique line. The group $\mathscr{D}_\infty$ can be generated by: two point reflections, line reflections in two ultra-parallels, or one point reflection and one line reflection. In the first two cases, the subgroup $\mathscr{C}_\infty$ is generated by a translation, while in the third it is generated by a glide reflection.

Finally, $\mathscr{C}_\infty \times \mathscr{D}_1$ can be realized as the symmetry group of a frieze pattern of type II if $\mathscr{C}_\infty$ is generated by a translation and $\mathscr{D}_1$ by reflection in the unique axis of the translation. $\mathscr{D}_\infty \times \mathscr{D}_1$ as a symmetry group is realized if $\mathscr{D}_\infty$ is generated by reflections in two ultra-parallels, and $\mathscr{D}_1$ by reflection in their unique common perpendicular.

The direct product of $\mathscr{D}_\infty$ generated by two point reflections (or by one point reflection and one line reflection) and of $\mathscr{D}_1$ generated by a commuting line reflection does not give a different frieze pattern. For the product of a point reflection and a line reflection, where the point and line are incident, is simply the reflection in the line through the point perpendicular to the given line ([1] p. 38). Thus we have again the above frieze pattern generated by reflections in two ultra-parallel lines and their common perpendicular.

This exhausts the list of hyperbolic isometries ([2] p. 201), and so:

There are exactly nine frieze patterns in the hyperbolic plane.

3. **Diagrams of the frieze patterns.** Two of the nine frieze patterns are illustrated in figures 2 and 3. We have used the conformal model of the hyperbolic plane, whose points are the interior points of a Euclidean circle (the *absolute*) and whose lines are the intersections of this interior with circles orthogonal to the absolute. Thus, reflection in a hyperbolic line is represented by inversion in the circle representing the line. A family of concentric horocycles is represented by a tangent coaxial family of circles, tangent to the absolute; this point of tangency represents their common "centre", being the point of concurrency of the parallel diameters. Equidistant curves are represented by members of an intersecting coaxial family of circles whose two points of intersection are on the absolute. Their diameters are represented by the orthogonal family of (non-intersecting) coaxial circles. The unique common perpendicular of these diameters is that member of the intersecting coaxial family which is also orthogonal to the absolute, and so is a hyperbolic line. See [5] Chapter IX where this model is discussed extensively.

In figures 2 and 3 the outer circle represents the absolute; the dots to the left and right indicate that the pattern continues indefinitely as in Table 1. The frieze patterns are marked in heavily.

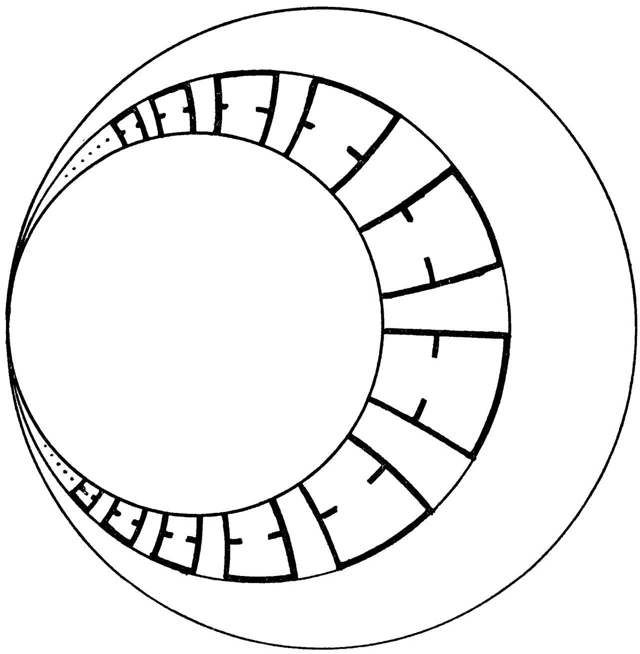

Figure 2

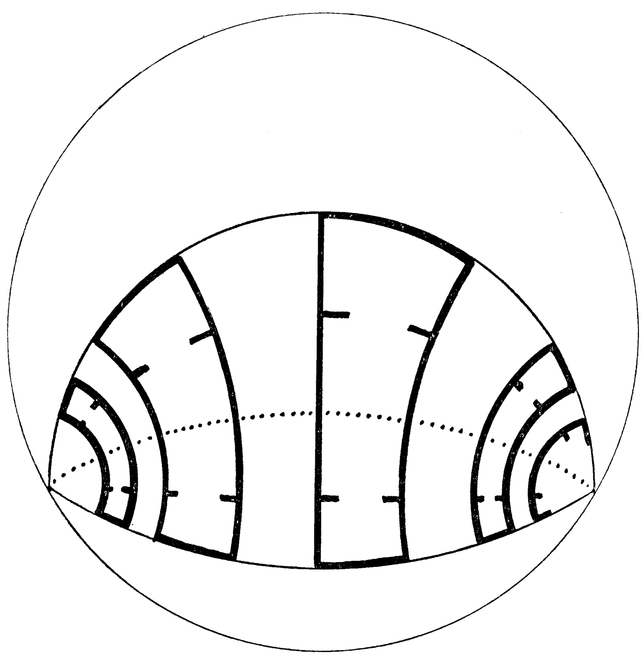

Figure 3

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Figure 2 illustrates one of the two frieze patterns bounded by two concentric horocycles (marked in lightly). The fundamental region is again the letter "F"; the stem lies on a diameter of the horocycles, while the bars lie on concentric horocycles.

Figure 3 illustrates one of the seven frieze patterns bounded by two concentric equidistant curves (again marked lightly). The fundamental region "F" has its stem on a diameter of the equidistant curves, while the bars lie on equidistant curves. The dotted line indicates the unique hyperbolic line which belongs to this family of equidistant curves, i.e. the line perpendicular to the ultra-parallel diameters. This line is also the axis of the translations and glide reflections, and reflection in it generates $\mathscr{D}_1$.

Figure 3 is the hyperbolic analogue of the seventh frieze pattern listed in Table 1. The analogue of the sixth pattern is similarly constructed. But to draw the hyperbolic analogues of the first five patterns of Table 1, the generating "F" should be positioned so that the smaller bar lies on the unique hyperbolic line which belongs to the family of equidistant curves.

Since this model of the hyperbolic plane is *conformal* but not *isometric*, these patterns in figures 2 and 3 decrease in size as they approach the absolute, but of course in the hyperbolic plane the F's are all congruent.

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