On Generating Functions.

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§ 1. *Introduction.*

It is well known that the polynomial in x,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n,$$

has the following properties:—

(A) it is the coefficient of t^n in the expansion of $(1 - 2xt + t^2)^{-\frac{1}{2}}$;

(B) it satisfies the three-term recurrence relation

$$(n+1) P_{n+1} - (2n+1) x P_n + n P_{n-1} = 0;$$

(C) it is the solution of the second order differential equation

$$(x^2 - 1) y_2 + 2xy_1 - n(n+1) y = 0;$$

(D) the sequence $P_n(x)$ is orthogonal for the interval $(-1, 1)$,

i.e. when $m \neq n$, $\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0.$

Several other familiar polynomials, *e.g.*, those of Laguerre, Hermite, Tschebyscheff, have properties similar to some or all of the above. The aim of the present paper is to examine whether, given a sequence of functions (polynomials or not) which has one of these properties, the others follow from it: in other words we propose to examine the inter-relation of the four properties. Actually we relate each property to the generating function.

§ 2. *Generating Functions and Recurrence Relations.*

2.1. *Given the generating function.*

Suppose that

$$F(x, t) = \sum_{n=0}^{\infty} L_n(x) t^n, \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

the series being assumed convergent in $|t| < K$. Such a function is called the generating function of $L_n(x)$.

Suppose further that, for any given x, $F(x, t)$ satisfies a linear differential equation in t, of order ν, whose coefficients are polynomials in t. Taking $\nu = 2$, let $F(x, t)$ satisfy

$$\sum_{m=0}^{k} t^m \left\{ p_m(x) F + q_m(x) \frac{\partial F}{\partial t} + r_m(x) \frac{\partial^2 F}{\partial t^2} \right\} = \sum_{m=0}^{k_1} \theta_m(x) t^m \quad \ldots\ldots\ldots(2)$$

where $k_1 < k$.

Then, substituting from (1) and equating coefficients of t^{n+k}, we see that, for $n \geqslant 0$,

$$\sum_{\lambda=2}^{k} \{p_{k-\lambda} + (n+\lambda) q_{k-\lambda+1} + (n+\lambda)(n+\lambda-1) r_{k-\lambda+2}\} L_{n+\lambda}$$
$$+ \{(n+k+1)q_0 + (n+k)(n+k+1)r_1\} L_{n+k+1} + (n+k+1)(n+k+2) r_0 L_{n+k+2}$$
$$+ p_k L_n + \{p_{k-1} + (n+1) q_k\} L_{n+1} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Hence the L_n satisfy a recurrence relation (3) in which the coefficients are polynomials of degree 2 in n, and the number of terms is in general $k+3$, but may be less; *e.g.* p_k may be zero. The coefficient of L_{n+r} in (3) may be written

$$\alpha_r(x) \, . \, n^2 + \beta_r(x) \, . \, n + \gamma_r(x),$$

so that the recurrence relation may be written

$$\sum_{r=0}^{k+2} L_{n+r} (\alpha_r n^2 + \beta_r n + \gamma_r) = 0 \quad \ldots\ldots\ldots\ldots\ldots(4)$$

where α_r, β_r, γ_r are functions of r and x only.

In addition to (3) there are, of course, relations governing the initial terms $L_0, L_1, \ldots . L_{k+1}$, namely

$$p_m L_0 + p_{m-1} L_1 + \ldots . + p_0 L_m$$
$$+ q_m L_1 + 2q_{m-1} L_2 + \ldots . + (m+1) q_0 L_{m+1}$$
$$+ 2r_m L_2 + 3 \, . \, 2 r_{m-1} L_3 + \ldots . + (m+2)(m+1) r_0 L_{m+2} = \theta_m \quad \ldots .(5)$$

for $m = 0, 1, 2, \ldots . k-1$.

Now it is clear from the method of establishing (4) that, if (2) is replaced by a differential equation of the same type but of order ν, then

(i) there is a recurrence relation with not more than $(\nu + k + 1)$ terms, and the coefficients of L_{n+r} are polynomials of degree ν in n,

(ii) the initial terms $L_0, L_1, \ldots . L_{k+\nu-1}$ satisfy k relations similar in form to (5).

Further, if in (2) $k_1 = k + \mu$, where $\mu \geqslant 0$, the recurrence relation (4) will be true, in general, only for $n \geqslant \mu + 1$.

Summing up, we have the following result:—

THEOREM 1. *If the generating function defined by* (1) *satisfy the differential equation*

$$P(x, t)\, F + Q(x, t)\, \frac{\partial F}{\partial t} + \ldots\ldots + Y(x, t)\, \frac{\partial^\nu F}{\partial t^\nu} = \Theta(x, t), \qquad \ldots\ldots\ldots(6)$$

where $P, Q, \ldots\ldots Y$ *are polynomials in* t *of degree* k, *and* Θ *is a polynomial in* t *of degree* k_1, *then a recurrence relation*

$$\sum_{r=0}^{k+\nu} L_{n+r}\,(\alpha_r n^\nu + \beta_r n^{\nu-1} + \ldots\ldots + \kappa_r) = 0, \qquad \ldots\ldots\ldots(7)$$

in which $\alpha_r, \beta_r, \ldots\ldots \kappa_r$ *are functions of* r *and* x, *is satisfied for* $n \geqslant 0$ *when* $k_1 < k$, *for* $n \geqslant k_1 - k + 1$ *when* $k_1 \geqslant k$.

Returning for a moment to the form (3) of the recurrence relation, we see that if the $L_n(x)$ are to be polynomials in x, then $P, Q, \ldots Y, \Theta$ must also be polynomials in x. But this condition is not sufficient. We have also that the last non-vanishing coefficient in (3) [or its analogue for $\nu \neq 2$] must be independent of x. It is easy to see in any given numerical example whether such a condition is satisfied, but the condition does not lend itself to the enunciation of any general theorem.

2 . 2. *Given the recurrence relation.*

Suppose now that $L_n(x)$ is a function of x which satisfies, for $n = 0, 1, 2 \ldots\ldots$, the $(N+1)$ term recurrence formula

$$\sum_{r=0}^{r=N} L_{n+r}\,(\alpha_r n^2 + \beta_r n + \gamma_r) = 0, \qquad \ldots\ldots\ldots\ldots\ldots(8)$$

in which the $\alpha_r, \beta_r, \gamma_r$ are functions of r and x only.

This may be written as

$$\sum_{r=0}^{r=N} L_{n+r}\{\alpha_r (n+r)(n+r-1) + \delta_r (n+r) + \epsilon_r\} = 0, \quad \ldots\ldots(9)$$

or replacing n by $n-2$ and rearranging the coefficients,

$$\sum_{r=2}^{N+2} L_{n+r}\{A_r (n+r)(n+r-1) + B_r (n+r) + C_r\} = 0. \quad \ldots(10)$$

Here A_r, B_r, C_r are functions of r and x only, and the relation is true for $n = -2, -1, 0, 1, 2, \ldots\ldots$

This again, for the purpose of comparing it with (3), may be written

$$0 \,.\, L_n + 0 \,.\, L_{n+1} + \sum_{r=2}^{N+2} L_{n+r}\{A_r (n+r)(n+r-1) + B_r (n+r) + C_r\}$$
$$+ 0 \,.\, L_{n+N+3} + 0 \,.\, L_{n+N+4} = 0.$$

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Now, putting $k = N + 2$,

$$p_k = p_{k-1} = q_k = 0; \qquad q_0 = r_0 = r_1 = 0,$$

and, for $\lambda = 2, 3, \ldots. k$,

$$r_{k-\lambda+2} = A_\lambda, \qquad q_{k-\lambda+1} = B_\lambda, \qquad p_{k-\lambda} = C_\lambda,$$

we see that polynomials in t of degree k $(= N + 2)$ or less,

$$P(x, t) = \sum_{m=0}^{k} p_m(x) t^m,$$

are defined in terms of the A_r, B_r, C_r of (10), or what is the same thing, in terms of the α_r, β_r, γ_r of (8). These polynomials are such that, if

$$F(x, t) = \Sigma L_n(x) t^n, \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

the series being assumed convergent for $|t| <$ some K, $F(x, t)$ satisfies the linear differential equation

$$P(x, t) F + Q(x, t) \frac{\partial F}{\partial t} + R(x, t) \frac{\partial^2 F}{\partial t^2} = \sum_{m=0}^{k-1} \theta_m(x) t^m. \ \ldots\ldots(11)$$

But, since the recurrence relation (10) holds for $n = -2, -1$, the values of θ_{k-1}, θ_{k-2} must be zero. Hence if $L_n(x)$ are defined by an $(N + 1)$ term recurrence relation, then $F(x, t)$, defined by (1), satisfies

$$P(x, t) F + Q(x, t) \frac{\partial F}{\partial t} + R(x, t) \frac{\partial^2 F}{\partial t^2} = \sum_{m=0}^{N=1} \theta_m(x) t^m. \ \ldots(11a)$$

Further, since $L_0, L_1, \ldots. L_{N-1}$ are arbitrary, the form of (11a) shews that, if they are chosen suitably, we may make the right hand side of (11a) zero.

2.21. *The question of convergence in* 2.2.

Trivial examples of (8), such as

$$(n + 2) L_{n+2} - (n + 2)^2 L_{n+1} - (n + 1) L_n = 0,$$

with $L_1 = 1$, $L_2 = 2$, so that $L_n > n!$, show that if (8) does contain n^2 among its coefficients, then convergence of $\Sigma L_n t^n$ will, in general, require $\alpha_N \neq 0$.

Suppose then that in (8), $|\alpha_N(x)| > K_1$ for all x of a certain region D of the x plane, and that $|\alpha_r|$, $|\beta_r|$, $|\gamma_r|$ are each $< K$, for $r = 0, 1, \ldots N$, and for all x in D. Then, when n is sufficiently large,

$$|\alpha_N n^2 + \beta_N n + \gamma_N| > K_1 n^2 / 2,$$

$$|L_{n+N}| < \frac{2}{n^2 K_1} \cdot 3Kn^2 \{|L_n| + |L_{n+1}| + \ldots + |L_{n+N-1}|\},$$

or, putting $\lambda = 6K / K_1$ and rewriting,

$$|L_n| < \lambda \{|L_{n-1}| + |L_{n-2}| + \ldots + |L_{n-N}|\}. \quad \ldots\ldots\ldots\ldots(12)$$

Suppose (12) is true for $n \geqslant m$. For $n = 0, 1, \ldots m-1$, let θ_n be an increasing sequence such that $\theta_n \geqslant |L_n|$. For $n = m, m+1, \ldots$ define θ_n by the formula

$$\theta_n = \lambda_1 (\theta_{n-1} + \theta_{n-2} + \ldots + \theta_{n-N}), \quad \ldots\ldots\ldots\ldots(13)$$

where λ_1 is greater than either λ or 1, we have

$$\theta_n > |L_n|, \quad n = 0, 1, 2, \ldots$$

But since $\lambda_1 > 1$, we have $\theta_n - \theta_{n-1} > 0$ and so, for $n \geqslant m$,

$$\theta_n < \lambda_1 N \theta_{n-1}.$$

Hence if we make $u_n = \theta_n$ for $n = 0, 1, \ldots m-1$, and define u_n for $n = m, m+1, \ldots$ by the formula

$$u_n = \lambda_1 N \cdot u_{n-1},$$

we have, for all values of n,

$$u_n \geqslant \theta_n > |L_n|.$$

But the series $\Sigma u_n t^n$ has a radius of convergence $1 / \lambda_1 N$, and so, for values of x in D, the series $\Sigma L_n t^n$ has a non-zero radius of convergence.

2.3. *Formal statement of the result.*

Summing up and extending slightly the previous work, we have the following

THEOREM II. *If a sequence of functions $L_n(x)$ satisfy an $(N+1)$ term recurrence formula*

$$\sum_{r=0}^{r=N} L_{n+r} (\alpha_r n^\nu + \beta_r n^{\nu-1} + \ldots + \kappa_r) = 0, \quad n = 0, 1, 2 \ \ldots\ldots(14)$$

in which $\alpha_r, \beta_r, \ldots \kappa_r$ are functions of r and x only, then

$$F(x, t) = \Sigma L_n(x) t^n, \ \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

assuming the expansion to be convergent for $|t| < K$, *satisfies a linear differential equation of order* ν *whose coefficients are polynomials in* t.

If the differential equation is

$$P(x, t) F + Q(x, t) \frac{\partial F}{\partial t} + \ldots . + Y(x, t) \frac{\partial^\nu F}{\partial t^\nu} = \Theta(x, t),$$

it is tolerably simple to shew that $P, Q, \ldots . Y$ are polynomials in t of degree $(N + 2\nu - 2)$ at most, and that Θ is a polynomial in t of degree $N + \nu - 3$ at most. In the particular case $\nu = 2$ a suitable choice of the arbitrary $L_0, L_1, \ldots . L_{n-1}$ will make $\Theta = 0$, but in the general case Θ cannot thus be made to vanish unless the coefficients in (14) have certain special values. Finally, *two distinct generating functions arising from different solutions of the same recurrence formula, satisfy differential equations which can differ only in the value of* Θ.

THEOREM III. *If in the recurrence relation* (14) *and for all* x *in a certain region* Δ *of the* x *plane,*

(i) $|\alpha_r|, |\beta_r|, \ldots . |\kappa_r|$ *are each* $< K$, *for* $r = 0, 1, \ldots . N$,

(ii) $|\alpha_N| > K_1$,

then the series (15) *defining* $F(x, t)$ *converges uniformly with regard to* x *in* Δ *over a circle* $|t| \leqslant K_2$.

§ 3. *Generating Functions and Differential Equations in* x.

Suppose now that $F(x, t)$ satisfies a differential equation

$$\sum_{r=0}^{h} t^r \frac{\partial^r}{\partial t^r} \left\{ \sum_{s=0}^{k} a_{r,s}(x) \frac{\partial^s F}{\partial x^s} \right\} = 0, \qquad \ldots\ldots\ldots\ldots(16)$$

and that $F(x, t) = \Sigma L_n(x) t^n$.

Suppose further, that for $|t| \leqslant K$ and $s = 1, 2, \ldots . k$, the series

$$\sum_{n=1}^{\infty} \frac{d^s L_n}{dx^s} t^n, \qquad \ldots\ldots\ldots\ldots(17)$$

converge uniformly with regard to x in some region Δ.

Then, substituting the appropriate series for F and its derivatives in (16), we have

$$\sum_{r=0}^{h} \sum_{s=0}^{k} a_{r,s}(x) t^r \sum_{n=r}^{\infty} \frac{d^s L_n}{dx^s} \frac{n! \, t^{n-r}}{(n-r)!} = 0.$$

From the coefficient of t^n, when $n \geqslant h$, we see that

$$\sum_{r=0}^{h} \sum_{s=0}^{k} \alpha_{r,s}(x) \frac{n!}{(n-r)!} \frac{d^s L_n}{dx^s} = 0,$$

or writing

$$\alpha_{0,s} + \sum_{r=1}^{h} n(n-1) \ldots . (n-r+1) \alpha_{r,s}(x) = A_s(x, n), \quad \ldots (18)$$

$$\sum_{s=0}^{k} A_s(x, n) \frac{d^s L_n}{dx^s} = 0, \quad \ldots \ldots \ldots \ldots (19)$$

where $A_s(x, n)$ is a polynomial of degree h at most in n.

If now $n < h$, we have from the coefficient of t^n

$$\sum_{r=0}^{n} \sum_{s=0}^{k} \alpha_{r,s}(x) \frac{n!}{(n-r)!} \frac{d^s L_n}{dx^s} = 0;$$

but this, for $n = 0, 1, 2, \ldots . h-1$, is the same as

$$\sum_{s=0}^{k} \frac{d^s L_n}{dx^s} \{ \sum_{r=0}^{h} n(n-1) \ldots . (n-r+1) \alpha_{r,s}(x) \} = 0.$$

Hence (19) is the form of a differential equation satisfied by $L_n(x)$ for $n = 0, 1, 2, \ldots .$. We have then

THEOREM IV. *If the generating function $F(x, t)$ satisfy a partial differential equation*

$$\sum_{r=0}^{h} t^r \frac{\partial^r}{dt^r} \left\{ \sum_{s=0}^{k} \alpha_{r,s}(x) \frac{\partial^s F}{\partial x^s} \right\} = 0 \quad \ldots \ldots \ldots \ldots (16)$$

then, subject to the uniform convergence of

$$\sum_{n=1}^{\infty} \frac{d^s L_n}{dx^s} t^n, \qquad (s = 1, 2 \ldots . k)$$

for x in some Δ and $|t| \leqslant K$, the $L_n(x)$ satisfy a linear differential equation of order k, namely

$$\sum_{s=0}^{k} A_s(x, n) \frac{d^s L_n}{dx^s} = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots (19)$$

where $A_s(x, n)$ is a polynomial of degree h or less in n.

COROLLARY. *Conversely, if a sequence of functions $L_n(x)$ satisfy differential equations of type* (19), *we can, by writing*

$$A_s(x, n) \equiv \sum_{r=0}^{h} n(n-1) \ldots . (n-r+1) \alpha_{r,}\ (x),$$

obtain the "generating differential equation" (16). Subject to the uniform convergence of the series

$$\sum_n \frac{d^s L_n}{dx^s} t^n,$$

the generating function of the L_n will be a solution of this differential equation.

It follows, of course, that if

$$L_0(x),\ L_1(x),\ L_2(x) \ldots.$$

be one set of solutions of the equations (19), and

$$M_0(x),\ M_1(x),\ M_2(x) \ldots.$$

be another set, both the generating functions satisfy the same partial differential equation (16).

§ 4. *Generating functions and orthogonal properties.*

4 . 1. *Preliminary questions of convergence.*

Suppose that $|L_n(x)| < \theta_n$ for $a \leqslant x \leqslant b$, and that for any definite t with $|t| < K$,

(i) $\Sigma\, |L_n(x)\, t^n|$ converges uniformly with regard to x in $a \leqslant x \leqslant b$.

Then it follows that, for $k = 1, 2, \ldots.,$

$$\text{(ii)} \quad \sum_n |n(n-1)\ldots.(n-k+1)\, L_n(x)\, t^{n-k}|, \qquad \ldots\ldots\ldots\ldots\ldots(20)$$

$$\text{(iii)} \quad \sum_{r,s} |L_r(x)\, L_s(x)\, t^{r+s}|, \qquad \ldots\ldots\ldots\ldots\ldots(21)$$

$$\text{(iv)} \quad \sum_{r,s} \left| \frac{r!\ s!}{(r-k)!\ (s-k)!} L_r(x)\, L_s(x)\, t^{r+s-2k} \right|, \qquad \ldots\ldots\ldots\ldots\ldots(22)$$

all behave in the same manner.

4 . 11. *Proof of* (ii).

If $|t| = r < R < K$, we can find a definite N, independent of x, such that $n > N$ implies, when $a \leqslant x \leqslant b$,

$$|L_n(x)\, R^n| < 1.$$

Hence the terms of (20), for $|t| = r$ and $n > N$, are less than those of

$$\sum_n n(n-1)\ldots.(n-k+1)\left(\frac{r}{R}\right)^{n-k}\left(\frac{1}{R}\right)^k,$$

which is a convergent series with terms independent of x. The uniform convergence of (20) follows at once.

4 . 12. *Proof of* (iii) *and* (iv).

We have used (21) as a convenient form to denote

$$|L_0^2| + |2L_0 L_1 t| + \ldots + |(L_0 L_n + L_1 L_{n-1} + \ldots + L_n L_0) t^n| + \ldots (23)$$

If N has the same meaning as in 4 . 11,

$$|L_0|,\ |L_1|,\ \ldots\ldots |L_N|,$$

are each less than some fixed K_1 for $a \leqslant x \leqslant b$. Hence when $n > 2N$, and $|t| = r < R < K$,

$$\begin{aligned} &|(L_0 L_n + L_1 L_{n-1} + \ldots + L_n L_0) t^n| \\ &< K_2 |L_n t^n + L_{n-1} t^{n-1} + \ldots + L_{n-N} t^{n-N}| \\ &\quad + |(L_{N+1} L_{n-N-1} + \ldots + L_{n-N-1} L_{N+1}) t^n| \end{aligned}$$

where $K_2 = \text{Max}\,(2KK_1,\ 2K^N K_1)$, each suffix exceeds N, and there are $(n - 2N - 1)$ terms in the second modulus.

Hence, for $|t| = r$,

$$\begin{aligned} &|(L_0 L_n + L_1 L_{n-1} + \ldots + L_n L_0) t^n| \\ &< K_2 (N+1) \left(\frac{r}{R}\right)^{n-N} + (n - 2N - 1) \left(\frac{r}{R}\right)^n. \end{aligned}$$

Accordingly, the terms of (23) for $n > 2N$, and $a \leqslant x \leqslant b$, are less than those of a convergent series whose terms are independent of x. Hence (iii) is proved.

Finally, (iv) follows from (ii) in the same way that (iii) follows from (i).

4 . 2. *Given the orthogonal property.*

Suppose we are given that, for $m \neq n$,

$$\int_a^b L_m(x) L_n(x)\, dx = 0,$$

and, further, that $\Sigma |L_n(x) t^n|$ converges uniformly with regard to x in $a \leqslant x \leqslant b$ for $|t| < K$.

Then, in virtue of the results of 4 . 1, we may write

$$\begin{aligned} I_0 = \int_a^b \{F(x, t)\}^2 dx &= \int_a^b (L_0 + L_1 t + \ldots)^2 dx \\ &= \int_a^b \sum_{r,s} L_r L_s t^{r+s} dx \\ &= \sum_{r,s} \int_a^b L_r L_s t^{r+s} dx \\ &= c_{0,0} + c_{1,1} t^2 + \ldots + c_{n,n} t^{2n} + \ldots \end{aligned}$$

where

$$c_{n,\,n} = \int_a^b \{L_n(x)\}^2\,dx.$$

Thus, "I_0 and, similarly,

$$I_k = \int_a^b \left(\frac{\partial^k F}{\partial t^k}\right)^2 dx$$

is an even function of t." ....(A)

Further, it is a direct consequence of the above calculations that "if $\dfrac{\partial^k F}{\partial t^k} = a_1^{(k)} + a_2^{(k)} t + \ldots. + a_n^{(k)} t^n + \ldots.$, then

$$\int_a^b \left\{\left(\frac{\partial^k F}{\partial t^k}\right)^2 - (a_1^{(k)} + a_2^{(k)} t + \ldots. + a_n^{(k)} t^n)^2\right\} dx$$

contains t^{2n+2} as a factor." ....(B)

4.3. *Given the properties* (A) *and* (B).

Much of the interest of the proof that the orthogonal property follows from (A) and (B) lies in seeing how much follows from (A) alone. Accordingly, we first assume (A) only, together with the convergence condition, "$\Sigma \,|\, L_n(x)\, t^n \,|$ converges uniformly with regard to x in $a \leqslant x \leqslant b$, for $|\,t\,| < K$."

Since I_k is an even function of t, the coefficient of t in I_k gives $c_{k,\,k+1} = 0$, where we have written

$$c_{r,\,s} \equiv \int_a^b L_r(x)\, L_s(x)\, dx.$$

Considering successively the coefficients of t^3 in I_{k-1}, t^5 in I_{k-2}, and so on, we obtain relations

$$\lambda_1 c_{k-1,\,k+2} + \lambda_2 c_{k,\,k+1} = 0$$
$$\mu_1 c_{k-2,\,k+3} + \mu_2 c_{k-1,\,k+2} + \mu_3 c_{k,\,k+1} = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where λ, μ are constants. Hence, since $c_{k,\,k+1} = 0$, we deduce that, for all values of r and k,

$$c_{r,\,2k+1-r} = 0.$$

Hence, assuming only the property (A), which is a property of $F(x, t)$ apart from its series development, all we can prove anent the orthogonal property of its coefficients is that

$$\int_a^b L_r(x)\, L_s(x)\, dx = 0 \text{ when } r + s \text{ is odd.}$$

When $r+s$ is even, all we can obtain from (A) is a series of equations

$$c_{k,\,k} = \gamma_k$$

$$\lambda_0 c_{0,\,2k} + \lambda_1 c_{1,\,2k-1} + \ldots + \lambda_k c_{k,\,k} = \mu_k\,,$$

where nothing is known of the values of γ_k and μ_k / λ_k.

If, however, we further assume the property (B) we have,
taking $n=1$ in (B) with $k=\lambda$, $c_{\lambda,\,\lambda+2} = 0$,
taking $n=3$ in (B) with $k=\lambda-1$; $\mu_1 c_{\lambda-1,\,\lambda+3} + \mu_2 c_{\lambda,\,\lambda+2} = 0$,
and so on. Hence we deduce that, for all even values of $r+s$,

$$\int_a^b L_r(x)\, L_s(x)\, dx = 0,$$

except when $r=s$.

4.4. *Summary.*

We may summarise our results in the following theorem.

THEOREM V. *Provided that the series $\Sigma\,|L_n(x)\,t^n|$ converges, for $|t| < K$, uniformly with regard to x in $a \leqslant x \leqslant b$, the necessary and sufficient condition for the $L_n(x)$ to be orthogonal over $(a,\,b)$ is that $F(x,t)$ should have the properties (A) and (B).*

The form of these conditions makes it clear that given a generating function $F(x,\,t)$, it is easy to see, without recourse to the integration of $L_r L_s$, whether or not its coefficients are partially orthogonal over some interval (a,b), *i.e.* whether the integral of $L_r L_s$ is zero when $r+s$ is odd. In order to see whether the L_n are fully orthogonal we must have recourse to integrals (B), which involve the series development of $F(x,\,t)$, and as this involves the evaluation of

$$\int_a^b L_r(x)\, L_s(x)\, dx$$

in most cases, our procedure is practically a formal verification of the orthogonal property.

§ 5. *Application to Legendre functions.*

As an example of how our results work out in a particular case we consider the Legendre functions. The functions $P_n(x)$, $Q_n(x)$ each satisfy the recurrence relation

$$(n+2)\,L_{n+2}(x) - x\,(2n+3)\,L_{n+1}(x) + (n+1)\,L_n(x) = 0 \quad \ldots\ldots (23)$$

The generating function of $P_n(x)$, namely,

$$F(x,\,t) \equiv X^{-1} \equiv (1 - 2xt + t^2)^{-\frac{1}{2}}$$

is a solution of

$$(1 - 2xt + t^2) \frac{\partial y}{\partial t} + (t - x) y = 0.$$

Accordingly, by 2.3, the generating function of the Q_n must satisfy a differential equation

$$(1 - 2xt + t^2) \frac{\partial y}{\partial t} + (t - x) y = A_0 + A_1 t.$$

Putting $A_0 = 1$, $A_1 = 0$ and solving the equation in the usual manner, we obtain as the generating function of some solution of (23),

$$\frac{1}{2X} \log \frac{x - t - X}{x - t + X} \qquad \text{.................(24)}$$

This reduces for $t = 0$, when $X = +\sqrt{1 - 2xt + t^2}$, to

$$\tfrac{1}{2} \log \frac{x - 1}{x + 1} = - Q_0(x).$$

Accordingly we have an elementary proof of the fact that $- Q_n(x)$ is generated by (24)[1].

Finally, the work of § 3 shows that, since $P_n(x)$ and $Q_n(x)$ are solutions of the differential equation

$$(x^2 - 1) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - y\{n(n - 1) + 2n\} = 0,$$

their generating functions

$$\frac{1}{X}, \quad \frac{1}{2X} \log \frac{x - t + X}{x - t - X},$$

are solutions of the partial differential equation[2]

$$(1 - x^2) \frac{\partial^2 F}{\partial x^2} - 2x \frac{\partial F}{\partial x} + 2t \frac{\partial F}{\partial t} + t^2 \frac{\partial^2 F}{\partial t^2} = 0. \qquad \text{......(25)}$$

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[1] Laurent, *Journal de Math.* (3) 1 (1875), 390.

[2] The direct verification of the fact that (24) is a solution of (25) is a rather heavy piece of calculation.