THE DIAMETERS OF THE GRAPHS OF SEMIRINGS

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1. Introduction

Let $\mathscr{F}$ be a family of sets, $\{F_\alpha | \alpha \in \Lambda\}$. By the graph $G(\mathscr{F})$ of the system $\mathscr{F}$, we mean the graph whose set of vertices is $\mathscr{F}$ and in which the vertices F_α, $F_\beta \in \mathscr{F}$ are adjacent (that is, are joined by an edge) if and only if $F_\alpha \neq F_\beta$ and $F_\alpha \cap F_\beta \neq \square$, where $\square$ denotes the empty set.

DEFINITION 1. Let $\mathscr{F}$ be a family of sets, a subfamily $\{F_1, F_2 \cdots, F_n\}$ of $\mathscr{F}$ forms a *path*, or a *chain*, between F_1 and F_n in the graph $G(\mathscr{F})$ if and only if $F_i \cap F_{i+1} \neq \square$ for all $i = 1, \cdots, n-1$. A graph is said to be *connected* provided, for every pair of vertices there is a path between them.

DEFINITION 2. The *distance* $d(F_\alpha, F_\beta)$ between two vertices F_α and F_β of a graph is the number of edges in a shortest path between these vertices (if no such path exists, we define $d(F_\alpha, F_\beta) = +\infty$; of course $d(F_\alpha, F_\alpha) = 0$). The *diameter* of a graph is the supremum of $d(F_\alpha, F_\beta)$, where (F_α, F_β) runs over all pairs of vertices of the graph.

DEFINITION 3. A *semiring* is a non-empty set R equipped with two binary operations, called addition $+$ and multiplication (denoted by juxtaposition), such that R is multiplicatively a semigroup, additively a commutative semigroup and multiplication is distributive across the addition.

We have the following well-known theorem.

THEOREM A [1]. *For any graph G there exists a system $\mathscr{F}$ of sets such that the graph G is isomorphic with the graph $G(\mathscr{F})$.*

Theorem A shows that the general case is not very interesting. It would be of interest to know more information about the graph $G(\mathscr{F})$, when the members of $\mathscr{F}$ have an algebraic structure. The first step in this direction was taken by Bosák [1].

Throughout this paper, let S be a given semigroup and $\mathscr{S}$ be the system of all proper subsemigroups of S; let R be a given semiring and $\mathscr{R}$ the family of all proper subsemirings of R.

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Bosák [1] proved the following theorem.

THEOREM B. *Let S be a periodic semigroup with more than two elements. Then its graph* $G(\mathscr{S})$ *is connected and the diameter* $D(S)$ *of this graph is equal to*:

(i) 0 *if S is a cyclic group of prime order*;

(ii) 1 *if S has a single idempotent, but S is not a cyclic group of prime order*;

(iii) 3 *if there exist in S two idempotents* $u \neq v$ *such that* $S = \langle u, v \rangle$ (*that is, S is the semigroup generated by the idempotents u, v as its generators*);

(iv) 2 *in the remaining cases.*

Bosák then raised the following open problem: Does there exist a semigroup with more than two elements whose graph is disconnected?

Lin [2] answered Bosák's problem by proving the following theorem:

THEOREM C. *The graph of every semigroup with more than two elements is connected.*

In [3] we discussed the graph $G(\mathscr{R})$ of a semiring R and posed the following

CONJECTURE. *The graph of every semiring with more than two elements is connected.*

Although we could not prove our conjecture for an arbitrary semiring R, we did prove it for the cases (i) R is left unital (ii) R is normal (iii) R is commutative (iv) R is uncountable.

In § 2 we prove that for some special semirings R the diameter $D(R)$ of the graphs $G(\mathscr{R})$ is $\leqq 3$.

2. The diameter of the graph of a semiring

In this section we discuss the diameter of the graphs of some special types of semirings.

THEOREM 1. *The diameter of the graph of a left unital semiring with more than two elements does not exceed three.*

PROOF. Let R be such a semiring with left unit e. Let R_1 and R_2 be any two disjoint proper subsemirings of R, and let $a \in R_1$ and $b \in R_2$ be two arbitrary fixed elements. We shall construct a path, in $G(\mathscr{R})$, of length at most three between R_1 and R_2. Clearly, either $\{R_1, \langle a, 2e \rangle, \langle 2e, b \rangle, R_2\}$ is a path, or else $\langle a, 2e \rangle = R$, or $\langle 2e, b \rangle = R$ (throughout this paper, $\langle x_1, \cdots, x_n \rangle$ means the subsemiring generated by $x_1, \cdots x_n$, as its generators). Let us assume $\langle a, 2e \rangle = R$; the case $\langle 2e, b \rangle = R$ can be handled similarly.

It is sufficient to construct a proper subsemiring R_α of R such that $R_1 \cap R_\alpha \neq \square$ and $R_\alpha \cap \langle e \rangle \neq \square$. Since if this has been established, then similarly there must exist R_β in $G(\mathscr{R})$ such that $R_\beta \cap \langle e \rangle \neq \square$ and $R_\beta \cap R_2 \neq \square$. Consequently,

$\{R_1, R_\alpha, R_\beta, R_2\}$ will be a path of length 3. To this end, we divide the rest of the proof into the cases (1) $2e = e$ and $a^2 = a$, (2) $2e = e$ and $a^2 \neq a$, and (3) $2e \neq e$.

CASE 1: $2e = e$ and $a^2 = a$.

(1.1) $a = e$. In this case we choose $R_\alpha = \langle a \rangle = \langle e \rangle$

(1.2) $a \neq e$. In this case we have semiring $R = \{e, a, ae, e+a, e+ae, a+ae, e+a+ae\}$, with the following multiplication table (the addition table, which may be constructed easily is omitted for the sake of space saving).

	e	a	ae	$e+a$	$e+ae$	$a+ae$	$e+a+ae$
e	e	a	ae	$e+a$	$e+ae$	$a+ae$	$e+a+ae$
a	ae	a	ae	$a+ae$	ae	$a+ae$	$a+ae$
ae	ae	a	ae	$a+ae$	ae	$a+ae$	$a+ae$
$e+a$	$e+ae$	a	ae	$e+a+ae$	$e+ae$	$a+ae$	$e+a+ae$
$e+ae$	$e+ae$	a	ae	$e+a+ae$	$e+ae$	$a+ae$	$e+a+ae$
$a+ae$	ae	a	ae	$a+ae$	ae	$a+ae$	$a+ae$
$e+a+ae$	$e+ae$	a	ae	$e+a+ae$	$e+ae$	$a+ae$	$e+a+ae$

From the above multiplication table, we find the following two 'master' proper subsemirings:

$$\{a, ae, e+ae, a+ae, e+a, e+a+ae\}$$

and

$$\{e, ae, e+ae, a+ae, e+a, e+a+ae\}$$

Since the union of these two proper subsemirings contains R, the graph $G(\mathscr{R})$ is connected and $D(R) = 3$.

CASE 2. $2e = e$ and $a^2 \neq a$. In this case we chose $R_\alpha = \langle e, a^3 \rangle$, unless $R_\alpha = R$. Assume $R = \langle e, a^3 \rangle$. Since $a \in R$, we have the following possibilities:

(2.1) $a = e$. This cannot happen, since $a^2 \neq a$.

(2.2) $a = a^{3l_1} + a^{3l_2} + \cdots + a^{3l_n}$, where l_i, $i = 1, \cdots, n$, are positive integers.

Let $p(a) = a^{3l_1 - 1} + \cdots + a^{3l_n - 1}$. Then (2.2) gives $a = ap(a)$; which implies $p^2(a) = p(a)$ and the proof follows from Case 1 by replacing a by $p(a)$.

(2.3) $a = e + q(a^3) + r(a^3)e$, for some $q(a^3)$ and $r(a^3)$ in $\langle a^3 \rangle$. Since $a \in R_1 \cap (e+R)$ and $e = 2e \in (e+R) \cap \langle e \rangle$; we may choose $R_\alpha = e+R$, unless $e+R = R$. Assume $e+R = R$. For $x \in R$, there exists $y \in R$ such that $x = e+y$; and $e+x = e+(e+y) = 2e+y = e+y = x$. Thus, e functions as the additive zero for R. Hence from (2.3) we get $a = q(a^3)+r(a^3)e$. By multiplying this last expression by a, we get $a^2 = a^2Q(a)$ where $Q(a) = a^{3l_1-1} + \cdots + a^{3l_n-1}$, for some integers $l_1, \cdots, l_n$.

This implies that $Q^2(a) = Q(a)$. Thus the proof follows from Case 1 by replacing a by $Q(a)$.

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(2.4) $a = e+q(a^3)$.

(2.5) $a = q(a^3)+r(a^3)e$.

(2.6) $a = r(a^3)e$

(2.7) $a = e+r(a^3)e$.

The subcases (2.4), (2.5), (2.6), and (2.7) are similar to the subcase (2.3), the proof for these cases is therefore omitted.

CASE 3. $e+e \neq e$. In this case we have $R = \langle 2e, a\rangle$, because otherwise we choose $R_\alpha = \langle 2e, a\rangle$. Since $e \in R$, e can be expressed as:

(3.1) $e = 2me$ for some integer $m > 1$.

Let $e_1 = (2m-1)e$, then $e_1+e_1 = e_1$ and $e_1^2 = e_1$. Since $(2m-1)a = e_1 a \in R_1 \cap e_1 R$ and $(2m-1)e \in e_1 R \cap \langle e\rangle$. We must have $R = e_1 R$ (if $R \neq e_1 R$, we choose $R_\alpha = e_1 R$). Since $e_1^2 = e_1$ and $e_1 R = R$, it is easily seen that e_1 is a left unit for R with $e_1+e_1 = e_1$, and the proof follows from cases 1 and 2.

(3.2) $e = f(a)$ for some $f(a) \in \langle a\rangle$.

Since $e = f(a)$, we have $R = \langle 2e, a\rangle \subset R_1 \neq R$, a contradiction.

(3.3) $e = p(a)e$ for some $p(a) \in \langle a\rangle$.

Since $p(a) \in R_1 \cap \langle 2e, p(a)\rangle$ and $2e \in \langle 2e, p(a)\rangle \cap \langle e\rangle$, we have $R = \langle 2e, p(a)\rangle$ (otherwise choose $R_\alpha = \langle 2e, p(a)\rangle$). Also the equation $e = p(a)e$ gives $a = p(a)a$, which implies that $p^2(a) = p(a)$. Since $e \in R = \langle 2e, p(a)\rangle$, we have the following possibilities:

(3.3.1) $e = 2me$. This case is the same as subcase (3.1) already discussed.

(3.3.2) $e = np(a)$ for some integer $n \geqq 1$. This case is similar to the subcase (3.2).

(3.3.3) $e = 2np(a)e = 2ne$. This is the subcase (3.1).

(3.3.4) $e = np(a)+2mp(a)e$. This equation gives $e = e^2 = np(a)e+2mp(a)e = (n+2m)e$, which is the subcase (3.1).

(3.3.5) $e = 2me+np(a)$. We again have $e = e^2 = 2me+np(a)e = (2m+n)e$, which is the subcase (3.1).

(3.3.6) $e = 2ne+mp(a)e$.

(3.3.7) $e = 2ne+mp(a)+2lp(a)e$.

The cases (3.3.6) and (3.3.7) can be similarly handled.

(3.4) $e = f(a)+p(a)e$, for some $f(a)$ and $p(a)$ in $\langle a\rangle$. In this case we have $e = e^2 = f(a)e+p(a)e = [f(a)+p(a)]e = h(a)e$, where $h(a) = f(a)+p(a) \in \langle a\rangle$. The subcase (3.3) now applies.

(3.5) $e = 2me+f(a)e$ for some integer $m \geqq 1$ and $f(a) \in \langle a\rangle$.

Let $d = (2m-1)e+f(a)e = d_1 e$ where $d_1 = (2m-1)e+f(a)$. With this notation, we get $e = e+d$, and hence

$$\begin{aligned} d+d &= (2m-1)e+f(a)e+d \\ &= [(2m-1)e+d]+f(a)e \\ &= (2m-1)e+f(a)e \\ &= d \end{aligned}$$

By squaring both sides of the equality $e = e+d$, we obtain $e = e+d^2$. Consequently,

$$d+d^2 = [f(a)e+(2n-1)e]+d^2 = f(a)e+(2n-1)e = d,$$

while

$$\begin{aligned} d+d^2 &= d+[(2m-1)e+f(a)e]^2 \\ &= d+(2m-1)^2e+2(2m-1)f(a)e+f^2(a)e \\ &= [d+(2m-1)^2e]+2(2m-1)f(a)e+f^2(a)e \\ &= (2m-1)^2e+2(2m-1)f(a)e+f^2(a)e \\ &= d^2 \end{aligned}$$

Thus $d = d^2$.

Let $R_{d_1} = \{x \mid x \in R$ and $(x+d_1)e = 2ke$ for some positive integer $k\}$. If $x_1, x_2 \in R_{d_1}$, then $x_1+d_1 = 2k_1 e$ and $x_2+d_1 = 2k_2 e$. Consequently

$$\begin{aligned} (x_1+x_2+d_1)e &= (x_1+x_2)e+d \\ &= (x_1+x_2)e+2d = (x_1+d_1)e+(x_2+d_1)e \\ &= 2(k_1+k_2)e, \end{aligned}$$

and

$$\begin{aligned} 4k_1k_2e &= [(x_1+d_1)e][(x_2+d_1)e] \\ &= (x_1e+d)(x_2e+d) \\ &= x_1x_2e+x_1d+dx_2e+d^2 \\ &= x_1x_2e+x_1d+d^2+dx_2e+d^2, \qquad \text{since } d^2 = 2d^2 \\ &= x_1x_2e+(x_1+d)d+dx_2e+d^2e, \qquad \text{since } d^2e = d^2 \\ &= x_1x_2e+(x_1+d)ed+d(x_2+d)e \\ &= x_1x_2e+2k_1ed+d(2k_2e) \\ &= x_1x_2e+2k_1d+2k_2d \\ &= x_1x_2e+d \\ &= (x_1x_2+d_1)e. \end{aligned}$$

Since $2e \in R_{d_1}$, $R_{d_1} \neq \square$. Consequently R_{d_1} is a subsemiring of R. Also

$$\begin{aligned}[d_1+f^2(a)]e &= d+f^2(a)e \\ &= (2m-1)[(2m-1)e+f(a)e]+f^2(a)e \\ &= (2m-1)^2e+f(a)[(2m-1)e+f(a)]e \\ &= (2m-1)^2e+f(a)d \\ &= (2m-1)^2e+(2m-1)d+f(a)d, \qquad \text{since } e+d = e \\ &= (2m-1)^2e+[(2m-1)e+f(a)e]d, \qquad \text{since } ed = d \\ &= (2m-1)^2e+d^2 \\ &= (2m-1)^2e.\end{aligned}$$

Thus,

$$\begin{aligned}(2f^2(a)+d_1)e &= 2f^2(a)e+d \\ &= 2f^2(a)e+2d \\ &= 2(f^2(a)e+d) \\ &= 2(2m-1)^2e.\end{aligned}$$

Therefore, $2f^2(a) \in R_{d_1}$.

Now $2f^2(a) \in R_1 \cap R_{d_1}$ and $2e \in R_{d_1} \cap \langle e \rangle$, we have $R_{d_1} = R$ (otherwise choose $R_\alpha = R_{d_1}$).

Since $e \in R = R_{d_1}$, we have $(e+d_1)e = 2ke$ for some positive integer k. Therefore,

$$2ke = (e+d_1)e = e+d_1e = e+d = e$$

and the proof follows from subcase (3.1).

(3.6) $e = 2me+f(a)$ for some positive integer m and $f(a) \in \langle a \rangle$. In this case $e = e^2 = 2me+f(a)e$, and which reduces to (3.5).

(3.7) $e = 2me+f(a)+2np(a)e$. We again have

$$\begin{aligned}e = e^2 &= 2me+(f(a)+2np(a))e \\ &= 2me+h(a)e, \quad \text{where } h(a) \in \langle a \rangle\end{aligned}$$

and this case also reduces to (3.5).

THEOREM 2. *The diameter of the graph of a commutative semiring R with more than elements does not exceed three.*

PROOF. Let R_1 and R_2 be two disjoint proper subsemirings of R, and let $a \in R_1$ and $b \in R_2$ be any two fixed elements. Then $\{R_1, aR, Rb, R_2\}$ is a path of length three between R_1 and R_2, unless $aR = R$ or $Rb = R$. Assume $aR = R$ (the case $Rb = R$ may be handled similarly). Since R is commutative, we have $R = aR = Ra$. It follows that there exists an element $e \in R$ such that $a = ea$. For each $x \in R$, $R = aR$ implies that there exists an element $y \in R$ such that $x = ay$.

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Thus,

$$ex = e(ay) = (ea)y = ay = x,$$

which shows that R is left unital and thus, by Theorem 1, the diameter of the graph $G(\mathscr{R})$ is $\leqq 3$.

THEOREM 3. *The graph of a semiring R with more than two elements, having ascending chain condition (A.C.C.) or descending chain condition (D.C.C.), is connected and the diameter $D(R)$ does not exceed three.*

PROOF. Let R_1 and R_2 be two disjoint proper subsemirings of R and let $a \in R_1$ and $b \in R_2$ be two fixed elements. We observe that $\{R_1, aR, Rb, R_2\}$ is a path of length at most 3 between R_1 and R_2 unless $aR = R$ or $Rb = R$. Assume $aR = R$ (the case $Rb = R$ may be similarly handled). Since $2a = a+a \in R_1 \cap (R+R)$ and $2b = b+b \in R_2 \cap (R+R)$, $\{R_1, R+R, R_2\}$ is a path between R_1 and R_2 unless $R+R = R$. Let us assume that $R+R = R$.

Let

$$A = \{x | x \in R \text{ and } xR = R\}.$$

It is easily seen that A is a subsemiring of R and $a \in A$.

Suppose $b \notin A$, i.e. $bR \in \mathscr{R}$. In this case $\{R_1, Ra, bR, R_2\}$ is a path between R_1 and R_2, unless $Ra = R$. Assume $Ra = R$. We then have

$$Ra = R = aR,$$

which implies that R is left unital and the proof follows from Theorem 2.

On the other hand if $b \in A$, then $\{R_1, A, R_2\}$ is a path between R_1 and R_2, unless $A = R$. Let us assume that $A = R$.

CASE 1. *R satisfies A.C.C.*

Since $aR = R$, there exists a sequence $\{x_i\} \subset R$ such that $a = ax_1 = a^2x_2 = \cdots$, where $x_i = ax_{i+1}$ for $i = 1, 2, \cdots$. We then have $Ra \subset Rx_1 \subset Rx_2 \subset \cdots$. Since R satisfies A.C.C. there exists an n such that $Rx_n = Rx_{n+1}$, which implies that $x_n = ax_{n+1} \in Rx_{n+1} = Rx_n$. Thus there exists an element $e \in R$ such that $x_n = ex_n$. We also have $x_n R = R$ (since $A = R$). Let $x \in R$. There exists $y \in R$ such that $x = x_n y$ and $ex = e(x_n y) = (ex_n)y = x_n y = x$. Thus e is a left unit for R and the proof follows from Theorem 1.

CASE 2. *R satisfies D.C.C.*

It R satisfies D.C.C., we see from $Ra \supset Ra^2 \supset Ra^3 \supset \cdots$, that for some m; $Ra^m = Ra^{m+1}$. Thus $a^{m+1} \in Ra^m = Ra^{m+1}$, which implies that there exists an e^* such that $a^{m+1} = e^*a^{m+1}$ and by the same argument as in Case 1 we can show that e^* is a left unit for R and the proof follows from Theorem 1.

REMARK. In [3] we showed that the graph of a semiring R with two elements is not necessarily connected.

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The following example shows that the diameter $D(R)$ of the graph of semiring R is equal to three.

EXAMPLE. Let $R = \{e, a, b\}$ be a semiring with the following addition and multiplication tables. The graph $G(\mathscr{R})$ is illustrated in Fig. 1.

+	e	a	b
e	e	b	b
a	b	a	b
b	b	b	b

.	e	a	b
e	e	a	b
a	a	a	a
b	b	a	b

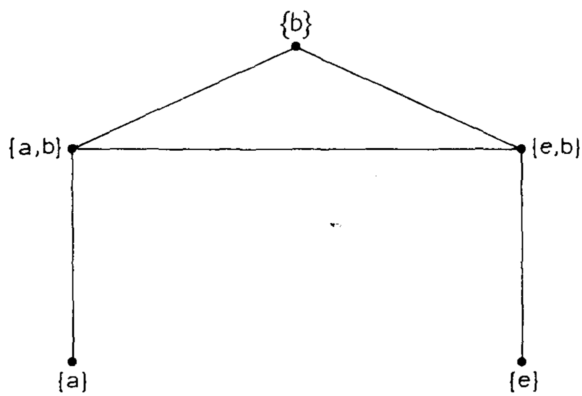

Figure 1

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