Canad. Math. Bull. Vol. 18 (2), 1975

L^1-SEQUENTIAL CONVERGENCE ON STONE SPACES AND STONE'S THEOREM ON UNITARY SEMIGROUPS

BY
J. B. COOPER

The following theorem is a well-known tool in the study of measurable functions:

THEOREM. *Let* $(M; \mu)$ *be a finite measure space and let* (x_n) *be a sequence of functions in* $L^1(M; \mu)$ *so that* $x_n \to 0$ *in the norm of* $L^1(M; \mu)$. *Then there is a subsequence so that* $x_{n_k} \to 0$ *pointwise almost everywhere on* M.

In this note, we show that under certain circumstances this theorem can be strengthened in the sense that a subsequence can be found which converges everywhere and we use this result to prove a strengthened form of Stone's theorem on one parameter groups of unitary operators in a Hilbert space.

1. DEFINITION. A *Stone space* is a compact topological space in which the closure of every open set is open.

2. NOTATION. If M is a compact topological space, $C(M)$ denotes the space of continuous, complex-valued functions on M. We regard $C(M)$ as a Banach space in the usual way.

3. PROPOSITION. *Let M be a Stone space, μ a positive Radon measure on M whose support is M. Then if (x_n) is a sequence in $C(M)$ so that $x_n \to 0$ in $L^1(M; \mu)$, (x_n) contains a subsequence (x_{n_k}) so that $x_{n_k} \to 0$ pointwise on M.*

Proof. We can construct a double sequence (x_{np}) so that each $(x_{np})_{p=1}^{\infty}$ is a subsequence of $(x_{n-1,p})_{p=1}^{\infty}$ $(n>1)$ and so that if $M_{np} := \{t \in M : |x_{np}(t)| \geq 2^{-n-p}\}$, then $\mu(M_{np}) \leq 2^{-n-p}$. Then $U_{np} := \text{clos}\{t \in M : |x_{np}(t)| > 2^{-n-p}\}$ is a closed–open set in M with $\mu(U_{np}) \leq 2^{-n-p}$ and so $U_n := \cup_{p=1}^{\infty} U_{np}$ is a closed–open set with $\mu(U_n) \leq 2^{-n}$. Clearly $(x_{nn})_{n=1}^{\infty}$ converges pointwise on the complement of $U := \cap_{n=1}^{\infty} U_n$ and U is an open set whose measure is zero. Hence U is the empty set.

Now let A be a commutative C^*-algebra of (bounded) linear operators in the Hilbert space H and denote by M the spectrum of A. We suppose in addition that A possesses a cyclic vector x_0, that is, Ax_0 is dense in H. Let

$$T \mapsto \hat{T}$$

denote the Gelfand-Neumark transform from A onto $C(M)$ and let μ be the positive Radon measure

$$\hat{T} \mapsto (Tx_0 \mid x_0)$$

Received by the editors November 14, 1973.

https://doi.org/10.4153/CMB-1975-038-6 Published online by Cambridge University Press

on M. Then there is a unitary mapping U from H onto $L^2(M; \mu)$ which transfers T onto the operator of multiplication by $\hat{T}$.

4. LEMMA. (i) *the support of* μ *is* M;

(ii) *if* A *is maximal commutative, then* M *is a Stone space.*

Proof. (i) We consider a positive continuous function on M, which we can take to be of the form $\hat{T}(T \in A)$, so that $\int \hat{T}\, d\mu = 0$ i.e. $(Tx_0 \mid x_0) = 0$. Then $T^{1/2}x_0 = 0$. Hence, for each $S \in A$,

$$T^{1/2}Sx_0 = S(T^{1/2}x_0) = 0$$

and so $T^{1/2} = 0$ since $\{Sx_0 : S \in A\}$ is dense in H.

(ii) It is sufficient to show that every two disjoint open sets in M have disjoint closures. We first remark that it follows immediately from the maximal commutativity of A that the injection from $C(M)$ into $L^\infty(M; \mu)$ is surjective. Let U, V be disjoint open sets in M. Then χ_U and χ_V, the characteristic sets of U and V, are in $L^\infty(M; \mu)$ and so there is an x_U (resp. an x_V) in $C(M)$ so that $\chi_U = x_U$ almost everywhere (resp. $\chi_V = x_V$ almost everywhere). Now

$$U \subseteq \{t \in M : x_U(t) > \tfrac{2}{3}\},\ V \subseteq \{t \in M : x_V(t) > \tfrac{2}{3}\}$$

and

$$\bar{U} \cap \bar{V} \subseteq W := \{t \in M : x_U(t) > \tfrac{1}{3} \wedge x_V(t) > \tfrac{1}{3}\}.$$

But W is open and $\mu(W) = 0$—hence $W = \phi$.

5. LEMMA. *Let* x *be a mapping from* $\mathscr{R}$ *into the circle group* T *so that*

(i) *if* (t_n) *is a null-sequence in* $\mathscr{R}$, *there is a subsequence* (t_{n_k}) *so that* $x(t_{n_k}) \to 1$;

(ii) $x(t_1 + t_2) = x(t_1)x(t_2)$ *for each* $t_1, t_2 \in \mathscr{R}$,

Then x *is continuous and so there is an* $h \in \mathscr{R}$ *so that*

$$x(t) = \exp(iht) \qquad (t \in \mathscr{R}).$$

6. NOTATION. Let $(M; \mu)$ be a measure space. Then if $x \in L^\infty(M; \mu)$, we denote by M_x the multiplication operator

$$y \mapsto xy$$

on the Hilbert space $L^2(M; \mu)$.

If h is a real-valued measurable function on M, then the mapping

$$\mathscr{T}_h : t \mapsto M_{\exp(iht)}$$

is a strongly continuous one parameter semigroup of unitary operators on $L^2(M;\mu)$.

7. PROPOSITION. *Let* $\mathscr{T}$ *be a unitary strongly continuous one parameter group of operators in a Hilbert space* H. *Then there is a locally compact space* M, *a positive Radon measure* μ *on* M, *a measurable real-valued function* h *on* M, *and a unitary mapping* U *from* H *onto* $L^2(M; \mu)$ *so that*

$$\mathscr{T}(t) = U\mathscr{T}_h(t)U^{-1} \qquad (t \in \mathscr{R})$$

Proof. Let A be a maximal commutative C^*-algebra of operators containing $\{\mathcal{T}(t):t\in\mathcal{R}\}$. We assume at first that A has a cyclic vector (which is always the case when H is separable). Let M and μ be as above and let U be the associated unitary operator from H onto $L^2(M;\mu)$. Then, for each $t\in\mathcal{R}$, there is a continuous function x_t on M so that

$$\mathcal{T}(t) = UM_{x_t}U^{-1}.$$

From the group condition, it follows that

$$x_{t_1+t_2}(m) = x_{t_1}(m)x_{t_2}(m) \qquad (m\in M,\ t_1,\ t_2\in\mathcal{R}).$$

Also, if $t_n\to 0$, then, since $\mathcal{T}(t_n)$ tends strongly to the identity, so does $M_{x_{t_n}}$. Hence x_{t_n} converges to the constant function 1 in $L^1(M;\mu)$. Thus, by Prop. 3, there is a subsequence $(x_{t_{n_k}})$ which converges pointwise to 1 on M. Hence, by Lemma 5, the mapping

$$t\mapsto x_t(m)$$

from $\mathcal{R}$ into T has the form

$$t\mapsto \exp(ith(m))$$

where $h(m)=\lim_{n\to\infty} n(x_{1/n}(m)-1)$. This implies that the function

$$m\mapsto h(m)$$

is measurable.

In the general case (i.e. where A does not have a cyclic vector), we can express H as the direct sum of subspaces, invariant under A, each of which has a cyclic vector with respect to A and so deduce the result using standard techniques.

The usual form of Stone's theorem can easily be deduced from the above result.

8. Bibliographical remark. The results on Hilbert space used in this note can be found in Segal and Kunze, *Integrals and operators* (McGraw-Hill, 1968). In particular, the form of Stone's theorem given here (Prop. 7) is given as Theorem 10.3 in this reference.

The proof of Prop. 3 is a simple variation of a standard type of argument used in integration theory. See, for example, the proof of IV, §5, No. 11, Lemme 4 in Bourbaki, *Intégration* (Hermann, 1965).

As a final remark, I would like to thank Kurt Plasser who pointed out an error in the original proof of Prop. 3.

Mathematisches Institut
Hochschule Linz
A4045 Linz
Austria